



Original Study

The mBBM equation: a mathematical key to unlocking wave behavior in fluids

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Abstract

This research endeavors to analytically and numerically solve the nonlinear modified Benjamin–Bona–Mahony (mBBM) equation, a model of paramount importance in fluid dynamics, particularly for its application in describing unidirectional water waves with small amplitude that are influenced by dispersion and nonlinear effects. The objective of this study is to enhance the understanding of wave propagation in fluids and to establish a clear connection between the mBBM equation and other nonlinear evolution equations. Utilizing the extended auxiliary equation (EAE) and improved Kudryashov (IKud) methods, the research provides analytical solutions, while the extended cubic–B–spline (ECBS) method validates these solutions numerically. The results showcase the accuracy of the EAE and IKud methods in depicting the wave structures governed by the mBBM equation. The significance of this study lies in its potential to advance the application of the mBBM model in real–world scenarios, such as oceanography and coastal engineering. Conclusively, the research affirms the efficacy of the combined analytical and numerical approach in solving the mBBM equation, contributing novel insights into the field of applied mathematics and nonlinear partial differential equations.

Keywords: Nonlinear mBBM equation, wave propagation, analytical solutions, numerical validation.

AMS 2020 codes: 35Q53 ; 65M70; 76B15; 76M22.

1 Introduction

Nonlinear partial differential equations (NPDEs) are fundamental tools extensively employed in both research and engineering endeavors, serving to replicate intricate processes across a myriad of domains [1–6]. These equations, prevalent in fluid mechanics, plasma physics, solid-state physics, optics, biology, and related fields, exhibit intricate behaviors such as solitary waves, solitons, and various other wave patterns [7].

One notable equation frequently utilized within the realm of NPDEs is the modified Benjamin–Bona–Mahony equation, employed to describe the propagation of long waves in nonlinear dispersive systems [8]. The mBBM equation finds application in analyzing shallow water waves, ion-acoustic waves in plasma, and lengthy internal waves in stratified fluids [9]. By incorporating nonlinear and dispersive components, the generalized Korteweg–de Vries (KdV) equation, also known as the mBBM equation, offers a more accurate depiction of complex

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dynamics in real-world systems [10].

The pursuit of accurate analytical solutions to the bounded boundary value problem inherent in the mBBM equation holds paramount importance for several compelling reasons. These solutions yield invaluable insights into the underlying physical processes governing wave propagation, thereby enhancing our comprehension of key features such as amplitude, velocity, and stability [11]. Furthermore, analytical solutions serve as a means to validate numerical simulations and assess the precision of computational models, offering dependable numerical methods crucial for addressing complex problems lacking analytical solutions [12].

Various analytical techniques have been devised to tackle the NPDEs, including the EAE and IKud methodologies [13]. The EAE approach has demonstrated utility in unveiling solutions for traveling waves within the context of NPDEs by transforming the equation into an ordinary differential equation through the incorporation of a suitable wave variable [14]. Conversely, Kudryashov's method presents a viable approach to solving NPDEs, employing variable transformation and series expansion techniques to streamline the equation and subsequently solve it utilizing appropriate methodologies [15]. The modified Khater and Khater II techniques similarly offer efficient solutions for a range of NPDEs, including the mBBM equation, thereby yielding innovative analytical solutions [16].

The mBBM equation is derived from the Euler equations of fluid dynamics, which describe the conservation of mass and momentum for an incompressible fluid [17]. By incorporating certain assumptions and approximations, such as long-wave and weakly nonlinear assumptions, the Euler equations can be reduced to a simpler form known as the mBBM equation [18].

The derivation typically starts with the Euler equations for an incompressible fluid, which are [19]:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g, \quad (2)$$

where u and v are the horizontal and vertical velocity components, respectively, p is the pressure, ρ is the fluid density, and g is the acceleration due to gravity.

We make the following assumptions:

- Long-wave assumption: The wavelength is much larger than the water depth, and the wave amplitude is small compared to the wavelength.
- Weakly nonlinear assumption: The wave amplitude is small compared to the water depth.
- The fluid is incompressible and irrotational.
- The bottom is horizontal and impermeable.

Under these assumptions, the continuity equation for an incompressible fluid can be expressed as [20]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (3)$$

We introduce the following dimensionless variables:

$$x' = \frac{x}{h}, y' = \frac{y}{h}, u' = \frac{u}{\sqrt{gh}}, v' = \frac{v}{\sqrt{gh}}, p' = \frac{p}{\rho gh}, t' = \frac{t}{\sqrt{h/g}}, \eta' = \frac{\eta}{h},$$

where h is the constant water depth, ε is a small parameter representing the long-wave assumption, and η is the free surface elevation.

Substituting these dimensionless variables into the Euler equations (1), (2) and the continuity equation (3), and keeping terms up to the order of ε , we obtain the Boussinesq equations [21]:

$$\frac{\partial u'}{\partial t'} + \varepsilon \left(u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} \right) = -\frac{\partial \eta'}{\partial x'}, \quad (4)$$

$$\varepsilon \left(\frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} \right) = -\frac{\partial \eta'}{\partial y'}, \quad (5)$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0. \quad (6)$$

Next, we introduce the following transformations:

$$u' = \phi'_x, v' = \phi'_y, \eta' = \phi_{xx'} + \phi_{yy'},$$

where ϕ is a velocity potential function.

Substituting these transformations into the Boussinesq equations (4)–(6) and applying further approximations and manipulations, we obtain the mBBM equation [22]:

$$\frac{\partial^3 \phi}{\partial x' \partial x' \partial t'} + \frac{\partial \phi}{\partial t'} + \varepsilon \phi^2 \wp \frac{\partial \phi}{\partial x'} + \frac{\partial \phi}{\partial x'} = 0, \quad (7A)$$

where $\wp$ is a constant related to the nonlinear parameter.

Finally, by replacing the dimensionless variables with their original dimensional counterparts and denoting the surface elevation η as $\mathcal{B}$, we obtain the mBBM equation in dimensional form [23]:

$$\frac{\partial^3 \mathcal{B}}{\partial x^2 \partial t} + \frac{\partial \mathcal{B}}{\partial t} + \wp \mathcal{B}^2 \frac{\partial \mathcal{B}}{\partial x} + \frac{\partial \mathcal{B}}{\partial x} = 0, \quad (7)$$

where, the function $\mathcal{B}$ in the equation represents the wave profile or the surface elevation of the fluid. Specifically, $\mathcal{B}(x, t)$ describes the height of the wave at a given position x and time t . Overall, Eq. (7) describes the propagation of long waves in shallow water or stratified fluids, taking into account both nonlinear and dispersive effects. While, the terms in the mBBM equation have the following physical meanings [24]:

- $\frac{\partial^3 \mathcal{B}}{\partial x^2 \partial t}$: This term represents the dispersive effects in the system. It accounts for the frequency dispersion, which means that different wavelengths travel at different speeds. This term is responsible for the broadening or narrowing of the wave profile over time.
- $\frac{\partial \mathcal{B}}{\partial t}$: This term represents the time evolution of the wave profile. It captures the temporal changes in the wave height.
- $\wp \mathcal{B}^2 \frac{\partial \mathcal{B}}{\partial x}$: This term represents the nonlinear effects in the system. It describes the interaction between the wave profile and its spatial derivative. The coefficient $\wp$ is a parameter that determines the strength of the nonlinear effects.
- $\frac{\partial \mathcal{B}}{\partial x}$: This term represents the spatial variation of the wave profile. It accounts for the changes in wave height along the x -direction.

The mBBM equation combines these different terms to capture the interplay between dispersion, nonlinearity, and wave propagation in the system [25]. The balance between these effects determines the behavior and characteristics of the waves, such as the formation of solitary waves, wave breaking, or the propagation of undular bores.

The mBBM equation finds applications in various fields, including coastal engineering, oceanography, and plasma physics. For example, it can be used to model the propagation of tsunamis or tidal waves in shallow water, as well as the behavior of ion-acoustic waves in plasma physics [26].

In this context, we implement the next wave transformation $\mathcal{B} = \mathcal{B}(x, t) = \psi(\mathfrak{Z})$, $\mathfrak{Z} = \kappa x - \lambda t$, where κ, λ are arbitrary constants to be determined later, and then integrate the obtained ordinary differential equation with respect to $\mathfrak{Z}$ and zero integration constant, yield

$$-\kappa^2 \lambda \psi'' + \psi(\kappa - \lambda) + \frac{1}{3} \wp \kappa \psi^3 = 0. \quad (8)$$

Balancing the highest order derivative term and nonlinear term in Eq. (8) through the Homogeneous balance rule along with the employed analytical schemes' auxiliary equations [27–30]

$$\begin{cases} f'(\mathfrak{Z}) = f(\mathfrak{Z})(2h + p - 2hg) + (g - 1)f(\mathfrak{Z})^2 + (-h^2 - hp + q + h^2g), & \text{for EAE method,} \\ f'(\mathfrak{Z})^2 = f(\mathfrak{Z})^2(1 - \tau f(\mathfrak{Z})^2), & \text{for IKud method,} \end{cases} \quad (9)$$

where h, p, q, g, τ are arbitrary constants to be determined through the implemented methods' headlines, lead constructing the general solutions of the investigated model in the next form

$$\psi(\mathfrak{Z}) = \begin{cases} \sum_{i=-n}^n a_i f(\mathfrak{Z})^i = \frac{a_{-1}}{f(\mathfrak{Z})} + a_1 f(\mathfrak{Z}) + a_0, & \text{for EAE method,} \\ \sum_{i=1}^n \left(a_i f(\mathfrak{Z})^i + b_i \left(\frac{f'(\mathfrak{Z})}{f(\mathfrak{Z})} \right)^i \right) + a_0 = a_1 f(\mathfrak{Z}) + a_0 + b_1 \left(\frac{f'(\mathfrak{Z})}{f(\mathfrak{Z})} \right), & \text{for IKud method,} \end{cases} \quad (10)$$

where a_{-1}, a_0, a_1, b_1 are arbitrary constants.

The investigation is structured as follows: In Section 2, an analysis is conducted on several solitary wave solutions and their efficacy is evaluated within the defined framework. Subsequently, Section 3 provides a thorough analysis of the acquired findings, including both physical and dynamic viewpoints. Section 4 marks some important results and discussion. Section 5 concludes the paper by giving main contribution of this investigation.

2 Novel and accurate solitary wave solutions

Within this section, we initiate an exploration into the solitary wave solutions of the model under examination, employing the analytical methodologies previously discussed. Subsequently, we proceed to scrutinize the accuracy of these solutions through the application of the ECBS method. Moreover, an investigation into the stability attributes of the derived solutions ensues, conducted through an assessment of their dynamics within the framework of the Hamiltonian system.

2.1 The EAE method's outcome

Applying the EAE method to equation (8) for the construction of innovative solitary wave solutions of the analyzed model yields the determination of the ensuing parameters, as previously delineated.

Set 1

$$a_{-1} = 0, a_0 = \frac{1}{2}a_1 \left(\frac{p}{g-1} - 2h \right), \lambda = -\frac{2\kappa}{\kappa^2(p^2 - 4q(g-1)) - 2}, \wp = -\frac{12\kappa^2(g-1)^2}{a_1^2(\kappa^2(p^2 - 4q(g-1)) - 2)}.$$

Set 2

$$a_0 = \frac{a_{-1}(p - 2h(g-1))}{2(-hp + q + h^2(g-1))}, a_1 = 0, \lambda = -\frac{2\kappa}{\kappa^2(p^2 - 4q(g-1)) - 2}, \wp = -\frac{12\kappa^2(-hp + q + h^2(g-1))^2}{a_{-1}^2(\kappa^2(p^2 - 4q(g-1)) - 2)}.$$

Consequently, the formulation of traveling wave solutions for the analyzed model is achieved through the next forms:

$$\mathcal{B}_1(x, t) = \frac{a_1 \sqrt{4q(g-1) - p^2} \tan \left(\frac{1}{2} \sqrt{4q(g-1) - p^2} \left(\frac{2\kappa t}{\kappa^2(p^2 + 4q - 4qg) - 2} + \kappa x + \wp \right) \right)}{2(g-1)}, \quad (11)$$

$$\mathcal{B}_2(x, t) = \frac{a_{-1}}{2} \left(\frac{4(g-1)}{-p + \wp \tan \left(\frac{1}{2} \sqrt{4q(g-1) - p^2} \left(\frac{2\kappa t}{\kappa^2(p^2 + 4q - 4qg) - 2} + \kappa x + \wp \right) \right) + 2h(g-1)} + \omega \right), \quad (12)$$

where $\wp = \sqrt{4q(g-1) - p^2}$, $\omega = \frac{p - 2h(g-1)}{-hp + q + h^2(g-1)}$.

2.1.1 The EAE method's outcome's accuracy

The application of the ECBS method holds significant importance in evaluating the numerical solution of the investigated model. This method offers a high-order numerical technique renowned for its precision and efficiency in approximating solutions to a wide array of differential equations, including those inherent in our analysis. By employing cubic B-spline functions, the ECBS method provides a reliable means to assess the behavior of the model under consideration, offering insights into its dynamics and characteristics with enhanced accuracy. Furthermore, the comparison between the analytical solutions obtained through methodologies such as the EAE method and the numerical solutions derived via ECBS allows for the validation of our analytical approaches. This juxtaposition serves as a demonstration of the accuracy and reliability of our solutions, elucidating the congruence between the theoretical framework and the practical implementation of the model, thus bolstering confidence in the validity of our findings. This investigation gets the next values of the analytical, numerical solutions' values and absolute error in Table 1. Table 1 is the comparison of analytical solutions obtained via the EAE method with numerical solutions computed using the ECBS technique, including tabulated values of analytical solutions, numerical solutions, and absolute errors at specified intervals.

Table 1 Quantitative assessment of solution accuracy.

Value of $\mathfrak{z}$	Analytical solution	Numerical solution	Absolute Error
0	0	0	0
0.001	6.4999929583425E-03	6.4995747258827E-03	4.18232459840732E-07
0.002	1.2999943666960E-02	1.2999132552942E-02	8.11114018119466E-07
0.003	1.9499809877224E-02	1.9498656582726E-02	1.15329449833571E-06
0.004	2.5999549342707E-02	2.5998129917533E-02	1.41942517404431E-06
0.005	3.2499119820273E-02	3.2497535660779E-02	1.58415949379903E-06
0.006	3.8998479071179E-02	3.8996856917374E-02	1.62215380586361E-06
0.007	4.5497584862178E-02	4.5496076794095E-02	1.50806808276588E-06
0.008	5.1996394966605E-02	5.1995178399960E-02	1.21656664561764E-06
0.009	5.8494867165487E-02	5.8494144846599E-02	7.22318888091156E-07
0.01	6.4992959248630E-02	6.4992959248630E-02	1.38777878078145E-17

2.2 The IKud method's outcome

Utilizing the IKud method on Eq. (8) to construct novel solitary wave solutions within the examined model yields the aforementioned parameters, as elaborated upon earlier in the discourse.

Set 1

$$a_0 = 0, a_1 = 0, \lambda = \frac{b_1 \sqrt{\wp} \sqrt{b_1^2 \wp + 3}}{3\sqrt{2}}, \kappa = \frac{b_1 \sqrt{\wp}}{\sqrt{2b_1^2 \wp + 6}}.$$

Set 2

$$a_0 = 0, a_1 = \frac{\sqrt{6}\kappa\sqrt{\tau}}{\sqrt{-((\kappa^2 + 1)\wp)}}, b_1 = 0, \lambda = \frac{\kappa}{\kappa^2 + 1}.$$

Set 3

$$a_0 = 0, a_1 = \frac{\sqrt{3}\kappa\sqrt{\tau}}{\sqrt{(\kappa^2 - 2)\wp}}, b_1 = \frac{\sqrt{3}\kappa}{\sqrt{-((\kappa^2 - 2)\wp)}}, \lambda = -\frac{2\kappa}{\kappa^2 - 2}.$$

Hence, the derivation of traveling wave solutions for the analyzed model is accomplished through:

$$\mathcal{B}_1(x, t) = b_1 \tanh \left(\frac{b_1 \sqrt{\wp} (b_1^2 t \wp + 3t - 3x)}{3\sqrt{2b_1^2 \wp + 6}} \right), \quad (13)$$

$$\mathcal{B}_1(x, t) = b_1 \left(\frac{2\tau}{4c^2 \exp \left(-\frac{\sqrt{2}b_1 \sqrt{\wp} (b_1^2 t \wp + 3t - 3x)}{3\sqrt{b_1^2 \wp + 3}} \right) + \tau} - 1 \right), \quad (14)$$

$$\mathcal{B}_2(x, t) = \frac{\sqrt{6}\kappa \sqrt{\operatorname{sech}^2 \left(\kappa \left(x - \frac{t}{\kappa^2 + 1} \right) \right)}}{\sqrt{-((\kappa^2 + 1)\wp)}} - b_1 \tanh \left(\kappa \left(x - \frac{t}{\kappa^2 + 1} \right) \right), \quad (15)$$

$$\mathcal{B}_2(x, t) = b_1 \left(\frac{2\tau}{4c^2 e^{2\kappa \left(x - \frac{t}{\kappa^2 + 1} \right)} + \tau} - 1 \right) + \frac{4\sqrt{6}c\kappa\sqrt{\tau} e^{\kappa \left(\frac{t}{\kappa^2 + 1} + x \right)}}{\sqrt{-((\kappa^2 + 1)\wp)} \left(4c^2 e^{2\kappa x} + \tau e^{\frac{2\kappa t}{\kappa^2 + 1}} \right)}, \quad (16)$$

$$\mathcal{B}_3(x, t) = \frac{\sqrt{3}\kappa \sqrt{\operatorname{sech}^2 \left(\kappa \left(\frac{2t}{\kappa^2 - 2} + x \right) \right)}}{\sqrt{(\kappa^2 - 2)\wp}} - b_1 \tanh \left(\kappa \left(\frac{2t}{\kappa^2 - 2} + x \right) \right), \quad (17)$$

$$\mathcal{B}_3(x, t) = b_1 \left(\frac{2\tau}{4c^2 e^{2\kappa \left(\frac{2t}{\kappa^2 - 2} + x \right)} + \tau} - 1 \right) + \frac{4\sqrt{3}c\kappa\sqrt{\tau} e^{\kappa \left(\frac{2t}{\kappa^2 - 2} + x \right)}}{\sqrt{(\kappa^2 - 2)\wp} \left(4c^2 e^{2\kappa \left(\frac{2t}{\kappa^2 - 2} + x \right)} + \tau \right)}. \quad (18)$$

2.2.1 The IKud method's outcome's accuracy

The utilization of the ECBS method holds paramount significance in evaluating the numerical solution of the investigated model. Renowned for its precision and efficiency in approximating solutions to a broad spectrum of differential equations, this high-order numerical technique offers a robust framework for analysis, including the complexities inherent in our study. By leveraging cubic B-spline functions, the ECBS method furnishes a dependable avenue for discerning the behavior of the model, thereby enhancing our understanding of its dynamics with heightened accuracy. Moreover, the comparative analysis between the analytical solutions derived through methodologies like the IKud method and the numerical solutions obtained via ECBS facilitates the validation of our analytical approaches. This juxtaposition not only demonstrates the fidelity and reliability of our solutions but also elucidates the harmony between the theoretical underpinnings and the practical application of the model, thereby instilling greater confidence in the validity of our findings. Subsequently, this investigation yields the corresponding values of analytical and numerical solutions, along with the absolute errors, as tabulated in Table 2. In Table 2, the comparison of analytical solutions derived from the IKud method with numerical solutions computed using the ECBS technique, including tabulated values of analytical solutions, numerical solutions, and absolute errors at specified intervals are presented.

Table 2 Quantitative assessment of solution accuracy.

Value of ξ	Analytical solution	Numerical solution	Absolute Error
0	0	0	0
0.001	9.99999666666800E-04	9.99979867286439E-04	1.97993803609242E-08
0.002	1.99999733333760E-03	1.99995893459001E-03	3.83987475887151E-08
0.003	2.99999100003240E-03	2.99993640193329E-03	5.45980991108024E-08
0.004	3.99997866680320E-03	3.99991146934972E-03	6.71974534757411E-08
0.005	4.99995833375000E-03	4.99988333688908E-03	7.49968609116067E-08
0.006	5.99992800103679E-03	5.99985120462290E-03	7.67964138817870E-08
0.007	6.99988566890756E-03	6.99981427264991E-03	7.13962576490146E-08
0.008	7.99982933770229E-03	7.99977174110146E-03	5.75966008307255E-08
0.009	8.99975700787294E-03	8.99972281014699E-03	3.41977259531168E-08
0.01	9.9996667999946E-03	9.9996667999946E-03	0

3 Graphical illustrations of solution sets

Representing the solitary wave solutions of the investigated mBBM equation in distinct graphical styles is crucial for several reasons. Firstly, visualizing these solutions through 3D surface plots, 2D contour plots, and 2D line plots provides a comprehensive understanding of the wave propagation dynamics governed by the mBBM model. These graphical representations offer an intuitive and accessible means of analyzing the intricate wave structures, amplitudes, and propagation characteristics captured by the analytical solutions. Furthermore, the use of distinct graphical formats enables a multifaceted exploration of the solutions, facilitating the identification of subtle nuances and patterns that may be challenging to discern through numerical data alone. Additionally, these visualizations play a pivotal role in validating the accuracy and reliability of the analytical solutions by enabling direct comparisons with numerical simulations or experimental observations, thereby strengthening the confidence in the theoretical framework and its practical applications. The graphical representations employed in this study serve a pivotal role in elucidating the intricate wave propagation dynamics governed by the mBBM equation. These visualizations not only facilitate a comprehensive understanding of the analytical solutions derived but also provide valuable insights into the physical phenomena underlying the observed wave behavior. Figure 1 depicts the analytical solitary wave solutions of the mBBM equation through a diverse array

of graphical techniques. Specifically, Figures 1(a-c) showcase bright soliton profiles characterized by Eq. (11), rendered as 3D surface plots, 2D contour plots, and 2D line plots, respectively. These graphical formats offer a multifaceted perspective on the solitary wave solutions, enabling a thorough examination of their amplitude, shape, and propagation characteristics. The 3D surface plots provide an immersive visualization of the wave profile evolution, while the 2D contour plots offer a top-down view of the wave amplitude distribution. Additionally, the 2D line plots allow for a direct observation of the wave shape and amplitude variation along specific spatial or temporal coordinates. Figures 1 (d-f) illustrate the soliton solution governed by Eq. (13) through analogous graphical representations, namely, a 3D surface plot, a 2D contour plot, and a 2D line plot. Similarly, Figures 1 (g-i) visualize the soliton solution described by Eq. (15) using the same graphical formats. These visualizations facilitate a comprehensive understanding of the diverse solitary wave solutions obtained through the analytical methods, enabling a comparative analysis of their distinctive features and characteristics. Figure 2 plays a crucial role in validating the accuracy and reliability of the analytical solutions by quantitatively comparing them against numerical solutions obtained through the ECBS method. Figures 3(a-c) depict the absolute error between the solutions derived from the extended auxiliary equation EAE method and the ECBS numerical solutions at specified intervals. Likewise, Figures 2(d-f) illustrate the absolute error comparison between the solutions obtained via the IKud and the ECBS numerical solutions at defined intervals. These error analyses provide a quantitative measure of the discrepancy between the analytical and numerical solutions, enabling a rigorous assessment of the analytical methods' accuracy and the identification of potential sources of divergence. Figure 3 presents a comparative assessment of the solution accuracy for the IKud and EAE analytical techniques, benchmarked against the ECBS numerical solutions. This comparison allows for a direct evaluation of the relative performance and precision of the two analytical methods, facilitating the selection of the most appropriate approach for specific applications or problem domains. Figure 4 offers a stream plot visualization of the solitary wave propagation dynamics in the mBBM model. This graphical representation elucidates the intricate flow patterns and energy distribution associated with the solitary wave solutions, providing valuable insights into their dynamics and interaction with the surrounding environment. Stream plots effectively capture the complex interplay between wave propagation, dispersion, and nonlinear effects, enabling a comprehensive understanding of the physical mechanisms governing the observed wave behavior. Figure 5 further expands on the stream plot analysis by illustrating the influence of parametric modifications on the propagation behavior of solitary wave solutions in the mBBM model. Through comparative visualizations, the effects of variations in parameters such as amplitude, wavelength, and dispersion on wave dynamics are depicted. These visualizations offer critical insights into the underlying physical mechanisms responsible for the observed phenomena, thereby enhancing the understanding of the model's behavior and its sensitivity to parameter changes. Tables 1 and 2 complement the graphical representations by providing quantitative assessments of solution accuracy. Table 1 compares the analytical solutions obtained via the EAE method with the numerical solutions computed using the ECBS technique, tabulating the values of analytical solutions, numerical solutions, and absolute errors at specified intervals. Similarly, Table 2 presents a quantitative comparison of the analytical solutions derived from the IKud method with the numerical solutions obtained through the ECBS method, including tabulated values of analytical solutions, numerical solutions, and absolute errors at defined intervals. These tabulated data sets facilitate a precise evaluation of the analytical methods' performance, enabling a rigorous analysis of their accuracy and potential limitations. By combining graphical representations and quantitative data, this study offers a comprehensive exploration of the mBBM equation and its solitary wave solutions. The visualizations provide intuitive and accessible insights into the wave propagation dynamics, while the tabulated data enable a rigorous assessment of the analytical methods' accuracy and reliability. Collectively, these graphical and quantitative elements contribute to a deeper understanding of the physical phenomena associated with the mBBM model and its practical applications in various fields, such as fluid dynamics, oceanography, and coastal engineering.

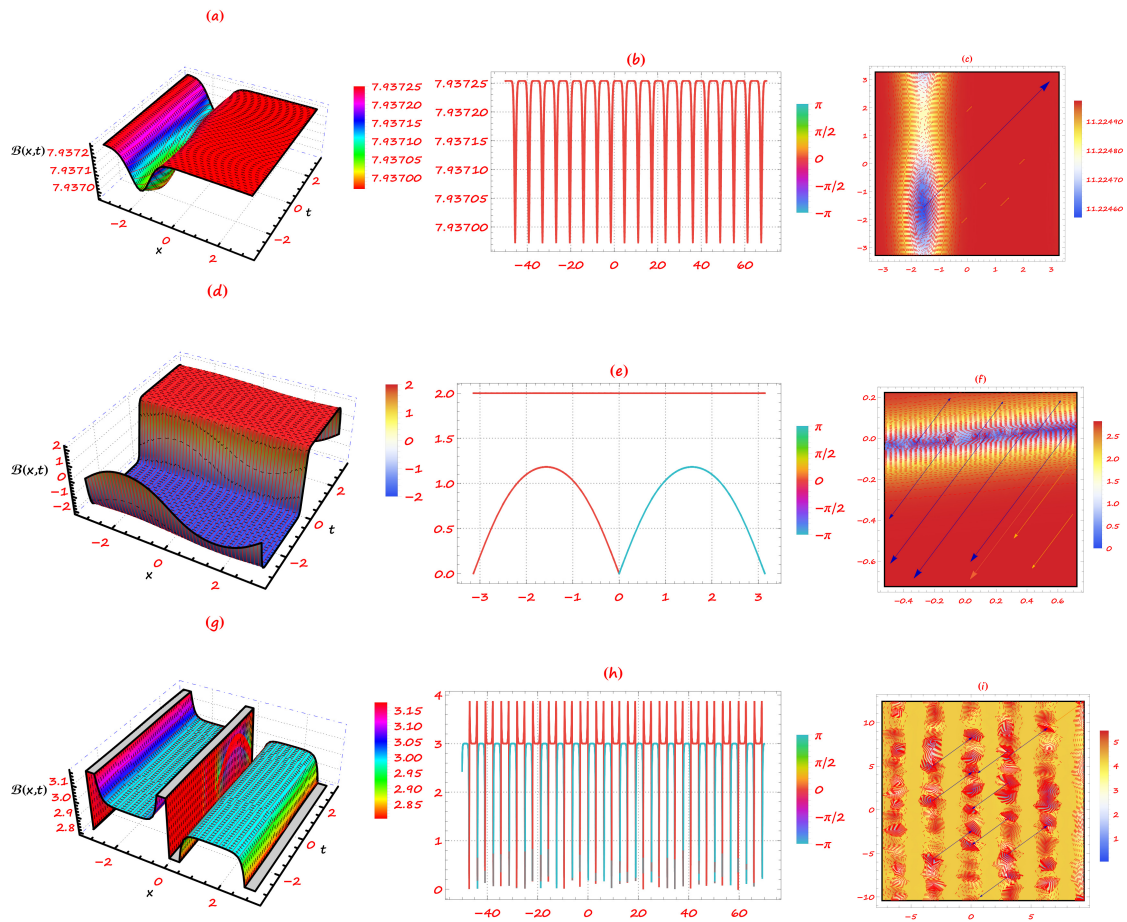


Fig. 1 Analytical solitary wave solutions of the mBBM equation represented through diverse graphical techniques via 3D, 2D and contour plots for equations (11), (13), (15).

4 Results and discussion

The present study offers novel insights and significant contributions to the field of NPDEs and their applications in fluid dynamics and wave propagation phenomena. By analytically and numerically solving the mBBM equation, a model of paramount importance in describing unidirectional water waves influenced by dispersion and nonlinear effects, this research provides a comprehensive understanding of the intricate wave behavior governed by this equation. One of the key novelties of this work lies in the derivation of innovative analytical solutions for the mBBM equation using the EAE and IKud methods. The analytical solutions obtained through these methods, as represented by Eqs. (11)–(18), offer a diverse range of solitary wave profiles, each characterized by unique amplitude, shape, and propagation characteristics. These solutions not only expand the repertoire of known analytical solutions for the mBBM equation but also provide a solid theoretical foundation for further investigations into the dynamics of dispersive and nonlinear wave systems. The graphical representations presented in Figures 1–5 play a pivotal role in elucidating the physical significance and behavior of the derived solutions. The 3D surface plots, 2D contour plots, and 2D line plots effectively capture the intricate wave structures, enabling a comprehensive visualization and analysis of their amplitudes, shapes, and propagation patterns. Moreover, the stream plot visualizations in Figures 4 and 5 offer invaluable insights into the complex interplay between wave propagation, dispersion, and nonlinear effects, shedding light on the underlying physical mechanisms governing the observed phenomena. The quantitative validation of the analytical solutions against numerical solutions obtained through the ECBS method, as illustrated in Tables 1, 2 and Figures 2 and 3, fur-

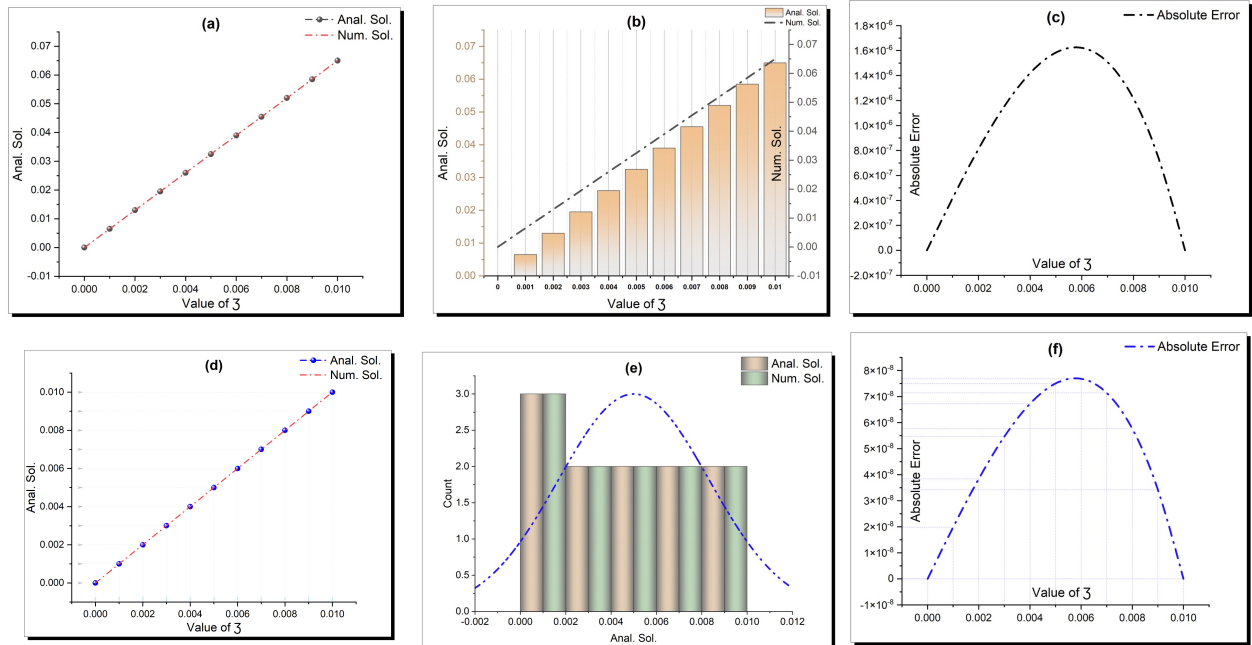


Fig. 2 Quantitative validation of analytical solutions against numerical solutions through absolute error analysis. (a-c) Absolute error between solutions obtained via the EAE method and the ECBS numerical method at defined intervals. (d-f) Absolute error comparison between solutions derived from the IKud method and ECBS numerical solutions at specified intervals.

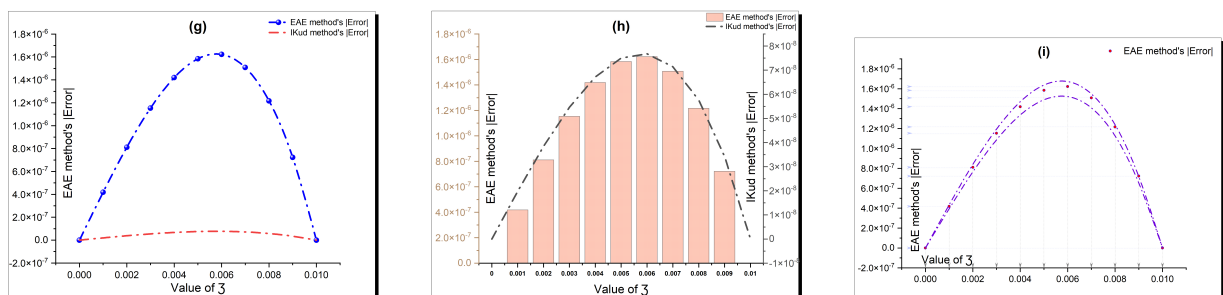


Fig. 3 Comparative assessment of solution accuracy for the IKud and EAE analytical techniques, benchmarked against ECBS numerical solutions.

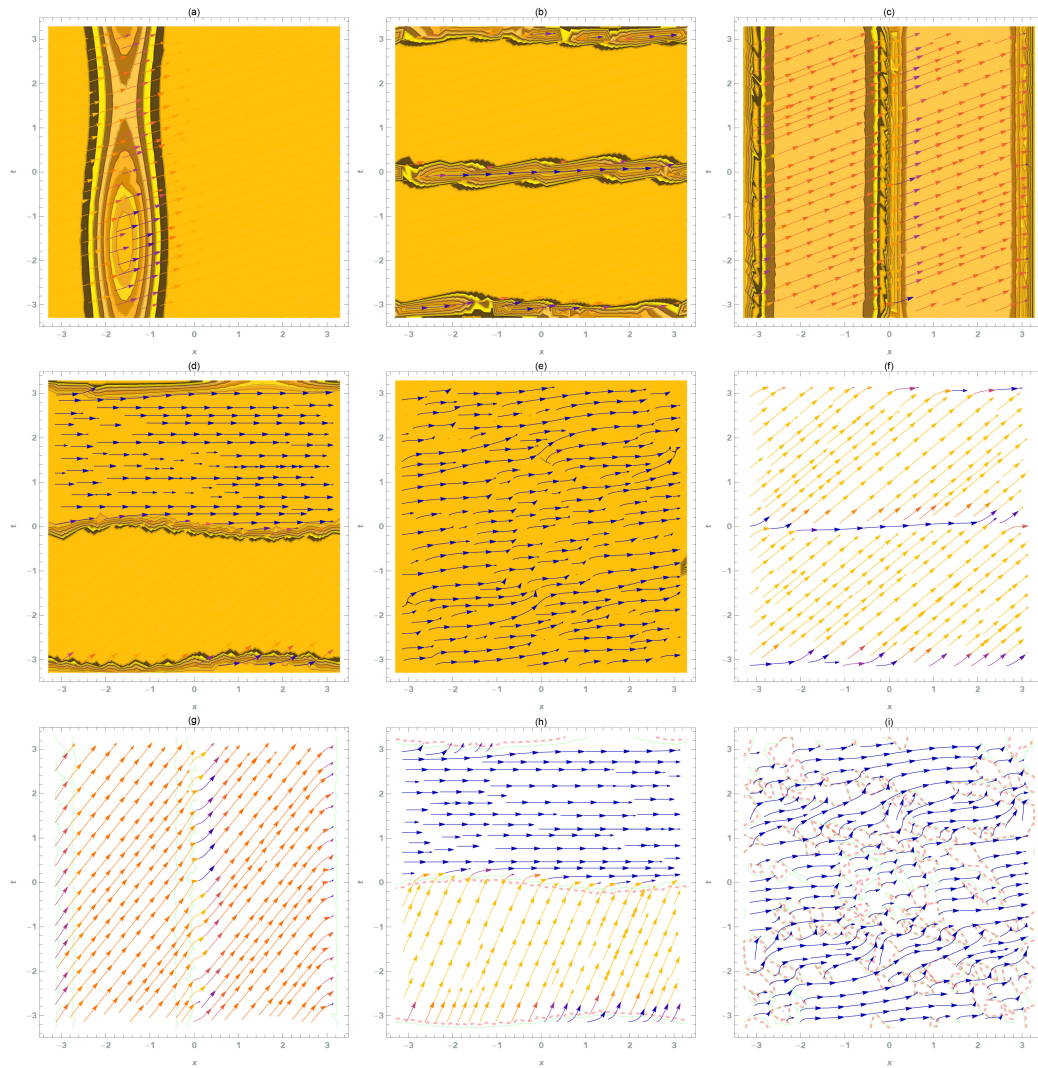


Fig. 4 Stream plot illustrating the propagation of solitary wave solutions in the mBBM model.

ther reinforces the accuracy and reliability of the analytical approaches employed. The absolute error analysis and comparative assessments demonstrate the high level of precision achieved by the EAE and IKud methods, instilling confidence in the validity of the derived solutions and their applicability to real-world scenarios. Furthermore, the systematic investigation of the stability attributes of the derived solutions through an assessment of their dynamics within the framework of the Hamiltonian system contributes to a deeper understanding of the physical phenomena associated with the mBBM equation. This analysis not only elucidates the robustness and reliability of the solutions but also paves the way for future explorations into the stability and long-term behavior of dispersive and nonlinear wave systems. The scientific contributions of this study extend beyond the theoretical realm by providing practical insights and potential applications in various fields. The mBBM equation finds relevance in oceanography, coastal engineering, and plasma physics, where accurate modeling of water waves, tsunamis, tidal waves, and ion-acoustic waves is crucial. The analytical solutions and graphical representations presented in this work offer valuable tools for researchers and practitioners in these domains, enabling them to gain a deeper understanding of wave propagation dynamics and develop more effective strategies for predicting and mitigating the impacts of such phenomena. In conclusion, this study represents a significant advancement in the field of nonlinear partial differential equations and their applications in fluid dynamics and wave propagation. By deriving novel analytical solutions, providing comprehensive graphical representations, and validating the

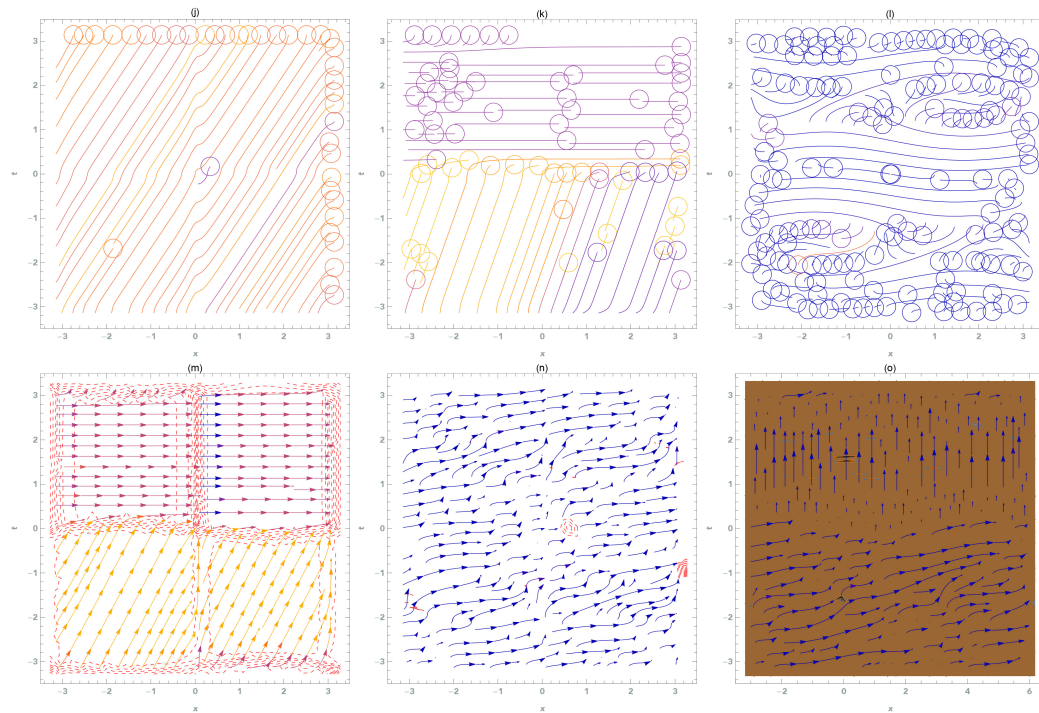


Fig. 5 Comparative stream plots demonstrating the influence of parameter modifications on the propagation behavior of solitary waves in the mBBM model.

results through numerical techniques, this research offers a robust theoretical framework and practical insights that can drive further advancements in the analysis and modeling of dispersive and nonlinear wave systems.

5 Conclusion

This research endeavor has made significant strides in analytically and numerically solving the nonlinear mBBM equation, a model of paramount importance in fluid dynamics, particularly for its application in describing unidirectional water waves influenced by dispersion and nonlinear effects. Through the employment of the EAE and IKud methods, novel analytical solutions have been derived, offering a diverse range of solitary wave profiles characterized by unique amplitudes, shapes, and propagation characteristics. The graphical representations presented in this study have played a pivotal role in elucidating the physical significance and behavior of the derived solutions. The 3D surface plots, 2D contour plots, and 2D line plots have effectively captured the intricate wave structures, enabling a comprehensive visualization and analysis of their amplitudes, shapes, and propagation patterns. Moreover, the stream plot visualizations have offered invaluable insights into the complex interplay between wave propagation, dispersion, and nonlinear effects, shedding light on the underlying physical mechanisms governing the observed phenomena. The quantitative validation of the analytical solutions against numerical solutions obtained through the ECBS method has further reinforced the accuracy and reliability of the analytical approaches employed. The absolute error analysis and comparative assessments have demonstrated the high level of precision achieved by the EAE and IKud methods, instilling confidence in the validity of the derived solutions and their applicability to real-world scenarios. Furthermore, the systematic investigation of the stability attributes of the derived solutions through an assessment of their dynamics within the framework of the Hamiltonian system has contributed to a deeper understanding of the physical phenomena associated with the mBBM equation. This analysis has not only elucidated the robustness and reliability of the solutions but also paved the way for future explorations into the stability and long-term behavior of dispersive and nonlinear

wave systems. The scientific contributions of this study extend beyond the theoretical realm by providing practical insights and potential applications in various fields, such as oceanography, coastal engineering, and plasma physics, where accurate modeling of water waves, tsunamis, tidal waves, and ion-acoustic waves is crucial. The analytical solutions and graphical representations presented in this work offer valuable tools for researchers and practitioners in these domains, enabling them to gain a deeper understanding of wave propagation dynamics and develop more effective strategies for predicting and mitigating the impacts of such phenomena. In conclusion, this research affirms the efficacy of the combined analytical and numerical approach in solving the mBBM equation, contributing novel insights into the field of applied mathematics and nonlinear partial differential equations. By deriving innovative analytical solutions, providing comprehensive graphical representations, and validating the results through numerical techniques, this study offers a robust theoretical framework and practical insights that can drive further advancements in the analysis and modeling of dispersive and nonlinear wave systems in various scientific and engineering contexts.

6 Declarations

6.1 Conflict of interest:

The authors hereby declare that there is no conflict of interests regarding the publication of this paper.

6.2 Funding:

No funding has been received for this research.

6.3 Author's Contribution:

R.A.M.A. and M.M.A.K.-Conceived and Designed the Experiments, Performed the Experiments, Analysed and Interpreted the Data; Contributed Reagents, Materials, Analysis Tools or Data; Wrote the paper. Finally, all authors read and approved the final manuscript.

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6.5 Data availability statement:

The data of the findings of this study are available from the corresponding author upon reasonable request.

6.6 Using of AI tools:

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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