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Original Study

Solving the generalized equal width wave equation via sextic B-spline collocation technique

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Abstract

This article applies the sextic B-spline collocation scheme to obtain the approximate solution of the generalized equal width (GEW) wave equation. The accuracy of the proposed technique is discussed over three test applications including the single soliton wave, interaction of soliton waves and Maxwellian initial problem while we are getting the three invariant A_1, A_2, A_3 and two error norms referred as to L_2 and L_∞. Applying the Von Neumann algorithm, the linearized technique is unconditionally stable. Our computational data show the superiority of results over those existing results in the literature review.

Keywords: GEW equation, sextic B-spline function, solitary waves, stability.
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1 Introduction

The GEW equation is taken from [1],

$$U_t + \varepsilon U^p U_x - \delta U_{xxt} = 0, \tag{1}$$

where p is set of integers. ε and δ are real constants with non zero. t and x are time and space derivatives with physical condition $U \to 0$ as $x \to \pm\infty$. In this work, boundaries and initial conditions are follows;

$$\begin{cases} U(a,t) = 0, & U_x(b,t) = 0, \quad t > 0 \\ U_x(a,t) = 0, & U_x(b,t) = 0, \quad t > 0 \\ U_{xx}(a,t) = 0, & U_{xx}(b,t) = 0, \quad t > 0 \\ U(x,0) = f(x), \qquad a \le x \le b. & \end{cases} \tag{2}$$

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GEW equation is a type of partial differential equations (PDEs) that is connected with Generalized Regularized Long Wave (GRLW) and Kortewege-de Vies (KdV) equations. A GEW equation is based on Equal Width (EW) equation. A GEW equation is a nonlinear equation with $(p+1)th$ non-linearity and these have solitary results because all general equations have solitary results. GEW was carried first time for small amplitude long wave on the surface of water channel by [2] and [3]. GEW equation is an alternative model to the GRLW and (GKdv) equation. Some researchers studied GEW equation using different methods seen in literature, [4–7]. The soliton wave results of proposed model has a significant performance in order to understand various phenomena. Numerically GEW equation can be solved by different techniques. In [1], authors presented the importance of the collocation algorithm with B-spline for GEW equation. In [8], authors worked on GEW equation by applying quadratic B-spline collocation algorithm and they obtained the analytical results for this model equation. In [9], authors implemented RBF method for GEW equation. In [10], authors studied the numerical result for GEW equation by applying Exponential B-spline algorithm. In [11], authors studied MLS method for GEW equation. In [12], authors solved the GEW equation by using the cubic B-spline Galerkin method for numerical frame. In [13], authors implemented the sextic B-spline function for modified Burger's equation. In [14], authors presented sextic B-spline collocation method for Euler-Bernoulli Beam Model. In [15], authors derived for the first time that if we put $p=1$, then equation 1 becomes EW equation. Many subsequent extensions and variations in technique have occurred, including [16–18]. Also some applications of collocation methods for solving engineering and basic science problems can be found in [19–25].

Several authors have worked on EW equation by applying different methods. Via [26], authors presented the EW equation by using the cubic B-spline collocation technique. In [27], authors presented numerical solution for EW equation by means of the finite difference method. Least square method for EW equation is studied by [28]. Many authors studied the MEW equation with the help of several algorithms. In [29], authors worked on MEW equation by means of solitary wave. Septic B-spline collocation technique for MEW equation is studied by [30]. Via [31], authors presented MEW equation by using the finite element method. In [32] and [33], experts investigated the different spline functions by using several algorithms. Recently, several important partial differential equations such as EW equation, MEW equation and GEW equation have been investigated by using different methods to get numerical solutions [34–36].

This paper is composed of five sections as follows. Section 2 presents the Sextic B-splines formulation. Section 3 deals with the stability analysis for the proposed algorithm. Section 4 computes the numerical results of the GEW equation applying various initial conditions. Some illustrative problems are provided to validate the concept defined in the paper. In the last section 5, the conclusions and future extension are reported.

2 Sextic B-spline function

For numerical solution, the region $[a,b]$ is split up into equal size of finite element of length $h=\frac{b-a}{N}$ by grid point such like, $a=x_0<x_1<x_2<\cdots<x_N=b$. The sextic function is denoted by $\phi_k(x)$, where $k=-3,-2,-1,\cdots,N+2$ at grid point is defined by [32];

$$\phi_k(x)=\frac{1}{h^6}\begin{cases} c_1=(x-x_{k-3})^6, & [x_{k-3},x_{k-2}],\\ c_2=a_1-7(x-x_{k-2})^6, & [x_{k-2},x_{k-1}],\\ c_3=a_2+21(x-x_{k-1})^6, & [x_{k-1},x_k],\\ c_4=a_3-35(x-x_k)^6, & [x_k,x_{k+1}],\\ (x-x_{k+4})^6-7(x-x_{k+3})^6+21(x-x_{k+2})^6, & [x_{k+1},x_{k+2}],\\ (x-x_{k+4})^6-7(x-x_{k+3})^6, & [x_{k+2},x_{k+3}],\\ (x-x_{k+4})^6, & [x_{k+3},x_{k+4}],\\ 0, & . \end{cases} \tag{3}$$

The sextic B-spline consist of 7 elements, so each element $x_k, x_m + 1$ is consisted of 8 splines and set region is $[a,b]$. The approximate results $U_N(x,t)$ can be explicit by sextic B-spline function as;

$$U_N(x,t) = \sum_{k=-3}^{N+3} \alpha_k(t)\phi_k(x), \tag{4}$$

where $\alpha_k(t)$ are time dependent values. By the combination of sextic B-spline function and approximate solution and nodal values that are U_x $U_x^{'}$, $U_x^{''}$; $U_x^{'''}$ of α_k which are following;

$$\begin{cases} U_x = u(x_k) = \alpha_{k-3} + 57\alpha_{k-2} + 302\alpha_{k-1} + 302\alpha_k + 57\alpha_{k+1} + \alpha_{k+2}, \\ U_x^{'} = u^{'}(x_k) = \frac{6}{h}(-\alpha_{k-3} - 25\alpha_{k-2} - 40\alpha_{k-1} + 40\alpha_k + 25\alpha_{k+1} + \alpha_{k+2}), \\ U_x^{''} = u^{''}(x_k) = \frac{30}{h^2}(\alpha_{k-3} + 9\alpha_{k-2} - 10\alpha_{k-1} - 10\alpha_k + 9\alpha_{k+1}) + \alpha_{k+2}, \\ U_x^{'''} = u^{'''}(x_k) = \frac{120}{h^3}(-\alpha_{k-3} - \alpha_{k-2} + 8\alpha_{k-1} - 8\alpha_k + \alpha_{k+1}) + \alpha_{k+2}. \end{cases} \tag{5}$$

To apply forward difference technique for time derivative and θ-weighted technique for space derivative in governing equation,

$$\frac{u^{n+1} - u^n}{\Delta t} + \varepsilon\theta(u^p u_x)^{n+1} + (1-\theta)(u^p u_x)^n - \delta\left(\frac{u_{xx}^{n+1} - u_{xx}^n}{\Delta t}\right) = 0. \tag{6}$$

In the above equation the Non linear term $(U^p U_x)^{n+1}$ can be expressed as [9]

$$(U^p U_x)^{n+1} \approx (U^p)^n U_x^{n+1} + p(U^{p-1})^n U^{n+1} U_x^n - p(U^p)^n U_x^n. \tag{7}$$

Δt is time adjacent level and put $\theta = \frac{1}{2}$, rearranging the above equation,

$$\begin{aligned} &2u^{n+1} + \Delta t\varepsilon p(u^{p-1})^n u^{n+1} u_x^n + \Delta t\varepsilon(u^p)^n u_{xx}^{n+1} - 2\delta u_{xx}^{n+1} \\ &= 2u^n + \Delta t\varepsilon p(u^p)^n u_x^n - \Delta t\varepsilon(u^p)^n u_x^n - 2\delta u_{xx}^n. \end{aligned} \tag{8}$$

For n^{th} time level u_k, $u_k^{'}$, $u_k^{''}$, and $u_k^{'''}$ at the grid point x_k, so

$$\begin{cases} U_{x_1} = \alpha_{k-3}^n + 57\alpha_{k-2}^n + 302\alpha_{k-1}^n + 302\alpha_k^n + 57\alpha_{k+1}^n + \alpha_{k+2}^n, \\ U_{x_2} = \frac{6}{h}(-\alpha_{k-3}^n - 25\alpha_{k-2}^n - 40\alpha_{k-1}^n + 40\alpha_k^n + 25\alpha_{k+1}^n + \alpha_{k+2}^n), \\ U_{x_3} = \frac{30}{h^2}(\alpha_{k-3}^n + 9\alpha_{k-2}^n - 10\alpha_{k-1}^n - 10\alpha_k^n + 9\alpha_{k+1}^n) + \alpha_{k+2}^n, \\ U_{x_4} = \frac{120}{h^3}(-\alpha_{k-3}^n - \alpha_{k-2}^n + 8\alpha_{k-1}^n - 8\alpha_k^n + \alpha_{k+1}^n) + \alpha_{k+2}^n. \end{cases} \tag{9}$$

Where grid point x_k, $k = 0,1,2,\cdots,N$ is a collocation point. The recursive scheme has been obtained at the point x_k, thus

$$\begin{aligned} &C_1\alpha_{k-3} + C_2\alpha_{k-2} + C_3\alpha_{k-1} + C_4\alpha_k + C_5\alpha_{k+1} + C_6\alpha_{k+2} \\ &= 2L_1 + \Delta t\varepsilon p(L_1)^p L_2 - \Delta t\varepsilon(L_1)^p L_2 - 2\alpha L_3, \end{aligned} \tag{10}$$

Where,

$$\begin{cases} C_1 = (a_1 - \frac{6b_1}{h} - \frac{60\alpha}{h^2}) \\ C_2 = (57a_1 - \frac{150b_1}{h} - \frac{540\alpha}{h^2}) \\ C_3 = (302a_1 - \frac{240b_1}{h} + \frac{600\alpha}{h^2}) \\ C_4 = (302a_1 + \frac{240b_1}{h} + \frac{600\alpha}{h^2}) \\ C_5 = (57a_1 + \frac{150b_1}{h} - \frac{540\alpha}{h^2}) \\ C_6 = (a_1 + \frac{6b_1}{h} - \frac{60\delta}{h^2}). \end{cases} \tag{11}$$

The general system contains linear system involving $(N+6)$ unknown parameters $\alpha_{-3}, \alpha_{-2}, \alpha_{-1}, \cdots \alpha_{N+1}, \alpha_{N+2}$. To get unique solution of general system. Using sextic b-spline function and collocation boundaries condition to get the unknown parameters values which are following;

$$\begin{cases} \alpha_{-1} = -\frac{1}{9}\alpha_0 + \alpha_1 + \frac{1}{9}\alpha_2, \\ \alpha_{-2} = \frac{20}{9}\alpha_0 - \alpha_1 - \frac{2}{9}\alpha_2, \\ \alpha_{-3} = -\frac{100}{9}\alpha_0 + 10\alpha_1 + \frac{19}{9}\alpha_2, \\ \alpha_{N+1} = \frac{1}{8}\alpha_{N-3} + \frac{17}{8}\alpha_{N-2} + \frac{15}{8}\alpha_{N-1} - \frac{25}{8}\alpha_N, \\ \alpha_{N+2} = -\frac{17}{8}\alpha_{N-3} - \frac{225}{8}\alpha_{N-2} - \frac{55}{8}\alpha_{N-1} + \frac{305}{8}\alpha_N. \end{cases} \tag{12}$$

By eliminating the unknown parameters, given in the above equation. So linear system $(N+1)$ and $(N+6)$ unknown parameters α_k, is obtained, which is solved by sexta-diagonal solver for α_k^n. The last approximated results of $U(x,t)$ is obtained. The initial parameters α_k^0 must be evaluated through the initial and boundaries conditions by following;

$$\begin{aligned}
&U^{'}(x_0,0) = U^{'}(x_0,0) = \frac{6}{h}(-\alpha_{-3}^0 - 25\alpha_{-2}^0 - 40\alpha_{-1}^0 + 40\alpha_1^0 + 25\alpha_2^0 + \alpha_3^0) = 0,\\
&U^{''}(x_0,0) = U^{''}(x_0,0) = \frac{30}{h^2}(\alpha_{-3}^0 + 9\alpha_{-2}^0 - 10\alpha_{-1}^0 - 10\alpha_0^0 + 9\alpha_1 + \alpha_2^0) = 0,\\
&U^{'''}(x_0,0) = U^{'''}(x_0,0) = \frac{120}{h^3}(-\alpha_{-3}^0 - \alpha_{-2}^0 + 8\alpha_{-1}^0 - 8\alpha_1^0 + \alpha_2^0 + \alpha_3^0 = 0,\\
&U(x_k,0) = (x_k,0) = \alpha_{k-3}^0 + 57\alpha_{k-2}^0 + 302\alpha_{k-1}^0 + 302\alpha_k^0 + 57\alpha_{k+1}^0 + \alpha_{k+2}^0 = f(x_k), k = 0,1,\cdots,N\\
&U^{'''}(x_N,0) = U^{'''}(x_N,0) = \frac{120}{h^3}(-\alpha_{N-3}^0 - \alpha_{N-2}^0 + 8\alpha_{N-1}^0 - 8\alpha_{N+1}^0 + \alpha_{N+2}^0 + \alpha_{N+3}^0 = 0,\\
&U^{''}(x_N,0) = U^{''}(x_N,0) = \frac{30}{h^2}(\alpha_{N-3}^0 + 9\alpha_{N-2}^0 - 10\alpha_{N-1}^0 - 10\alpha_N^0 + 9\alpha_{N+1} + \alpha_{N+2}^0) = 0,\\
&U^{'}(x_N,0) = U^{'}(x_N,0) = \frac{6}{h}(-\alpha_{N-3}^0 - 25\alpha_{N-2}^0 - 40\alpha_{N-1}^0 + 40\alpha_{N+1}^0 + 25\alpha_{N+2}^0 + \alpha_{N+3}^0) = 0.
\end{aligned}$$

The above equation is consisted of $(N+1)\times(N+1)$ which is a linear system of equation which can be solved by a sexta-diagonal system.

3 Stability analysis

The stability of GEW equation can be linearized by the Von-Numann process. The non-linear term U^p can be expressed by constant term k and also replace $\alpha_k^n = \xi^n e^{ikkh}$, $i = \sqrt{-1}$ and k represents mode number and h represents element size. So, in linearization a recurrence relationship between the two time levels at iteration n and $n+1$ relating the two unknown functions α_k^{n+1} and α_k^n, see [7–10]

$$\begin{aligned} C_1\xi^{n+1}e^{i(k-3)kh} + C_2\xi^{n+1}e^{i(k-2)kh} + C_3\xi^{n+1}e^{i(k-1)kh} + C_4\xi^{n+1}e^{ikkh} + C_5\xi^{n+1}e^{i(k+1)kh} + C_6\xi^{n+1}e^{i(k+2)kh} \\ = C_1\xi^n e^{i(k-3)kh} + C_2\xi^n e^{i(k-2)kh} + C_3\xi^n e^{i(k-1)kh} + C_4\xi^n e_k^{ikkh} + C_5\xi^n e^{i(k+1)kh} + C_6\xi^n e^{i(k+2)kh}. \end{aligned} \tag{13}$$

By using Eular method, $e^{i\theta} = \cos\theta + i\sin\theta$, in calculated equation which carried to the factor of ξ, we have

$$\xi = \frac{M - iN}{M + iN}, \tag{14}$$

$$\xi = |\frac{M - \imath N}{M + \imath N}| = \frac{M^2 + N^2}{M^2 + N^2} = 1 \tag{15}$$

Where,

$$M = C_6 \cos 3kh + (C_5 + C_1)\cos 2kh + (C_4 + C_2)\cos kh + C_3,$$
$$N = C_6 \sin 3kh + (C_5 - C_1)\sin 2kh + (C_4 - C_2)\sin kh. \tag{16}$$

Thus the modulus of $|\xi| = 1$. Hence, the given scheme of GEW equation is unconditionally stable.

4 Computational experiments

To examine the accuracy and validity of the numerical result of the GEW equation, the sextic B-spline method is implemented. The present scheme is checked by means of three problems. The solution of the proposed scheme is compared with other techniques which are given in literature. The error norm L_2 and L_∞ is simplified and compared with [1]. The Three conservative invariants A_1, A_2, and A_3 are obtained by using numerical scheme. The error norms L_2 and L_∞ are defined as;

$$L_2 = \|U^{\text{exact}} - U_N\|_2 \simeq \sqrt{h\sum_{j=0}^{n}|U_j^{exact} - (U_N)_j|^2}. \tag{17}$$

$$L_\infty = \|U^{\text{exact}} - U_N\|_\infty \simeq max_j|U_j^{exact} - (U_N)_j|. \tag{18}$$

The analytical results of GEW equation (1) is taken by using the modification of $u(x,t) = f(x-ct)$ is;

$$U(x,t) = \sqrt[p]{\frac{c(p+1)(p+2)}{2\varepsilon}\sec h^2[\frac{p}{2\sqrt{\delta}}(x-ct-x_0)]}, \tag{19}$$

Where x_0 is arbitrary constant and c is constant velocity which moving toward x-axis direction. Three invariants of motion can be defined as;

$$A_1 = \int_a^b U dx, \quad A_2 = \int_a^b [U^2 + \delta U_x^2]dx, \quad A_3 = \int_a^b U^{p+2}dx. \tag{20}$$

4.1 The single soliton wave

To use the general equation for numerical results, let's suppose five parameters that have various values of p, c and

$$amp = \sqrt[n]{\frac{c(p+1)(p+2)}{2\varepsilon}}.$$

Take other parameters for five sets that are $h = 0.1, \Delta t = 0.2, \varepsilon = 3, \delta = 1, x_0 = 30$ and $0 \le x \le 80$.

Example 1. *Put $p = 2$ and $c = 1/2$ and total time run up to [0, 20]. Therefore, the amplitude of soliton wave is $amp = 0.25$ and three invariants A_1, A_2, A_3 and error norms L_2 and L_∞ has been simplified by the proposed scheme. The results of all invariants are shown in Table 1 and Table 2. Its clarify that the values of invariants are smaller then given results of literature and error norms is less than earlier methods.*

Example 2. *Choose, $p = 3$, $c = 0.3$ and numerical time run up to [0, 20]. Therefore, the amplitude of soliton wave is $amp = 1$ and three invariants A_1, A_2, A_3 and error norms L_2 and L_∞ have been simplified by the proposed scheme. The results of all invariants are shown in Table 3. Its clarify that invariants values are smaller then given results of literature and error norms is less than earlier methods.*

Example 3. *For $p = 4$, $c = 0.2$ and numerical time run up to $[0,20]$. Therefore, the amplitude of soliton wave is $amp = 1$ and three invariants A_1, A_2, A_3 and error norms L_2 and L_∞ have been simplified by the proposed scheme. The results of all invariants are shown in Table 4. Its clarify that the values of invariants are smaller then given results of literature and error norms is less than earlier methods.*

4.2 The interaction of two soliton waves

In this problem, some initial conditions are applied in order to find numerical results of interaction of two solitary waves. These are given as;

$$U(x,0)=\sum_{j=1}^{2}\sqrt[p]{\frac{c_j(p+1)(p+2)}{2\varepsilon}}\ \sec h^2[\frac{p}{2\sqrt{\delta}}(x-x_j)], \tag{21}$$

Where, x_j, c_j, $j=1,2$ are arbitrary constants. Choose three set of parameters of p,c for different amplitudes values and also choose other parameters values for three set which are ; $x_1=15$, $x_2=30$, $\Delta t=0.025$, $h=0.1$, $0\leq x\leq 80$ and two amplitude for separate soliton wave are $[0.5,1]$.

Example 4. *To choose,* $c_1=0.5,\quad c_2=0.125,\quad p=2$ *and programming time goes up to* $t=[0,60]$. *The three invariants values are recorded in Table 5. In initial step the invariant values are smaller respectively. The results are more accurate than other techniques given in literature. These figures clarify two solitary waves by different times. The increasing time overlapping procedure is occurred by the starting of interaction and waves start to resume their original shape after* $t=50$.

Example 5. *To choose* $p=3,\quad c_1=0.3,\quad c_2=0.0275,$ *and computing time goes up to* $t=[0,100]$. *The three invariants are shown in Table 6. In initial step the invariant values are smaller respectively. The results are more accurate than other techniques given in literature. These figures clarify two solitary waves at different times. The increasing of time overlapping process is occurred by the starting of interaction and waves start to resume their original shape after* $t=50$.

Example 6. *To select* $p=4,\ c_1=0.2,\ c_2=0.0125$ *and computing time goes up to* $t=[0,120]$. *The three invariants are shown in Table 7. In initial step the invariant values are lesser respectively. The results are more accurate than other techniques given in literature. These figures clarify two solitary waves by various times. The increasing of time overlapping process is occurred by the starting of interaction and waves start to resume their original shape after* $t=70$.

4.3 A maxwellian initial problem

The final step, an approximate result of major equation with Maxwellian initial problem is taken from [31]

$$U(x,0)=\exp(-x^2)\quad -20\leq x\leq 20. \tag{22}$$

Example 7. *In problem, the results of Maxwellian initial problem are dependent on different values of* δ. *So, to take various values of* δ *that are;* $\delta=0.01,\ \delta=0.025,\ \delta=0.05,\ \delta=0.1$ *where* $p=2,3,4$ *and computing time run to* $t=12$. *The three conservative invariants and there values are recorded in Table 8, by different values of* δ. *The results of Maxwellian initial condition at* $t=12$ *are illustrated from various values of* δ *are plotted in figures. Its clear from all figures that when* δ *quantities are decreased then number of stable solitary wave are increased.*

Table 1 Solution for single soliton wave $c = \frac{1}{32}$, $p = 2$ and $0 \leq x \leq 80$

t	$L_2 \times 10^5$	$L_\infty \times 10^5$	A_1	A_2	A_3
0	0.0	0.0	0.78539	0.16666	0.00520
5	0.02930	0.01793	0.78539	0.16666	0.00520
10	0.06091	0.03719	0.78539	0.16666	0.00520
15	0.09201	0.05597	0.78539	0.16666	0.00520
20	0.12236	0.07425	0.78539	0.16666	0.00520
$t = 20$ [1]	0.14404	0.10853	0.78539	0.16666	0.00520

Table 2 Solution for single soliton wave $c = \frac{1}{2}$, $p = 2$ $0 \leq x \leq 80$

t	$L_2 \times 10^5$	$L_\infty \times 10^5$	A_1	A_2	A_3
0	0.0	0.0	3.14159	2.66666	1.33333
5	0.00665	0.00430	3.14159	2.66639	1.33305
10	0.01383	0.00883	3.14159	2.66610	1.33277
15	0.02135	0.01349	3.14159	2.66582	1.33249
20	0.02911	0.01827	3.14159	2.66554	1.33220
$t = 20$ [1]	0.05132	0.03416	3.14167	2.66690	1.33357

Table 3 Solution for single soliton wave $c = 0.3$, $p = 3$ and $0 \leq x \leq 80$

t	$L_2 \times 10^3$	$L_\infty \times 10^3$	A_1	A_2	A_3
0	0.0	0.0	2.80436	2.46399	0.98555
5	0.00307	0.00234	2.80436	2.46374	0.98539
10	0.00695	0.00483	2.80436	2.46356	0.98522
15	0.01076	0.00741	2.80436	2.46339	0.98504
20	0.01472	0.01007	2.80436	2.46322	0.98487
$t = 20$, [1]	0.00801	0.00538	2.80435	2.46390	0.98556

Table 4 Solution for single soliton wave $c = 0.2$, $p = 4$ and $0 \leq x \leq 80$

t	$L_2 \times 10^3$	$L_\infty \times 10^3$	A_1	A_2	A_3
0	0.0	0.0	2.62205	2.35619	0.78539
5	0.00183	0.00133	2.62205	2.35610	0.78530
10	0.00374	0.00274	2.62205	2.35600	0.78521
15	0.00577	0.00419	2.62205	2.35599	0.78511
20	0.00789	0.00570	2.62205	2.35582	0.78502
$t = 20$, [1]	0.004216	0.002979	2.62205	2.35619	0.78539

Table 5 Solution for two soliton waves

t	A_1	A_2	A_3
0	4.712383	3.333335	1.4166697
10	4.712386	3.333334	1.4166686
20	4.712389	3.333333	1.4166670
30	4.712389	3.333331	1.4166469
40	4.712389	3.333332	1.416654
50	4.712389	3.333331	1.416650
60	4.712389	3.333330	1.416664
$t = 60$, [1]	4.71237	3.33332	1.41666

Table 6 Solution for two soliton waves

t	A_1	A_2	A_3
0	4.206545	3.079893	1.016366
10	4.206546	3.079893	1.016365
20	4.206546	3.079892	1.016364
40	4.206546	3.079891	1.016361
60	4.206546	3.079889	1.016361
80	4.206546	3.079889	1.016361
90	4.206546	3.079888	1.016360
100, [1]	4.206546	3.079887	1.016360

Table 7 Solution for two soliton waves

t	A_1	A_2	A_3
0	3.933085	2.945244	0.797613
10	3.933086	2.945243	0.797670
20	3.933086	2.945242	0.797669
40	3.933086	2.945242	0.797667
60	3.933086	2.945240	0.797667
80	3.933086	2.945239	0.797667
100	3.933086	2.945237	0.797664
120	3.933086	2.945235	0.797662
$t = 120$, [1]	3.93307	2.94523	0.79766

Table 8 Solution for maxwellian initial problem

t	A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
$\delta = 0.01$	$p = 2$	$p = 2$	$p = 2$	$p = 3$	$p = 3$	$p = 3$	$p = 4$	$p = 4$	$p = 4$
0	1.772453	1.265994	0.886226	1.772453	1.265998	0.792665	1.772453	1.265991	0.723601
4	1.772453	1.265851	0.886244	1.772453	1.265850	0.792686	1.772453	1.265849	0.723624
8	1.772453	1.265851	0.886244	1.772453	1.265850	0.792686	1.772453	1.265849	0.723624
12	1.772453	1.265851	0.886244	1.772453	1.265850	0.792686	1.772453	1.265849	0.723624
$t = 12$	1.7739	1.2711	0.9123	1.7813	1.2901	0.8664	1.8156	1.3901	1.2707
$\delta = 0.025$									
0	1.772453	1.284879	0.886226	1.772453	1.284872	0.792665	1.772453	1.284851	0.723601
4	1.772453	1.284649	0.886234	1.772453	1.284648	0.792672	1.772453	1.284648	0.723607
8	1.772453	1.284649	0.886234	1.772453	1.284648	0.792672	1.772453	1.284648	0.723607
12	1.772453	1.284649	0.886234	1.772453	1.284648	0.792672	1.772453	1.284648	0.723607
$t = 12$	1.7725	1.2846	0.8881	1.7730	1.2837	0.7946	1.7791	1.3056	0.8198
$\delta = 0.05$									
0	1.772453	1.316236	0.886226	1.772453	1.316218	0.792665	1.772453	1.316188	0.723601
4	1.772453	1.315981	0.886229	1.772453	1.315980	0.792667	1.772453	1.315980	0.723602
8	1.772453	1.315981	0.886229	1.772453	1.315980	0.792667	1.772453	1.315980	0.723602
12	1.772453	1.315981	0.886229	1.772453	1.315980	0.792667	1.772453	1.315980	0.723602
$t = 12$	1.7724	1.3159	0.8864	1.7725	1.3160	0.7940	1.7735	1.3188	0.7345
$\delta = 0.1$									
0	1.772453	1.378858	0.886226	1.772453	1.378839	0.792665	1.772453	1.378809	0.723601
4	1.772453	1.378646	0.886227	1.772453	1.378646	0.792665	1.772453	1.378646	0.723601
8	1.772453	1.378646	0.886227	1.772453	1.378646	0.792665	1.772453	1.378646	0.723601
12	1.772453	1.378646	0.886227	1.772453	1.378646	0.792665	1.772453	1.378646	0.723601
$t = 12$	1.7724	1.3786	0.8862	1.7724	1.3786	0.7928	1.7725	1.3786	0.7243

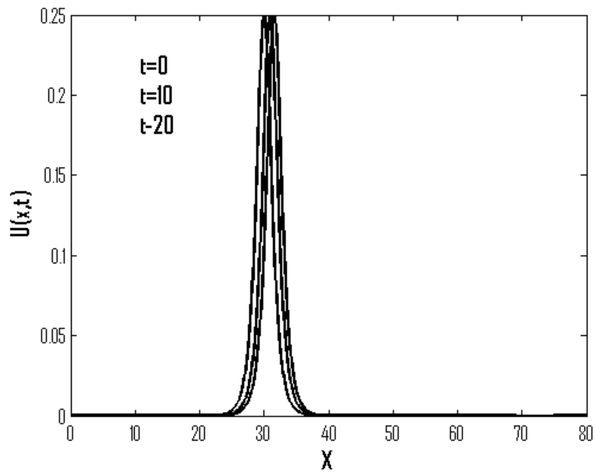

Fig. 1 Single solitary wave at $p = 2$.

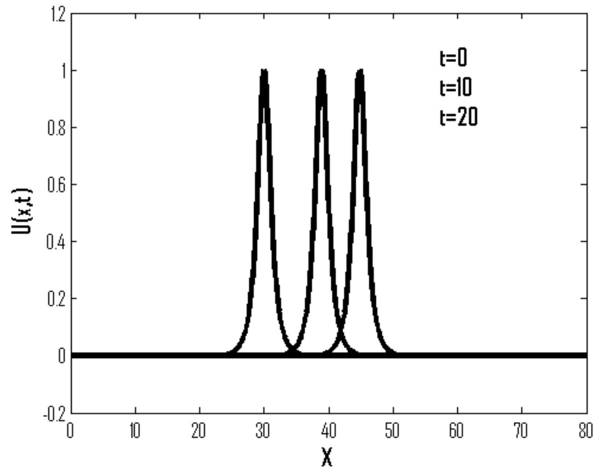

Fig. 2 Single solitary wave at $p = 3$.

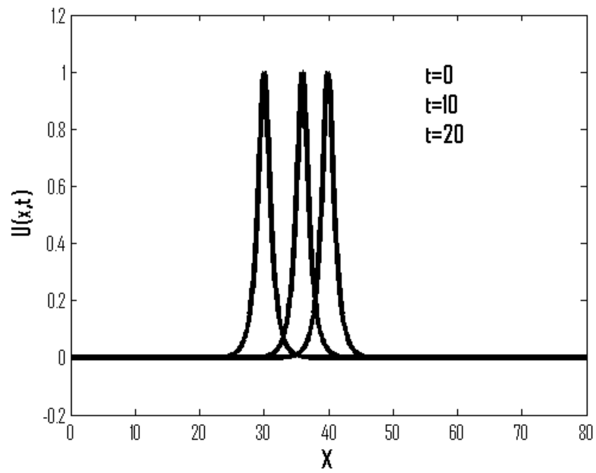

Fig. 3 Single solitary wave at $p = 4$.

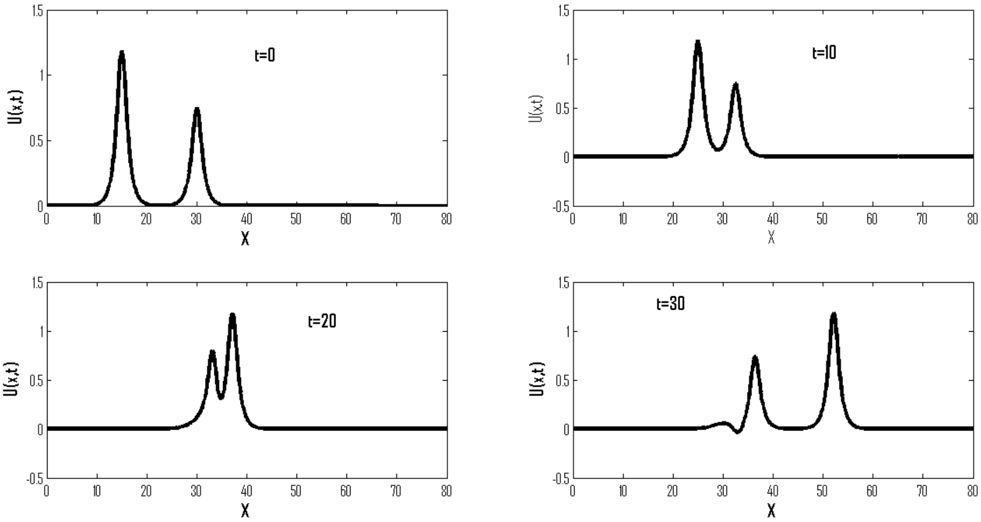

Fig. 4 Interaction of two soliton waves at $p = 3$.

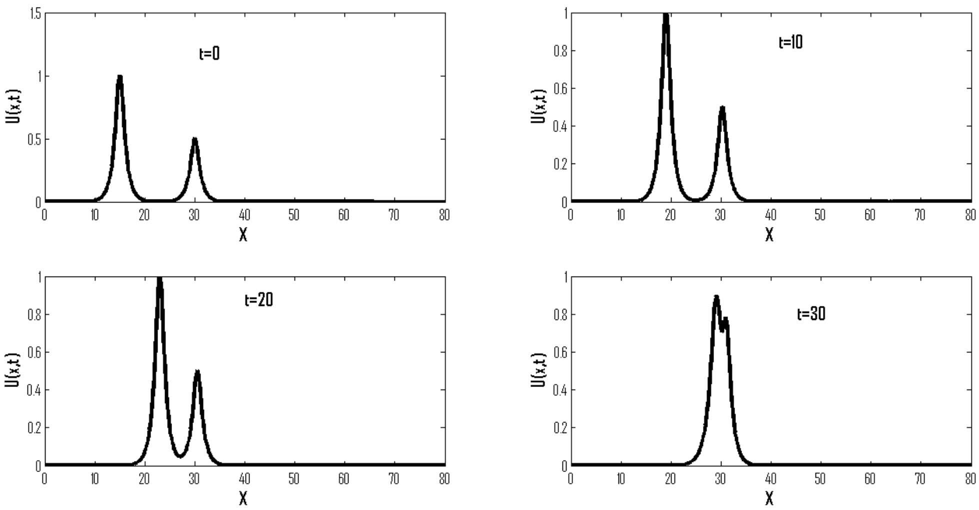

Fig. 5 interaction of two soliton waves at $p = 4$.

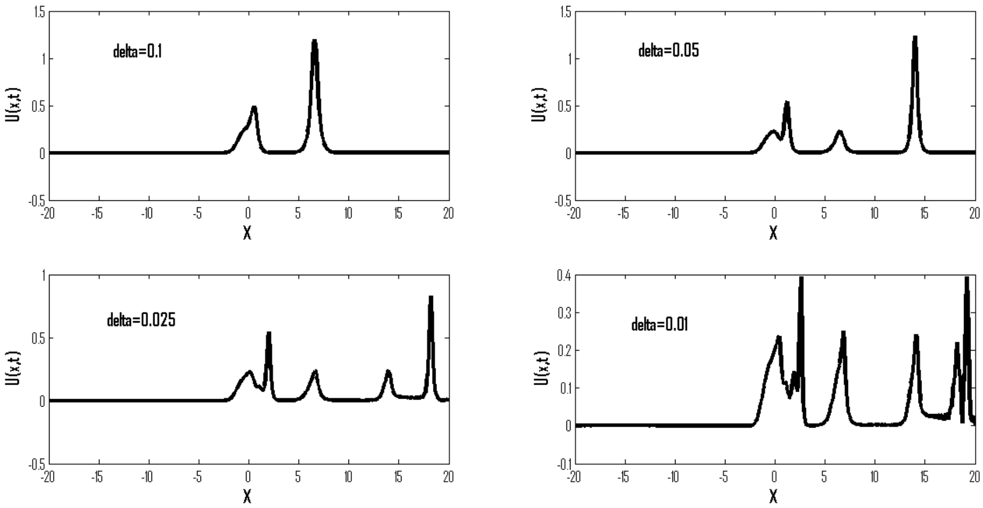

Fig. 6 Maxwellian initial problem $p = 3$ at $t = 12$.

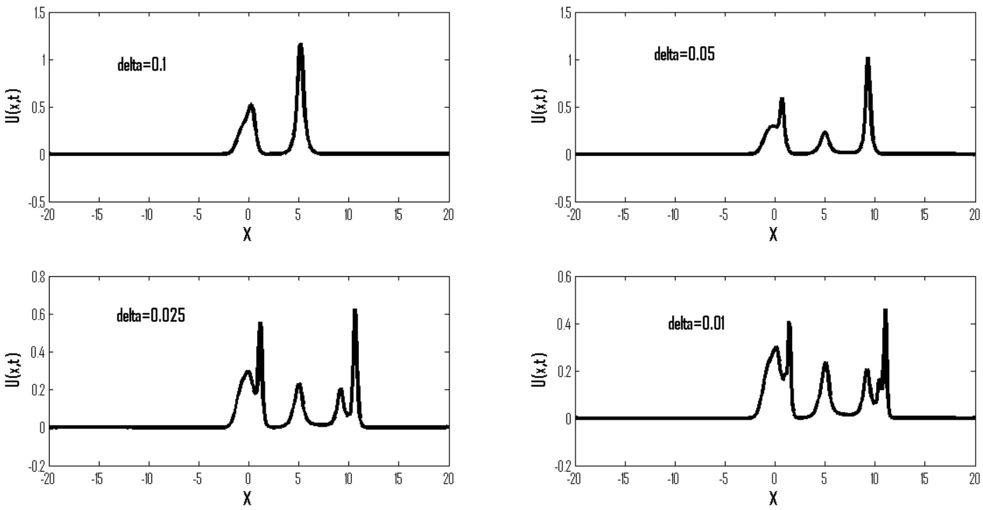

Fig. 7 Maxwellian initial problem $p = 4$, at $t = 12$.

5 Conclusion

In this study, the recursive method based on sextic B-spline collocation algorithm has been executed to get the approximated results of GEW equation. The accuracy of proposed algorithm has been verified by various problems like soliton wave, interaction of soliton waves and final Maxwellian initial problem. Three conservative invariants such as A_1, A_2 and A_3 as well as the two error norms L_2 and L_∞ are obtained by the proposed algorithm. By linearization of the presented method, the different tables show that the invariants are sufficiently less and stable with earlier methods. The accuracy of error norms are smaller than [1]. Various figures show that approximate results reproduce the motion of soliton waves. For future work, the same result can be computed by using different methods such as analytical, numerical and trigonometric methods.

6 Declarations

6.1 Conflict of interest:

The authors declare that there is no conflict of interest regarding the publication of this paper.

6.2 Authors' contributions:

M.N.-Methodology, Writing-Review and Editing, Supervision; S.J.-Resources, Writing-Original Draft, Methodology; F.A.-Validation, Conceptualization, Formal Analysis; A.Z.-Data Curation, Investigation, Visualization. The paper has been submitted with the knowledge and consent of all authors.

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6.5 Data availability statement:

All data that support the findings of this study are included within the article.

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