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Seepage-Induced Spatially-Variied Flow over the Downstream Face of a Rockfill Dam

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Abstract. The combined effects of dynamic pressure and steep bed slope on seepage-induced spatially-varied flow have so far received little research attention. In this study, a depth-averaged equation, which incorporates a higher-order Boussinesq-type correction for these effects, along with its simplified versions, was applied to investigate the two-dimensional problem of steady spatially-varied flow along the seepage face of a rockfill dam. The numerical solutions for these non-linear equations were obtained by means of an implicit finite-difference scheme. The results of the proposed equation were compared with those obtained from the simplified equations and experimental data, resulting in a more accurate agreement with the measurements than the simplified versions. Furthermore, the results of the comparison demonstrated that such a higher-order, depth-averaged equation is sufficiently accurate to capture the complex features of the supercritical spatially-varied flow field. This study concludes that adopting a mild slope approximation greatly affects the accuracy and reliability of the numerical results related to free-surface and bed pressure profiles, especially for such a type of flow problem, where the flow's dynamic characteristics are significantly influenced by the steepness of the channel bed slope.

Key words: finite-difference method, rockfill dam, seepage-face overflow, turbulent seepage flow, spatially-varied flow, supercritical flow

Notation

- dh – thickness of a horizontal layer (L),
- g – gravitational acceleration (LT^{-2}),
- H – flow depth normal to the channel bed (L),
- H_p – piezometric or hydraulic head (L),
- i – computational node index (–),
- K_C – depth-averaged pressure correction coefficient (–),
- L_C – crest length of an embankment dam (L),
- L_S – length of the seepage face in the streamwise direction (L),
- m – exponent for the lateral inflow discharge equation (–),
- n – Gauckler-Manning roughness coefficient ($L^{-1/3}T$),

p	– pressure ($\text{ML}^{-1}\text{T}^{-2}$),
p_b	– total bed pressure ($\text{ML}^{-1}\text{T}^{-2}$),
p_0	– hydrostatic component of the total bed pressure ($\text{ML}^{-1}\text{T}^{-2}$),
q	– discharge per unit width (L^2T^{-1}),
q_N	– total lateral inflow discharge per unit width (L^2T^{-1}),
R	– hydraulic radius (L),
S_0	– slope of the channel bed (–),
u	– velocity parallel to the channel bed (LT^{-1}),
u_*	– bulk velocity in the horizontal direction (LT^{-1}),
U_L	– lateral inflow or outflow velocity in the streamwise direction (LT^{-1}),
V	– average lateral inflow or outflow velocity (LT^{-1}),
w	– velocity normal to the channel bed (LT^{-1}),
w_H	– bed-normal velocity at the free surface (LT^{-1}),
x, y, z	– local Cartesian coordinates parallel and normal to the channel bed (L),
x_*, y_*, z_*	– Cartesian coordinates in the horizontal and vertical directions (L),
Z_b	– channel bed elevation (L),
β or β_L	– Boussinesq coefficient (–),
Γ	– dimensionless parameter associated with the lateral inflow momentum flux (–),
Δx	– spatial step size (L),
η	– free-surface elevation (L),
θ	– slope of the channel bed (deg),
κ	– curvature of the free surface (L^{-1}),
κH	– relative curvature of the free surface (–),
λ	– effective porosity of the porous media (–),
Λ	– non-dimensional height normal to the channel bed (–),
μ_2	– dimensional constant (L^mT^{-m}),
ρ	– density of the fluid (ML^{-3}),
τ_b	– bed shear stress ($\text{ML}^{-1}\text{T}^{-2}$),
τ_{xz}	– shear stress ($\text{ML}^{-1}\text{T}^{-2}$),
φ	– lateral inflow or outflow angle relative to the channel bed (deg),
X	– non-dimensionalized horizontal distance (–),
Ω	– non-dimensionalized free-surface elevation (–).

1. Introduction

The seepage flow through the rockfill dam induces hydraulic and drag forces within the body of the dam and along the seepage face. The magnitudes of these forces change considerably with the level of turbulence in the seepage flow. Such a turbulent behavior is more directly affected by the intensity of the seepage rate. An increase in seepage discharge results in the entrainment of rocks on the seepage face, which is subsequently followed by mass sliding as a result of slope instability. The height of

the phreatic surface at the exit point is a key factor in determining the stability of this face of the dam. Under free-flow conditions, the seepage through the dam emerges above the toe and passes over the downstream face, with seepage from the rock mass further augmenting the free-surface flow. Hence, the flow situation over the seepage face demonstrates a case of spatially-varied flow with an increasing discharge. The dynamic characteristics of this type of open-channel flow are strongly influenced by the effects of the spatially-varied discharge and steep slope of the downstream face of the dam. Such a two-dimensional (2D) vertical plane flow problem necessitates the use of a higher-order dynamic equation that inherently accounts for these effects, thereby enabling a comprehensive characterization of the complex flow field.

The problems of spatially-varied flows with an increasing or a decreasing discharge have been investigated using lower- and higher-order open-channel flow equations. Using the principle of the conservation of linear momentum, along with the assumption of a hydrostatic pressure distribution, several researchers presented lower-order equations to analyze the flow problems (e.g., Favre 1933, Camp 1940, Li 1955, Liggett 1959, Hansen 1992, Kells 1993). The presence of lateral inflow or outflow may induce a change in flow conditions from subcritical to supercritical, with a non-hydrostatic pressure distribution in the vicinity of the critical section. The effects of a non-hydrostatic pressure distribution on flow patterns must be considered for accurately treating the flow problems (Mostkow 1957, Utami and Nakagawa 1967, 1968). This reveals that those equations based on a hydrostatic pressure approach are less effective in solving such types of problems. By applying the principles of the conservation of energy and linear momentum, Nakagawa (1969) systematically extended the lower-order equations by incorporating the effects of dynamic pressure. His approach resulted in higher-order equations that are suitable for solving spatially-varied flow problems. He also examined the suitability of the equations and obtained a satisfactory agreement with the experimental data. On the basis of the conservations of energy and linear momentum, Yen and Wenzel (1970) also developed the spatially-varied flow equations for the case of a steady open-channel flow with a lateral inflow or outflow. However, the numerical solutions of the resulting equations require a prior knowledge of the distributions of the correction coefficients. Utami and Nakagawa (1968) previously indicated the importance of these coefficients for accurately analyzing the curvilinear flow problem. Following the approach established by Nakagawa (1969), Castro-Orgaz and Hager (2011) presented Boussinesq-type equations for flows with spatially increasing or decreasing discharges. Both studies focused exclusively on simulating spatially-varied flows in a mild-slope channel. By applying the linear momentum principle and the Boussinesq theory, Zerihun (2015, 2019) recently developed spatially-varied flow equations that incorporate a higher-order dynamic pressure correction for the effects of the dynamic pressure. The equations are capable of analyzing the mean flow characteristics of an open-channel flow with a spatially increasing or decreasing discharge in a moderately sloping channel. Despite its lower computational efficiency, a computational fluid dynamics model utilizing the

Reynolds-averaged Navier–Stokes (RANS) equations was implemented to thoroughly investigate the internal structure of the turbulent, spatially-varied flow (Carrillo et al 2018, Chan et al 2018, Bina et al 2024). In addressing this type of large-scale open-channel flow problem, the 2D shallow-water model was also adopted to achieve a compromise solution and enhance efficiency by considerably reducing computational time (see, e.g., Arega et al 2019).

As previously noted, limited research has been dedicated to employing a higher-order approach to investigate the problem of spatially-varied flow. Additionally, there has been no attempt so far to develop spatially-varied flow equations that incorporate the effects of both the dynamic pressure and steep bed slope (to the author’s knowledge). These effects significantly affect the structure of the flow field and modify the distributions of velocity and pressure. In this study, the effects of the dynamic pressure, spatially-varied discharge, and steep bed slope are considered to develop the dynamic equation. Unlike those equations developed based on the hydrostatic pressure approach, the proposed equation does not require a prior knowledge of the location of the critical flow section. It has the capacity to predict the transition of a flow from a subcritical to a supercritical state and to analyze the interaction between seepage and overtopping flows. Using the weakly curved flow approximation and neglecting the effect of the dynamic pressure, the simplified versions of the proposed equation are also developed. The applicability of the full and the simplified equations is thoroughly investigated by simulating a seepage-induced spatially-varied flow over the face of a rockfill dam. Prior researches have not explored this particular flow problem using a higher-order equation. Additionally, a comparison is made between the results of these equations and the experimental data to evaluate how dynamic pressure and steep bed slope affect the overall performance of the equations. From practical perspective, investigating the spatially-varied flow phenomenon along the seepage face of a rockfill dam is a pre-requisite to placing the design of such a type of dam on a sounder theoretical basis. It is important to note that this study focuses exclusively on the analysis of a steady spatially-varied flow with a lateral inflow through the bottom of the channel. In subsequent sections, the derivation of the dynamic equations, the development of the numerical solution procedure, and the results of the investigation will be presented.

2. Development of the Mathematical Model

2.1. Dynamic Equations

A 2D spatially-varied flow in a steep-slope channel is considered for analysis, as illustrated in Fig. 1. A Cartesian coordinate system, with the x -axis along the channel bed of constant slope, the y -axis in transverse direction, and the z -axis normal to the channel bed, is also depicted in this figure. The flow under consideration is assumed to be a fully developed turbulent flow so that the Reynolds stress, varying gradually in the z -direction, is the primary contributor compared to the viscous shear stress

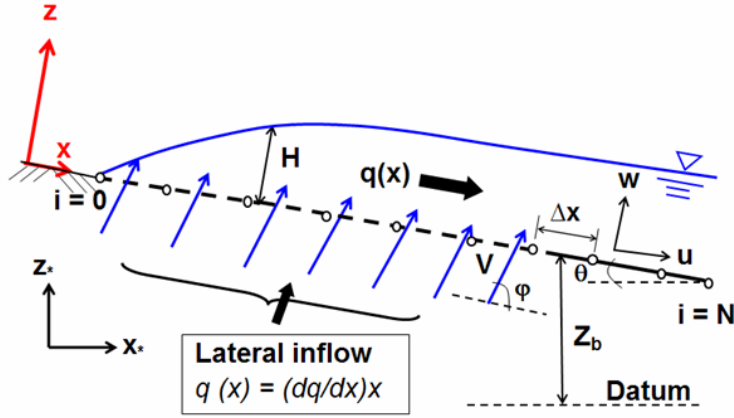


Fig. 1. Schematic representation of the problem of a spatially-varied flow with an increasing discharge. The computational nodes are indicated by a geometric symbol of circle

(Tennekes and Lumley 1972, pp. 79–85). Additionally, the gradient of the Reynolds stress along the streamwise x -direction is relatively small to be neglected. Using these approximations, the RANS equations are simplified to give the following equations for steady flow in a rectangular channel:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = g \sin \theta - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \frac{\partial \tau_{xz}}{\partial z}, \quad (2)$$

$$u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -g \cos \theta - \frac{1}{\rho} \frac{\partial p}{\partial z}, \quad (3)$$

where u and w are the velocities parallel and perpendicular to the channel bed, respectively; g is the acceleration due to gravity; ρ is the density of the fluid; p is the pressure; τ_{xz} refers to the shear stress; and θ is the slope of the channel bed. Implementing the aforementioned approximations in the RANS equations does not introduce significant errors in the resulting equations. In this study, the effects of the dynamic pressure and spatially-varied discharges are taken into account by using a pre-defined distribution of velocity across the flow depth in Equation (1). Hence, the velocity parallel to the channel bed is given by its average value over the depth as follows:

$$u = \frac{q}{H}, \quad (4)$$

where q is the discharge per unit width, and H is the flow depth normal to the bed. For the type of spatially-varied flow problem considered, the inflow or outflow takes

place along the channel bed. This implies that the channel bed serves as a surface, where streamlines either begin or terminate, highlighting that the kinematic boundary condition can be specified solely on the free surface. Substituting Equation (4) in Equation (1) and integrating from z to H gives the equation for the distribution of the velocity normal to the bed after employing the free-surface kinematic boundary condition, $w_H = (dH/dx) u$,

$$w = \frac{dq}{dx}(1 - \Lambda) - \frac{q}{H} \frac{dH}{dx}(1 - \Lambda) + \frac{q}{H} \frac{dH}{dx}, \quad (5a)$$

$$\Lambda = \frac{z}{H}, \quad (5b)$$

where Λ is a non-dimensional height normal to the bed; w_H is the velocity normal to the bed at the free surface; and z is the coordinate of a point in the flow field. The first term on the right-hand side (RHS) of this equation accounts for the effect of the spatially-varied discharge.

By differentiating Equation (5a) with respect to x and substituting the resulting expression into Equation (3), followed by integrating from an arbitrary point z to the free surface H , the pressure equation is obtained as follows:

$$\begin{aligned} \frac{p}{\rho g} = & H \cos \theta (1 - \Lambda) + \frac{q^2}{2gH} \frac{d^2 H}{dx^2} (1 - \Lambda^2) + \frac{q^2}{2gH^2} \left(\frac{dH}{dx} \right)^2 (\Lambda^2 - 1) \\ & + \frac{q}{2g} \frac{d^2 q}{dx^2} (1 - \Lambda)^2 - \frac{1}{2g} \left(\frac{dq}{dx} \right)^2 (1 - \Lambda)^2 + \frac{q}{gH} \frac{dq}{dx} \frac{dH}{dx} (1 - \Lambda). \end{aligned} \quad (6)$$

The implicit nature of Equation (6) allows for the inclusion of the effects of the dynamic pressure and steep bed slope. This equation clearly indicates that a substantial deviation from the hydrostatic pressure distribution is likely to arise only in the super-critical flow region, as previously noted (Hager and Hager 1985, Montes 1998, p. 26). The last three terms on the RHS of the equation account for the changes in pressure as a result of the spatially-varied discharge. Notably, the first term distinguishes this equation from the earlier pressure equations proposed by Zerihun (2015, 2019). For a gradually-varied flow in a steep-slope channel with a spatially-constant discharge, Equation (6) can be expressed in a simplified form as a lower-order pressure equation (Chow 1959, p. 32), i.e.,

$$\frac{p}{\rho g} = H \cos \theta (1 - \Lambda). \quad (7)$$

By making use of Equations (4) and (5a), Equation (2) is integrated over the flow depth. Following this, the resulting expression is simplified using the kinematic and dynamic boundary conditions, leading to the depth-averaged equation as follows:

$$\frac{d}{dx} \left(\beta \frac{q^2}{gH} \right) - \beta_L \frac{U_L}{g} \frac{dq}{dx} = H \sin \theta - \frac{d}{dx} \left(\int_0^H \frac{p}{\rho g} dz \right) - \frac{\tau_b}{\rho g}, \quad (8)$$

where τ_b is the bed shear stress; β or β_L refers to the Boussinesq coefficient; $U_L (= V \cos \varphi)$ is the streamwise velocity component of a lateral inflow or an outflow; V is the average lateral inflow or outflow velocity; and φ is the lateral inflow or outflow angle relative to the channel bed (see Fig. 1). It is important to note that dq/dx takes on a positive value in the case of lateral inflow, while it assumes a negative value for lateral outflow.

The pressure gradient term in Equation (8) is obtained by integrating Equation (6) with respect to z and subsequently differentiating the result with respect to x . Inserting the resulting expression into Equation (8) and rearranging the terms leads to the following equation for a steady, spatially-varied flow in a rectangular channel:

$$\begin{aligned} & \frac{q^2}{3} \frac{d^3 H}{dx^3} + \left(\frac{7q}{6} \frac{dq}{dx} - \frac{2q^2}{3H} \frac{dH}{dx} \right) \frac{d^2 H}{dx^2} + \left(gH \cos \theta - \beta \frac{q^2}{H^2} \right) \frac{dH}{dx} \\ & + \Gamma \frac{q}{H} \frac{dq}{dx} + \left(\frac{2q}{3} \frac{dH}{dx} - \frac{H}{6} \frac{dq}{dx} \right) \frac{d^2 q}{dx^2} - gH \sin \theta + \frac{\tau_b}{\rho} = 0, \end{aligned} \quad (9)$$

where $\Gamma = 2\beta - \beta_L U_L H/q$, and its values rely on the magnitudes of the change in momentum of the main flow and of the momentum flux of the inflow or outflow in the streamwise direction. It is clear from its definition that $\Gamma \leq 2$ for $\beta = \beta_L = 1$.

For a spatially-varied flow, the lateral inflow has a notable impact on the magnitude of the bed shear stress, especially when the ratio of the velocity of the lateral inflow to the streamwise velocity of the main flow is substantial (Turcotte 1960, Oldenziel and Brink 1974, Hansen 1992). Here, the Gauckler–Manning equation is applied to estimate the bed shear stress while ignoring the effect of inflow discharge. Thus,

$$\frac{\tau_b}{\rho} = \frac{gn^2 q^2}{R^{1/3} H^2}, \quad (10)$$

where R is the hydraulic radius of the flow section, and n is the Gauckler–Manning roughness coefficient. The resistance to flow is primarily dominated by the grain resistance of the material used in the dam face. Therefore, the empirical equation proposed by Strickler (1923) is employed to estimate the value of n . For the small lateral inflow discharge considered, the aforementioned approximation regarding τ_b does not substantially affect the quality of the numerical results.

Equation (9) is a depth-averaged higher-order equation for steady, 2D spatially-varied flows with increasing discharges or a lateral outflow from the bottom of the channel. This equation not only accounts for dynamic pressure effects but also incorporates the effects of steep bed slope and spatially-varied discharge, making it suitable for analyzing strongly curved open-channel flow problems with relative curvatures (κH) between -1 and 1 . For a weakly curved flow in a mild-slope channel ($|\kappa H| \leq 0.25$, $\cos \theta \cong 1$, and $\sin \theta = S_0$, where S_0 and κ are the channel bed slope and curvature of the free surface, respectively), the contributions of the second and the

fifth terms of the equation are insignificant compared to the other terms. Excluding these terms from Equation (9) results in the following simplified equation:

$$\frac{q^2}{3} \frac{d^3 H}{dx^3} + \left(gH - \beta \frac{q^2}{H^2} \right) \frac{dH}{dx} + \Gamma \frac{q}{H} \frac{dq}{dx} - gHS_0 + \frac{\tau_b}{\rho} = 0. \quad (11a)$$

Likewise, the pressure equation for this flow situation is obtained from Equation (6) by ignoring the small terms associated with the slope of the free surface as follows:

$$\begin{aligned} \frac{p}{\rho g} = & H(1 - \Lambda) + \frac{q^2}{2gH} \frac{d^2 H}{dx^2} (1 - \Lambda^2) \\ & + \frac{q}{2g} \frac{d^2 q}{dx^2} (1 - \Lambda)^2 - \frac{1}{2g} \left(\frac{dq}{dx} \right)^2 (1 - \Lambda)^2. \end{aligned} \quad (11b)$$

For a similar type of flow with a spatially-constant discharge in a mild-slope channel, the Boussinesq (1877, p. 192) equation can be recovered from Equation (11a). In the case of gradually-varied flows with spatially-varied discharges, the contributions from the curvature of the free surface diminish. Given this flow condition, Equation (9) can be expressed in its simplified form as the steep slope variant of the spatially-varied flow equation (Utami and Nakagawa 1967, Montes 1998, p. 301), i.e.,

$$\left(gH \cos \theta - \beta \frac{q^2}{H^2} \right) \frac{dH}{dx} + \Gamma \frac{q}{H} \frac{dq}{dx} - gH \sin \theta + \frac{\tau_b}{\rho} = 0. \quad (12)$$

By ignoring the inflow momentum flux ($\Gamma = 2\beta$), a more simplified version of the above equation was previously reported (e.g., Sharp and James 1962, Hansen 1992). For a hydrostatic flow with a spatially-increasing discharge in a mild-slope channel, this equation degenerates to the lowest-order form of the gradually-varied flow equation (see Chow 1959, p. 331). Besides, the steep slope form of the shallow-water equation (Utami and Nakagawa 1968, Berger 1994) can be retrieved from Equation (12) by setting $dq/dx = 0$ as follows:

$$\left(gH \cos \theta - \beta \frac{q^2}{H^2} \right) \frac{dH}{dx} - gH \sin \theta + \frac{\tau_b}{\rho} = 0. \quad (13)$$

The above analysis reveals that Equation (9) is different from the earlier equations (e.g., Nakagawa 1969, Castro-Orgaz and Hager 2011, Zerihun 2019) and includes a higher-order correction factor for the effects of the dynamic pressure and steep bed slope. This equation, together with Equations (11a) and (12), will be applied to analyze the salient features of the problems of spatially-varied flow along the seepage face of a rockfill dam. A comparison will be made between the numerical solutions of these equations and the experimental data to systematically analyze the contribution of the correction factor of the proposed dynamic equation.

2.2. Inflow Discharge

In the absence of measured data, the lateral inflow discharge from the turbulent seepage zone can be estimated by using a seepage flow model (e.g., van Gent 1995, Zeri-hun 2025) or empirical equations. Using an exponential equation to account for inertial effects and non-linear flow behavior and assuming a hydrostatic pressure distribution in the seepage and spatially-varied flow regions (for details, see the Appendix), the following expression is obtained:

$$\frac{dq}{dx} = \frac{\mu_1}{\mu_2^m} \left(\tan \theta - \frac{dH}{dx} \right)^m, \quad (14a)$$

where $\mu_1 = \lambda \sin \theta$; λ is the effective porosity of the porous media; m is an exponent; and μ_2 is a dimensional constant. The exponent of the above equation is a function of the grain size of the dam material; it is approximately equal to 0.54 for grain size greater than 12.7 mm (see Parkin et al 1966). If the contribution of the term dH/dx is ignored, then Equation (14a) reduces to the seepage discharge equation proposed by Sharp and James (1962). Differentiating this equation with respect to x gives the required closure equation as follows:

$$\frac{d^2q}{dx^2} = \frac{m\mu_1}{\mu_2^m} \left(\tan \theta - \frac{dH}{dx} \right)^{m-1} \frac{d^2H}{dx^2}. \quad (14b)$$

The lateral inflow discharge equations are numerically combined with the above dynamic equations in order to solve the problem of spatially-varied flow with curved streamlines.

2.3. Boundary Conditions

The numerical solutions of the previously mentioned higher-order dynamic equations require three boundary conditions. In contrast, Equation (12), being a first-order differential equation, can be solved by imposing only one boundary condition. For the supercritical flow problems considered here, these conditions are specified at the upstream end of the lateral-inflow zone (indicated by node $i = 0$ in Fig. 1). The flow depth and the slope of the free surface are specified as boundary conditions. In addition, a zero-flux boundary condition (no overtopping flow over the rockfill dam) is imposed at this section, which yields the following expression for the free-surface slope after simplifying Equation (9) or (11a) using this specified condition:

$$\left(\frac{dH}{dx} \right)_{i=0} - \tan \theta = 0. \quad (15)$$

3. Numerical Method

The dynamic equation describing the steady spatially-varied flow is a third-order non-linear ordinary differential equation. Even for the simplest flow cases, an analytical solution is unattainable for this equation. In this study, an implicit finite-difference scheme is applied to obtain a numerical solution for the seepage-induced flow problem. Following the approach of Zerihun (2015), Equations (9) and (11a) are rewritten in the following general form by making the coefficients of the leading terms of the equations unity:

$$\frac{d^3 H}{dx^3} + \xi_0 \frac{d^2 H}{dx^2} + \xi_1 \frac{dH}{dx} + \xi_2 = 0, \quad (16)$$

where ξ_0 , ξ_1 and ξ_2 are the non-linear coefficients associated with the equations, and the corresponding expressions can be obtained by comparing this equation with the respective dynamic equations. Since the flow depth is an unknown variable, the spatial derivative terms in the above equation are approximated using the central finite-difference equations (Bickley 1941). For computational nodes situated near the outflow boundary, these derivative terms are discretized using the three-point backward finite differences. The implementation of the discretized form of Equation (16) at every computational node results in a system of non-linear algebraic equations. An iterative method is employed here to solve this system of equations in conjunction with the given boundary conditions, starting from the presumed initial position of the free surface. The non-linear algebraic equations are initially linearized using the Newton–Raphson method with a numerical Jacobian matrix and subsequently solved by the lower-upper (LU) decomposition technique. The assessment of the convergence of the numerical solutions is based on the relative change in the solution criterion, which has a convergence tolerance set at 10^{-6} . To reduce numerical inaccuracies resulting from spatial discretization, the step size (Δx) was maintained within 0.6 to 2% of the streamwise length of the computational domain.

4. Results of the Numerical Tests

This section examines the influences of the dynamic pressure and steep bed slope on the overall performance of the dynamic equations. Such a vital modeling task requires measured data for verification. There is a limited availability of field data regarding steady spatially-varied flow along the seepage face of a rockfill dam. Hence, experimental data from previous studies were used to achieve the objectives of the numerical investigation. These data were collected from experiments conducted on rockfill dams with different geometrical configurations. The properties of the rockfill material and the geometry of the dam primarily affect the magnitude of the seepage discharge and hence, the nature of the supercritical open-channel flow with a spatially-increasing discharge.

The numerical results of the present study are systematically summarized by using non-dimensional parameters. The horizontal length of the seepage face $L_S \cos \theta$ (L_S is the length of the seepage face in streamwise direction) and the elevation of the free surface at a nodal point $i = 0$ served as baseline parameters for non-dimensionalizing the horizontal distance x_* and the free-surface elevation η ($= H \cos \theta + Z_b$, where Z_b is the elevation of the channel bed), respectively, with $x = x_* \sec \theta$.

4.1. Effect of the Lateral Inflow Momentum Flux

The application of the dynamic equation of a spatially-varied flow based on a linear momentum method requires prior information about the magnitude and direction of the inflow momentum flux. In this study, the mean value of the flux was determined by comparing the results of Equation (9), which were computed for different values of Γ , with the experimental data provided by Sharp and James (1962) and evaluating the errors. Sharp and James conducted the experiments in a rectangular flume of 305 mm in width, 457 mm in depth, and 914 mm in length. A triangular embankment dam model with a downstream-face slope of 25.2° was constructed using rockfill material with a median grain size of 19.1 mm and a porosity of 0.422. The dam model stood at a height of 318 mm, with a base length measuring 744 mm. The flow depth and velocity at various sections were measured using a sealed pointer and a Pitot tube, respectively. As shown in Fig. 2a, the non-dimensional free-surface profile was marginally affected by the value of the inflow momentum flux parameter Γ . Employing Γ values of 0.5, 1.0, and 2.0 resulted in relatively less accurate results (see Fig. 2b), with the values of the mean absolute relative error (MARE) varying from 1.79 to 2.61%. A notably better result was attained by using a Γ value of 1.5, leading to a MARE value of 1%. Even though the experimental data were dispersed, an analogous analysis result for the discharge profile was obtained, as illustrated in Fig. 2c. A typical Γ value of 1.5 was considered as a mean value to account for the effects of the inflow momentum flux for the seepage-induced spatially-varied flow problem.

4.2. Effect of the Dynamic Pressure

The experimental data of Roshanfekr (2013) were chosen to quantitatively substantiate the numerical results. He conducted the experiments in a horizontal flume with glass walls, measuring 310 mm wide, 450 mm deep, and 9 m long. Models of embankment dam were constructed from rockfill material with a median grain size of 17.2 mm and were installed in the test section located 7.2 m downstream of the inlet of the flume. The tested models had a constant height of 300 mm, and their crest lengths (L_C) varied from 0 to 750 mm. Also, the upstream face of each model maintained a slope of 45° , while the downstream faces were inclined at 18.43° , 26.57° , and 45° . The experiments used a material with a porosity of 0.42. For each test, the total discharge was measured at the downstream end section using an electromagnetic field distortion meter. The elevations of the free surface along the seepage face of the

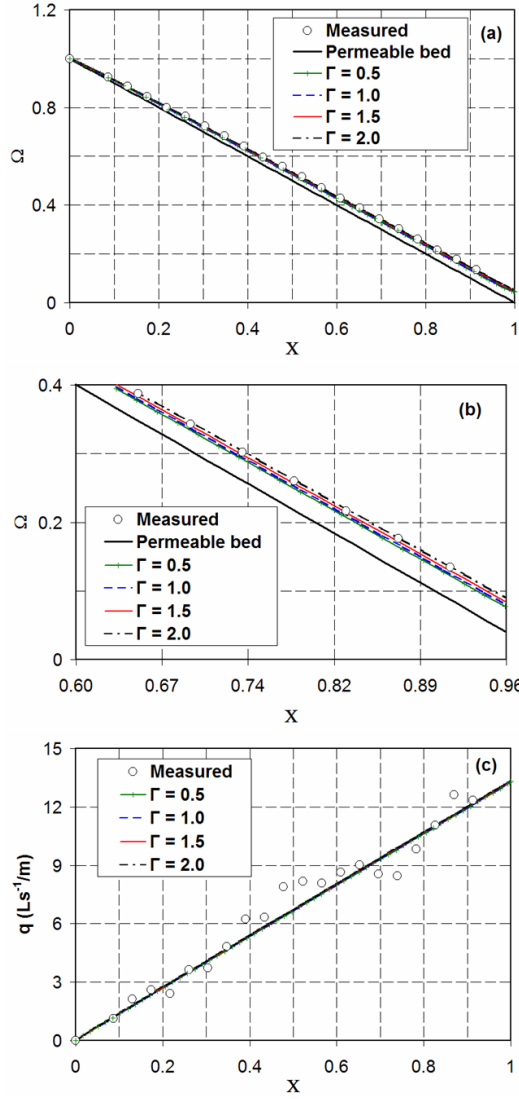


Fig. 2. Comparison of the computed results with the experimental data for various values of Γ : (a) free-surface profile; (b) a detailed view near the toe of the dam; and (c) discharge profile. The non-dimensional free-surface elevation and horizontal distance are denoted by Ω and X , respectively

dam were obtained by placing rectangular grids on the glass surface of the flume. For additional details about the experimental system, see Roshanfekr (2013).

The role of dynamic pressure in affecting the results of Equation (9) was examined by comparing its results with those of Equation (12) and experimental data. As mentioned before, the latter equation was developed without considering the effects

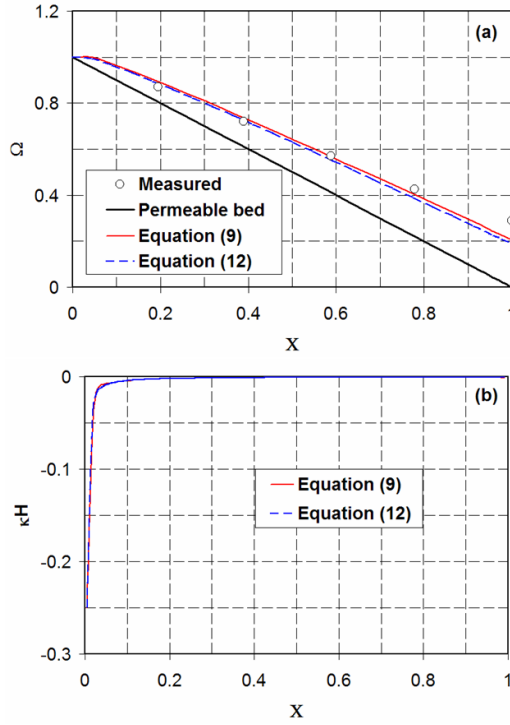


Fig. 3. (a) Comparison of the computed results for free-surface profile with the experimental data ($\theta = 45^\circ$, $q_N = 12.32$ (L/s)/m, $L_C = 450$ mm); and (b) variation of the relative curvature of the free surface κH with the non-dimensionalized horizontal distance X

of the streamline vertical curvature. Fig. 3a presents a comparison between the results of these equations and the experimental data for a total lateral inflow discharge (q_N) of 12.32 (L/s)/m. Although the numerical results of the two equations differ slightly, the results obtained from Equation (9) reveal a better concordance with the experimental data. The computed MARE values for Equations (9) and (12) are 7.6 and 9.5%, respectively. The numerical results were further analyzed to examine the nature of the curved flow problem by simulating the relative curvature of the free surface, κH . Fig. 3b shows the streamwise variation of the relative curvature of the flow. Both equations accurately predicted the convex nature of the free-surface profile ($\kappa < 0$) in the vicinity of the inflow section ($0 < X \leq 0.3$). Downstream of this region, the curvature of the streamline remains insignificant ($\kappa \approx 0$). In addition to this, the magnitude of this parameter ($|\kappa H| \leq 0.25$) evidently implies that the analyzed spatially-varied flow problem represents a form of weakly curved flow problem. For this type of problem, the effect of the dynamic pressure is not significant compared to the pronounced effect of the steep slope of the channel bed. Thus, Equation (12) satisfactorily captured the complex attributes of the flow field, exhibiting a performance level that was compara-

ble to Equation (9). This result is in agreement with the findings presented by Berger (1994) regarding transcritical open-channel flow with a spatially-constant discharge.

4.3. Weakly Curved Flow Approximation

4.3.1. Free-Surface Profile

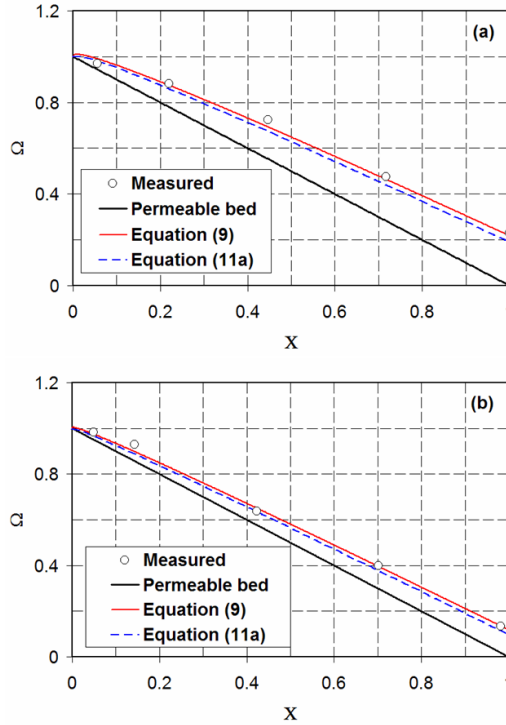


Fig. 4. Free-surface profiles for spatially-varied flows along the seepage face of a triangular rockfill dam: (a) $\theta = 26.57^\circ$, $q_N = 8.65$ (L/s)/m; and (b) $\theta = 18.43^\circ$, $q_N = 8.65$ (L/s)/m

In order to examine the impact of the weakly curved flow approximation, the solutions of Equation (9) were compared with those of Equation (11a) and experimental data. As previously stated, Equation (11a) incorporates a lower-order correction for the effect of the dynamic pressure. Fig. 4 illustrates that this equation underestimates the elevations of the free surface throughout the computational domain, with a maximum MARE value of 6.5%. For this quasi-hydrostatic flow problem, the steep slope of the channel plays a dominant role in affecting the dynamic characteristics of the supercritical flow with a spatially-varied discharge. However, this equation does not have any terms that take into account the effect of a hydraulically steep bed slope. Consequently, its application is limited to an open-channel flow with slightly sloped

and curved streamlines in a mild-slope channel. Compared to Equation (11a), Equation (9) accurately predicts the free-surface profiles, and its results agree reasonably well with the measurements. The computed maximum MARE value for this equation is 2.9%.

4.3.2. Bed Pressure Profile

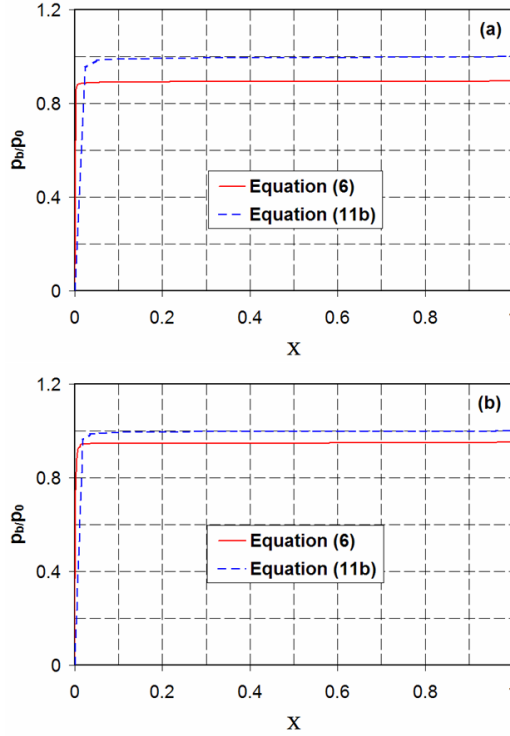


Fig. 5. Bed pressure profiles for seepage-induced spatially-varied flows: (a) $\theta = 26.57^\circ$, $q_N = 8.65$ (L/s)/m; and (b) $\theta = 18.43^\circ$, $q_N = 8.65$ (L/s)/m

For thoroughly analyzing the impact of the weakly curved flow approximation, the bed pressure profiles of Equation (6) were compared with the results of Equation (11b). Fig. 5 shows the variation of the non-dimensionalized bed pressure in the streamwise horizontal direction. In this figure, the total bed pressure and its hydrostatic component are indicated by p_b and p_0 , respectively. Due to the lack of a correction term for the effect of the steep slope of the channel bed, Equation (11b) overestimates the magnitude of the bed pressure compared to the results of Equation (6). In the flow region downstream of the inflow section ($X \geq 0.2$), it predicts a hydrostatic bottom pressure. Ignoring the minor effect of the streamline vertical curvature, the disparity between the results of these pressure equations is proportional to the cosine

of the longitudinal slope of the channel, with a notable discrepancy observed in case of a steeper slope. The result of this investigation reveals that employing a mild slope approximation has a substantial impact on the accuracy of the numerical results associated with this type of spatially-varied flow problem, in agreement with that of Zerihun (2016) for a short-crested weir flow with a spatially-constant discharge.

5. Evaluation of the Overall Performance

As noted earlier, various assumptions have been employed in the formulation of Equations (9), (11a), and (12). Consequently, these equations incorporate different degrees of approximation for the dynamic effect of the flow. The validity of these assumptions plays a crucial role in determining the accuracy of the results pertaining to the local and global characteristics of the flow. For the spatially-varied flow problem considered, the effects of dynamic pressure and the channel's steep gradient significantly alter the structure of the supercritical flow. The incorporation of these intricate effects into the dynamic equation does not render it overly complex, but rather enhances its performance and range of applicability. The combined effects of the applied approximations were examined, and the results are summarized below.

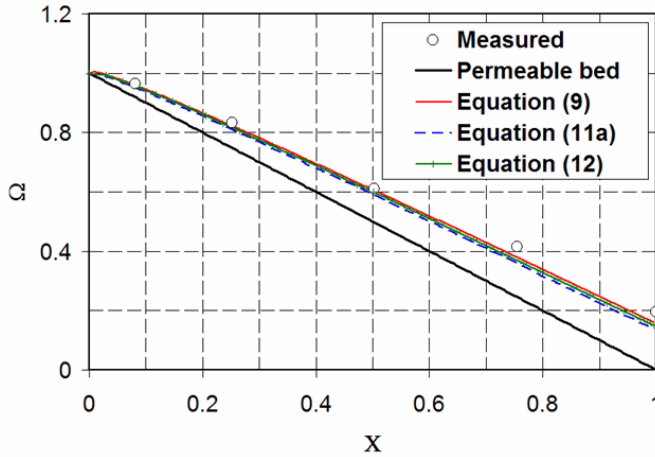


Fig. 6. Free-surface profiles predicted by various spatially-varied flow equations. The values of the key parameters for triangular embankment dam are $\theta = 45^\circ$ and $q_N = 12.32$ (L/s)/m

For a typical spatially-varied flow problem, the numerical results obtained from the aforementioned three equations were compared with the experimental data in Fig. 6. As a consequence of being a weak curvature approximation to Equation (9), Equation (11a) predicts shallower flow depths that diverge from the measured data, as depicted in this figure. For this quasi-hydrostatic flow in a steeply inclined channel, the inclusion of the steep slope effect while omitting the dynamic pressure effect

marginally enhances the results of Equation (12). Equation (9), being a higher-order equation compared to these two equations, yields noticeably improved results. The computed MARE values for Equation (9) and those of Equations (11a) and (12) are 5.9, 9.8, and 7.7%, respectively. As demonstrated in Section 4.3, the results of the bed pressure profile exhibit greater sensitivity to the applied approximations compared to the results of the free-surface profile. The overall results of the comparison suggest that it is sufficient to apply Equation (11a) for analyzing the problem of spatially-varied flow with small curvatures of the streamline. Furthermore, Equation (12) is deemed more appropriate for a similar flow problem in a steep-slope channel. For moderately and strongly curved flow problems, Equation (9) is preferred over these simplified equations (see also Zerihun (2016) for a similar investigation result related to constant channel flow conditions). As a higher-order, depth-averaged equation, it is capable of offering valuable information regarding the mean flow characteristics of an open-channel flow with a non-uniform discharge.

6. Conclusions

The problem of spatially-varied flow along the seepage face of a rockfill dam was numerically investigated using three different types of depth-averaged open-channel flow equations. The proposed Equation (9) incorporates a higher-order Boussinesq-type correction to account for the combined effects of dynamic pressure and steep bed slope, while Equations (11a) and (12) are the simplified versions of this equation for slightly curved flows in channels with mild and steep slopes, respectively. An implicit finite-difference scheme was employed to acquire numerical solutions for these non-linear equations. Their predicted results were compared with the measurements obtained from experiments conducted on rockfill dams with different geometrical configurations. The impacts of the simplifying assumptions on the results of the equations were thoroughly examined.

In all simulation cases, the results of Equation (9) for free-surface profile were marginally better than those of Equations (11a) and (12). For the quasi-hydrostatic flow problem considered here, the effect of the dynamic pressure is insignificant compared to the marked influence of the steep slope of the channel. Consequently, Equation (12) adequately captured the complex features of the spatially-varied flow field, demonstrating a performance level that was comparable to Equation (9). However, the results of Equation (11a) were not quite so accurate due to the mild slope approximation used to derive this equation. Such an approximation was also affected the accuracy of the results of the bed pressure profile. Although Equation (9) is relatively complex, the significance of its terms was clearly demonstrated through the results of the comparison. This equation is therefore the preferred choice for modeling open-channel flow that exhibits moderate to strong streamline curvature, rather than utilizing its simplified versions. In conclusion, Equation (9) could be coupled

with a turbulent seepage flow model to analyze the problem of unraveling failure in rockfill dam.

Appendix: Derivation of the Inflow Discharge Equation

For a nearly horizontal flow through a coarse-grained porous media, the piezometric head at the exit point is approximated by

$$H_p = K_C H \cos \theta + Z_b, \quad (17)$$

where H_p is the piezometric or hydraulic head, and K_C is the depth-averaged pressure correction coefficient for the effect of the streamline curvature. The horizontal gradient (i_H) of the piezometric head can be obtained by differentiating Equation (17) with respect to x_* as follows:

$$i_H = \frac{dH_p}{dx_*} = \left(K_C \frac{dH}{dx_*} + H \frac{dK_C}{dx_*} \right) \cos \theta + \frac{dZ_b}{dx_*} = K_C \frac{dH}{dx} + H \frac{dK_C}{dx} - \tan \theta, \quad (18)$$

where $x_*(= x \cos \theta)$ is the local Cartesian coordinate in the horizontal direction (see Fig. 1). Using the exponential equation (Polubarinova-Kochina 1962, p. 25) for non-linear flow through porous media, the seepage velocity can be expressed as

$$u_* = \lambda \left(\frac{|i_H|}{\mu_2} \right)^m, \quad (19)$$

where u_* is the bulk velocity of the turbulent seepage flow in the horizontal direction. The seepage discharge through a horizontal layer of thickness dh is given by

$$dq = u_* dh = u_* dx \sin \theta. \quad (20)$$

Inserting Equations (18) and (19) into Equation (20) and simplifying the resulting expression using the hydrostatic pressure approximation ($K_C = 1.0$) leads to

$$\frac{dq}{dx} = \frac{\mu_1}{\mu_2^m} \left(\tan \theta - \frac{dH}{dx} \right)^m, \quad (21)$$

$$\mu_1 = \lambda \sin \theta. \quad (22)$$

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