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Strong b-Metric Spaces and Common Fixed Points [1]

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Abstract

This contribution's aim is to find unique common fixed points using our concept of occasionally weakly biased mappings of type $(\mathbb{A})$, in complete strong b-metric spaces, and by minimal conditions. To convince the readers of the validity and pertinently of our results, we provided for each theorem its suitable example. Our theorems improve some existing results in metric as well as strong b-metric spaces.

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1 Introduction

We all know that, a standard metric space is a set of points say for instance $\mathbf{X}$ equipped with a standard distance function d, called a standard metric, that satisfies the next conditions:

1. **non-negativity:** for all ι_1 and ι_2 belong to $\mathbf{X}$, $d(\iota_1, \iota_2) \geq 0$,

2. **identity of indiscernible:** for all ι_1 and ι_2 belong to $\mathbf{X}$, $d(\iota_1, \iota_2) = 0 \Leftrightarrow \iota_1 = \iota_2$,

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3. **symmetry:** for all ι_1 and ι_2 belong to $\mathbf{X}$, $d(\iota_1, \iota_2) = d(\iota_2, \iota_1)$,
4. **triangle inequality:** for all ι_1, ι_2, ι_3 belong to $\mathbf{X}$, $d(\iota_1, \iota_2) \leq d(\iota_1, \iota_3) + d(\iota_3, \iota_2)$.

Also, we all know that Banach fixed-point theorem states that: if we have a complete standard metric space say for instance $(\mathbf{X}, d)$ and a contraction mapping $\mathbf{T} : \mathbf{X} \to \mathbf{X}$, then, $\mathbf{T}$ possesses a unique fixed point.

To improve and generalize this "star" result, a huge number of mathematicians concentrated on the standard metric space. They made some significant changes especially in the triangle inequality. In particular, Bakhtin [2] and Czerwik [17] proposed the notion of b-metric space as an extension of the standard metric space where the triangle inequality is relaxed to a weakened form; i.e., $d(x, y) \leq b[d(x, z)+d(z, y)]$ where $b \geq 1$ is a real element. Again, Kirk and Shahzad [19] gave a novel notion called strong b-metric space as another generalization of the standard metric space, where the triangle inequality is again relaxed to a weakened form; i.e., $d(x, y) \leq bd(x, z) + d(z, y)$ or $d(x, y) \leq d(x, z) + bd(z, y)$ where $b \geq 1$ is a real point. For this end, we note that, several researchers used strong b-metric spaces to prove the existence and uniqueness of fixed and common fixed points, for example see [1], [18], [19], [20], [21], [22], and [23]. We also mention that, a lot of papers treated the existence and uniqueness of fixed and common fixed points for single and multi-valued mappings in different spaces, see for instance [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14] and [15].

Now, to establish minimal conditions under which pairs of mappings will have common fixed points, we suggested a novel notion of occasionally weakly biased mappings of type $(\mathbb{A})$ (see [3]). It refers to, if we have pairs of mappings in a mathematical space where there is at least one coincidence point where the mappings outputs are equal, and at this specific point, the mappings commute. Our notion is weaker than some previous concepts.

In this article, we will use our notion to prove the existence and uniqueness of common fixed points in strong b-metric spaces, and with convinced examples, we will show the pertinently of our results.

2 Needed Statements

Definition 1 *Let $\mathbf{X} \neq \emptyset$ be a set. A mapping $d : \mathbf{X} \times \mathbf{X} \to [0, +\infty)$ is said to be standard metric if for all ϱ_1, ϱ_2, ϱ_3 belong to $\mathbf{X}$, the next axioms hold:*

a_1**:** $d(\varrho_1, \varrho_2) = 0 \Leftrightarrow \varrho_1 = \varrho_2$,

a_2**:** $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1)$,

a_3**:** $d(\varrho_1, \varrho_2) \leq d(\varrho_1, \varrho_3) + d(\varrho_3, \varrho_2)$, *the pair* $(\mathbf{X}, d)$ *is called* ***standard metric space.***

Definition 2 *([2], [17]) Let* $\mathbf{X} \neq \emptyset$ *be a set and* $b \geq 1$ *be any real number. A mapping* $d : \mathbf{X} \times \mathbf{X} \to [0, +\infty)$ *is said to be b-metric with coefficient b if for all* ϱ_1, ϱ_2, ϱ_3 *belong to* $\mathbf{X}$*, the next axioms hold:*

a_1**:** $d(\varrho_1, \varrho_2) = 0 \Leftrightarrow \varrho_1 = \varrho_2$,

a_2**:** $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1)$,

a_3**:** $d(\varrho_1, \varrho_2) \leq b[d(\varrho_1, \varrho_3) + d(\varrho_3, \varrho_2)]$,

the triplet $(\mathbf{X}, d, b)$ *is called* ***b-metric space.***

Definition 3 *([19]) Let* $\mathbf{X} \neq \emptyset$ *be a set and* $b \geq 1$ *be any real number. A mapping* $d : \mathbf{X} \times \mathbf{X} \to [0, +\infty)$ *is said to be strong b-metric with coefficient b if for all* ϱ_1, ϱ_2, ϱ_3 *belong to* $\mathbf{X}$*, the next axioms hold:*

a_1**:** $d(\varrho_1, \varrho_2) = 0 \Leftrightarrow \varrho_1 = \varrho_2$,

a_2**:** $d(\varrho_1, \varrho_2) = d(\varrho_2, \varrho_1)$,

a_3**:** $d(\varrho_1, \varrho_2) \leq bd(\varrho_1, \varrho_3) + d(\varrho_3, \varrho_2)$ *or* $d(\varrho_1, \varrho_2) \leq d(\varrho_1, \varrho_3) + bd(\varrho_3, \varrho_2)$,

the triplet $(\mathbf{X}, d, b)$ *is called* ***strong b-metric space.***

Definition 4 *([3]) Self-mappings* $\mathbf{L}$ *and* $\mathbf{M}$ *of a strong b-metric space* $(\mathbf{X}, d, b)$ *are* ***occasionally weakly L-biased (respectively M-biased) of type*** $(\mathbb{A})$*, if and only if, there is an element* $\xi \in \mathbf{X}$ *which satisfies* $\mathbf{L}\xi = \mathbf{M}\xi$*, this implies*

$$d(\mathbf{LL}\xi, \mathbf{M}\xi) \leq d(\mathbf{ML}\xi, \mathbf{L}\xi),$$
$$d(\mathbf{MM}\xi, \mathbf{L}\xi) \leq d(\mathbf{LM}\xi, \mathbf{M}\xi),$$

respectively.

3 Our Main Theorems

Theorem 1 *Consider four mappings* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{S}$ *and* $\mathbf{T}$ *from a complete strong b-metric space* $(\mathbf{X}, d, b)$ *into itself which satisfying the two conditions:*

1. $\mathbf{A}$ *and* $\mathbf{S}$ *as well as* $\mathbf{B}$ *and* $\mathbf{T}$ *are occasionally weakly* $\mathbf{A}$*-biased (respectively* $\mathbf{B}$*-biased) of type* $(\mathbb{A})$*,*

2. *for all x and y belong to* $\mathbf{X}$*, we have*

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) \quad \leq \quad & \delta_1 \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y)\} \\ & +\delta_2 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\} \end{aligned}$$

where $(b + 1)\delta_1 + \delta_2 < 1$.

Then, the four mappings have a unique common fixed point.

Proof. In the following, we will prove the existence and uniqueness of the common fixed point of the four mappings.

Existence: From the first condition, there are two elements (say α and β) belong to $\mathbf{X}$ such that $\mathbf{A}\alpha = \mathbf{S}\alpha$, $\mathbf{B}\beta = \mathbf{T}\beta$ and

$$\begin{aligned} d(\mathbf{AA}\alpha, \mathbf{S}\alpha) &\leq d(\mathbf{SA}\alpha, \mathbf{A}\alpha), \\ d(\mathbf{BB}\beta, \mathbf{T}\beta) &\leq d(\mathbf{TB}\beta, \mathbf{B}\beta). \end{aligned}$$

To prove the existence,we must go through three inevitable steps:

First step: Assume that $\mathbf{A}\alpha \neq \mathbf{B}\beta$, then, we have

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) \quad \leq \quad & \delta_1 \max\{d(\mathbf{A}\alpha, \mathbf{B}\beta), d(\mathbf{A}\alpha, \mathbf{S}\alpha), d(\mathbf{B}\beta, \mathbf{T}\beta)\} \\ & +\delta_2 \max\{d(\mathbf{S}\alpha, \mathbf{B}\beta), d(\mathbf{A}\alpha, \mathbf{T}\beta)\} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) \quad &\leq \quad \delta_1 \max\{d(\mathbf{S}\alpha, \mathbf{T}\beta), d(\mathbf{S}\alpha, \mathbf{S}\alpha), d(\mathbf{T}\beta, \mathbf{T}\beta)\} \\ & \qquad +\delta_2 \max\{d(\mathbf{S}\alpha, \mathbf{T}\beta), d(\mathbf{S}\alpha, \mathbf{T}\beta)\} \\ &= \quad (\delta_1 + \delta_2) d(\mathbf{S}\alpha, \mathbf{T}\beta) \\ &< \quad d(\mathbf{S}\alpha, \mathbf{T}\beta) \end{aligned}$$

which is a contradiction, hence, $\mathbf{S}\alpha = \mathbf{T}\beta$.

Second step: Suppose that $\mathbf{SS}\alpha \neq \mathbf{S}\alpha$, then, we have

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{T}\beta) \quad \leq \quad & \delta_1 \max\{d(\mathbf{AS}\alpha, \mathbf{B}\beta), d(\mathbf{AS}\alpha, \mathbf{SS}\alpha), d(\mathbf{B}\beta, \mathbf{T}\beta)\} \\ & +\delta_2 \max\{d(\mathbf{SS}\alpha, \mathbf{B}\beta), d(\mathbf{AS}\alpha, \mathbf{T}\beta)\} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \quad &\leq \quad \delta_1 \max\{d(\mathbf{AA}\alpha, \mathbf{S}\alpha), d(\mathbf{AA}\alpha, \mathbf{SS}\alpha), d(\mathbf{S}\alpha, \mathbf{S}\alpha)\} \\ & \qquad +\delta_2 \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), d(\mathbf{AA}\alpha, \mathbf{S}\alpha)\} \\ &= \quad \delta_1 \max\{d(\mathbf{AA}\alpha, \mathbf{S}\alpha), d(\mathbf{AA}\alpha, \mathbf{SS}\alpha)\} \\ & \qquad +\delta_2 \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), d(\mathbf{AA}\alpha, \mathbf{S}\alpha)\} \\ &\leq \quad \delta_1 \max\{d(\mathbf{AA}\alpha, \mathbf{S}\alpha), bd(\mathbf{AA}\alpha, \mathbf{S}\alpha) + d(\mathbf{S}\alpha, \mathbf{SS}\alpha)\} \\ & \qquad +\delta_2 \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), d(\mathbf{AA}\alpha, \mathbf{S}\alpha)\} \\ &\leq \quad \delta_1 \max\{d(\mathbf{SA}\alpha, \mathbf{A}\alpha), bd(\mathbf{SA}\alpha, \mathbf{A}\alpha) + d(\mathbf{S}\alpha, \mathbf{SS}\alpha)\} \\ & \qquad +\delta_2 \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), d(\mathbf{SA}\alpha, \mathbf{A}\alpha)\} \\ &= \quad \delta_1 \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), (b+1) d(\mathbf{SS}\alpha, \mathbf{S}\alpha)\} + \delta_2 d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\ &= \quad [(b+1)\delta_1 + \delta_2] d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\ &< \quad d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \end{aligned}$$

which is a contradiction, hence, $\mathbf{SS}\alpha = \mathbf{S}\alpha$ and therefore $\mathbf{AS}\alpha = \mathbf{S}\alpha$.

Third step: Now, assume that $\mathbf{TT}\beta \neq \mathbf{T}\beta$, then, we have

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{TT}\beta) &\leq \delta_1 \max\{d(\mathbf{A}\alpha, \mathbf{BT}\beta), d(\mathbf{A}\alpha, \mathbf{S}\alpha), d(\mathbf{BT}\beta, \mathbf{TT}\beta)\} \\ &\quad +\delta_2 \max\{d(\mathbf{S}\alpha, \mathbf{BT}\beta), d(\mathbf{A}\alpha, \mathbf{TT}\beta)\} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\mathbf{T}\beta, \mathbf{TT}\beta) &\leq \delta_1 \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), d(\mathbf{BB}\beta, \mathbf{TT}\beta)\} \\ &\quad +\delta_2 \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), d(\mathbf{T}\beta, \mathbf{TT}\beta)\} \\ &\leq \delta_1 \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), bd(\mathbf{BB}\beta, \mathbf{B}\beta) + d(\mathbf{B}\beta, \mathbf{TT}\beta)\} \\ &\quad +\delta_2 \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), d(\mathbf{T}\beta, \mathbf{TT}\beta)\} \\ &\leq \delta_1 \max\{d(\mathbf{B}\beta, \mathbf{TB}\beta), bd(\mathbf{TB}\beta, \mathbf{B}\beta) + d(\mathbf{T}\beta, \mathbf{TT}\beta)\} \\ &\quad +\delta_2 \max\{d(\mathbf{B}\beta, \mathbf{TB}\beta), d(\mathbf{T}\beta, \mathbf{TT}\beta)\} \\ &= \delta_1 \max\{d(\mathbf{T}\beta, \mathbf{TT}\beta), (b+1)d(\mathbf{T}\beta, \mathbf{TT}\beta)\} + \delta_2 d(\mathbf{T}\beta, \mathbf{TT}\beta) \\ &= [(b+1)\delta_1 + \delta_2] d(\mathbf{T}\beta, \mathbf{TT}\beta) \\ &< d(\mathbf{T}\beta, \mathbf{TT}\beta) \end{aligned}$$

which is a contradiction, thus, $\mathbf{T}\beta = \mathbf{TT}\beta$ and therefore $\mathbf{T}\beta = \mathbf{BT}\beta$. Consequently, $\eta = \mathbf{A}\alpha = \mathbf{S}\alpha = \mathbf{B}\beta = \mathbf{T}\beta$ is a common fixed point of the four mappings.

Uniqueness: Finally, assume that there exists another fixed point than η say for instance μ, then, we have

$$\begin{aligned} d(\mathbf{S}\eta, \mathbf{T}\mu) &\leq \delta_1 \max\{d(\mathbf{A}\eta, \mathbf{B}\mu), d(\mathbf{A}\eta, \mathbf{S}\eta), (\mathbf{B}\mu, \mathbf{T}\mu)\} \\ &\quad +\delta_2 \max\{d(\mathbf{S}\eta, \mathbf{B}\mu), d(\mathbf{A}\eta, \mathbf{T}\mu)\} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\eta, \mu) &\leq \delta_1 \max\{d(\eta, \mu), d(\eta, \eta), d(\mu, \mu)\} + \delta_2 \max\{d(\eta, \mu), d(\eta, \mu)\} \\ &= [\delta_1 + \delta_2] d(\eta, \mu) \\ &< d(\eta, \mu) \end{aligned}$$

this contradiction requires the equality between η and μ.

Example 1 *Let* $\mathbf{X} = \{0, 1, 2\}$. *Define*

$$\begin{aligned} d: \quad & \mathbf{X} \times \mathbf{X} \to [0, +\infty) \\ & (x, y) \to (x - y)^2. \end{aligned}$$

Now, Consider four mappings as follows:

$$\mathbf{A}x = \begin{cases} 0 \text{ when } x = 0 \\ 2 \text{ when } x \in \{1, 2\}, \end{cases} \quad \mathbf{B}x = \begin{cases} 0 \text{ when } x = 0 \\ x \text{ when } x \in \{1, 2\}, \end{cases}$$

$$\mathbf{S}x = \mathbf{T}x = 0 \text{ when } x \in \{0, 1, 2\}.$$

In the following lines, we will prove three required axioms:

First axiom: $(\mathbf{X}, d, b = 3)$ *is a complete strong b-metric space.*

Second axiom: *Mappings* $\mathbf{A}$ *and* $\mathbf{S}$ *as well as* $\mathbf{B}$ *and* $\mathbf{T}$ *are occasionally weakly* $\mathbf{A}$*-biased (respectively* $\mathbf{B}$*-biased) of type* $(\mathbb{A})$*.*

Third axiom: *The inequality of the above theorem is satisfied.*

Indeed, we have

1) $(\mathbf{X}, d, b = 3)$ *is a complete strong b-metric space because* $\mathbf{X}$ *is finite and discrete, and we have*

$$\begin{aligned} d(0,0) &= d(1,1) = d(2,2) = 0, \\ d(0,1) &= d(1,0) = 1, \\ d(0,2) &= d(2,0) = 4, \\ d(1,2) &= d(2,1) = 1, \end{aligned}$$

and

$$\begin{aligned} 1 &= d(0,1) \leq 3.d(0,2) + d(2,1) = 13, \\ 4 &= d(0,2) \leq 3.d(0,1) + d(1,2) = 4, \\ 1 &= d(1,2) \leq 3.d(1,0) + d(0,2) = 7. \end{aligned}$$

2) *It is easy to see that mappings* $\mathbf{A}$ *and* $\mathbf{S}$ *as well as* $\mathbf{B}$ *and* $\mathbf{T}$ *are occasionally weakly* $\mathbf{A}$*-biased (respectively* $\mathbf{B}$*-biased) of type* $(\mathbb{A})$*.*

3) *Taking* $\delta_1 = \delta_2 = \frac{1}{6}$*, then,* $(b+1)\delta_1 + \delta_2 = \frac{5}{6} < 1$*. We have*

Stage 1: *for* $x = 0 = y$*,* $\mathbf{A}x = \mathbf{S}x = 0 = \mathbf{B}y = \mathbf{T}y$ *and*

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \delta_1 \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y)\} \\ &\quad + \delta_2 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\} = 0. \end{aligned}$$

Stage 2: *For* x *and* y *belong to* $\{1,2\}$*,* $\mathbf{A}x = 2$*,* $\mathbf{B}y = y$*,* $\mathbf{S}x = 0 = \mathbf{T}y$*. For this stage, we have two cases:*

1. *By* $= 1$*, then, we have*

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{6}\max\{1,4,1\} + \frac{1}{6}\max\{1,4\} \\ &= \frac{4}{3} \\ &= \delta_1 \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y)\} \\ &\quad + \delta_2 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}, \end{aligned}$$

2. $By = 2$, then, we have

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{6}\max\{0,4,4\} + \frac{1}{6}\max\{4,4\} \\ &= \frac{4}{3} \\ &= \delta_1 \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y)\} \\ &\quad +\delta_2 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}. \end{aligned}$$

Stage 3: *For $x = 0$ and y belongs to $\{1, 2\}$, $\mathbf{A}x = 0 = \mathbf{S}x = \mathbf{T}y$, $\mathbf{B}y = y$. For this stage, we have two cases:*

1. $By = 1$, then, we have

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{6}\max\{1,0,1\} + \frac{1}{6}\max\{1,0\} \\ &= \frac{1}{3} \\ &= \delta_1 \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y)\} \\ &\quad +\delta_2 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}, \end{aligned}$$

2. $By = 2$, then, we have

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{6}\max\{4,0,4\} + \frac{1}{6}\max\{4,0\} \\ &= \frac{4}{3} \\ &= \delta_1 \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y)\} \\ &\quad +\delta_2 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}. \end{aligned}$$

Stage 4: *For $y = 0$ and x belongs to $\{1, 2\}$, $\mathbf{A}x = 2$, $\mathbf{S}x = 0 = \mathbf{B}y = \mathbf{T}y$, and*

$$\begin{aligned} 0 &= d(\mathbf{S}x, \mathbf{T}y) \\ &\leq \frac{1}{6}\max\{4,4,0\} + \frac{1}{6}\max\{0,4\} \\ &= \frac{4}{3} \\ &= \delta_1 \max\{d(\mathbf{A}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{S}x), d(\mathbf{B}y, \mathbf{T}y)\} \\ &\quad +\delta_2 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}, \end{aligned}$$

so, the second condition of the theorem is satisfied, therefore, the four mappings admit "zero" as the unique common fixed point.

Theorem 2 *Consider four mappings* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{S}$ *and* $\mathbf{T}$ *from a complete strong b-metric space* $(\mathbf{X}, d, b)$ *into itself which satisfying the two conditions:*

1. $\mathbf{A}$ *and* $\mathbf{S}$ *as well as* $\mathbf{B}$ *and* $\mathbf{T}$ *are occasionally weakly* $\mathbf{A}$*-biased (respectively* $\mathbf{B}$*-biased) of type* $(\mathbb{A})$*,*

2. *for all* x *and* y *belong to* $\mathbf{X}$*, we have*

$$\begin{aligned} d(\mathbf{S}x, \mathbf{T}y) &\leq \delta_1 d(\mathbf{A}x, \mathbf{B}y) + \delta_2 d(\mathbf{A}x, \mathbf{S}x) + \delta_3 d(\mathbf{B}y, \mathbf{T}y) \\ &\quad + \delta_4 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\} \end{aligned}$$

where $\delta_1 + (b+1)(\delta_2 + \delta_3) + \delta_4 < 1$*.*

Then, the four mappings have a unique common fixed point.

Proof. As in the proof of the first theorem, according to condition one, there are two elements (say α and β) belong to $\mathbf{X}$ such that $\mathbf{A}\alpha = \mathbf{S}\alpha$, $\mathbf{B}\beta = \mathbf{T}\beta$ and

$$d(\mathbf{A}\mathbf{A}\alpha, \mathbf{S}\alpha) \leq d(\mathbf{S}\mathbf{A}\alpha, \mathbf{A}\alpha),$$
$$d(\mathbf{B}\mathbf{B}\beta, \mathbf{T}\beta) \leq d(\mathbf{T}\mathbf{B}\beta, \mathbf{B}\beta).$$

To prove the existence and uniqueness of the common fixed point, we need four steps.

Step one: We will prove that $\mathbf{A}\alpha = \mathbf{B}\beta$. Suppose we are the contrary, then, we get

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) &\leq \delta_1 d(\mathbf{A}\alpha, \mathbf{B}\beta) + \delta_2 d(\mathbf{A}\alpha, \mathbf{S}\alpha) + \delta_3 d(\mathbf{B}\beta, \mathbf{T}\beta) \\ &\quad + \delta_4 \max\{d(\mathbf{S}\alpha, \mathbf{B}\beta), d(\mathbf{A}\alpha, \mathbf{T}\beta)\} \end{aligned}$$

which is equivalent to

$$\begin{aligned} d(\mathbf{S}\alpha, \mathbf{T}\beta) &\leq [\delta_1 + \delta_4] d(\mathbf{S}\alpha, \mathbf{T}\beta) \\ &< d(\mathbf{S}\alpha, \mathbf{T}\beta) \end{aligned}$$

which is a contradiction, consequently $\mathbf{S}\alpha = \mathbf{A}\alpha = \mathbf{B}\beta = \mathbf{T}\beta$.

Step two: Now, we will prove that $\mathbf{S}\alpha = \mathbf{T}\beta$ is a common fixed point of mappings $\mathbf{A}$ and $\mathbf{S}$. Indeed, we have

$$\begin{aligned} d(\mathbf{S}\mathbf{S}\alpha, \mathbf{T}\beta) &\leq \delta_1 d(\mathbf{A}\mathbf{S}\alpha, \mathbf{B}\beta) + \delta_2 d(\mathbf{A}\mathbf{S}\alpha, \mathbf{S}\mathbf{S}\alpha) + \delta_3 d(\mathbf{B}\beta, \mathbf{T}\beta) \\ &\quad + \delta_4 \max\{d(\mathbf{S}\mathbf{S}\alpha, \mathbf{B}\beta), d(\mathbf{A}\mathbf{S}\alpha, \mathbf{T}\beta)\} \end{aligned}$$

which is equivalent to

$$\begin{aligned}
d(\mathbf{SS}\alpha, \mathbf{S}\alpha) &\leq \delta_1 d(\mathbf{AA}\alpha, \mathbf{S}\alpha) + \delta_2 d(\mathbf{AA}\alpha, \mathbf{SS}\alpha) \\
&\quad + \delta_4 \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), d(\mathbf{AA}\alpha, \mathbf{S}\alpha)\} \\
&\leq \delta_1 d(\mathbf{AA}\alpha, \mathbf{S}\alpha) + b\delta_2 d(\mathbf{AA}\alpha, \mathbf{S}\alpha) + \delta_2 d(\mathbf{S}\alpha, \mathbf{SS}\alpha) \\
&\quad + \delta_4 \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), d(\mathbf{AA}\alpha, \mathbf{S}\alpha)\} \\
&\leq \delta_1 d(\mathbf{SA}\alpha, \mathbf{A}\alpha) + b\delta_2 d(\mathbf{SA}\alpha, \mathbf{A}\alpha) + \delta_2 d(\mathbf{S}\alpha, \mathbf{SS}\alpha) \\
&\quad + \delta_4 \max\{d(\mathbf{SS}\alpha, \mathbf{S}\alpha), d(\mathbf{SA}\alpha, \mathbf{A}\alpha) \\
&= [\delta_1 + (b+1)\delta_2 + \delta_4] d(\mathbf{SS}\alpha, \mathbf{S}\alpha) \\
&< d(\mathbf{SS}\alpha, \mathbf{S}\alpha)
\end{aligned}$$

the above contradiction implies that $\mathbf{SS}\alpha = \mathbf{S}\alpha$ and in consequence $\mathbf{AS}\alpha = \mathbf{S}\alpha$.

Step three: In the next, we will prove that the common fixed point of $\mathbf{A}$ and $\mathbf{S}$ is also the common fixed point of $\mathbf{B}$ and $\mathbf{T}$. Indeed, we have

$$\begin{aligned}
d(\mathbf{S}\alpha, \mathbf{TT}\beta) &\leq \delta_1 d(\mathbf{A}\alpha, \mathbf{BT}\beta) + \delta_2 d(\mathbf{A}\alpha, \mathbf{S}\alpha) + \delta_3 d(\mathbf{BT}\beta, \mathbf{TT}\beta) \\
&\quad + \delta_4 \max\{d(\mathbf{S}\alpha, \mathbf{BT}\beta), d(\mathbf{A}\alpha, \mathbf{TT}\beta)\}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
d(\mathbf{T}\beta, \mathbf{TT}\beta) &\leq \delta_1 d(\mathbf{T}\beta, \mathbf{BB}\beta) + \delta_3 d(\mathbf{BB}\beta, \mathbf{TT}\beta) \\
&\quad + \delta_4 \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), d(\mathbf{T}\beta, \mathbf{TT}\beta)\} \\
&\leq \delta_1 d(\mathbf{T}\beta, \mathbf{BB}\beta) + b\delta_3 d(\mathbf{BB}\beta, \mathbf{T}\beta) + \delta_3 d(\mathbf{T}\beta, \mathbf{TT}\beta) \\
&\quad + \delta_4 \max\{d(\mathbf{T}\beta, \mathbf{BB}\beta), d(\mathbf{T}\beta, \mathbf{TT}\beta)\} \\
&\leq \delta_1 d(\mathbf{B}\beta, \mathbf{TB}\beta) + b\delta_3 d(\mathbf{TB}\beta, \mathbf{B}\beta) + \delta_3 d(\mathbf{T}\beta, \mathbf{TT}\beta) \\
&\quad + \delta_4 \max\{d(\mathbf{B}\beta, \mathbf{TB}\beta), d(\mathbf{T}\beta, \mathbf{TT}\beta)\} \\
&= [\delta_1 + (b+1)\delta_3 + \delta_4] d(\mathbf{T}\beta, \mathbf{TT}\beta) \\
&< d(\mathbf{T}\beta, \mathbf{TT}\beta),
\end{aligned}$$

the above contradiction implies that $\mathbf{T}\beta = \mathbf{TT}\beta$, in consequence, $\mathbf{T}\beta = \mathbf{BT}\beta$, therefore, $\mathbf{A}\alpha = \mathbf{S}\alpha = \mathbf{B}\beta = \mathbf{T}\beta = \chi_1$ is a common fixed point of the four mappings.

Step four: Assume that there is another common fixed point say χ_2, then, we have

$$\begin{aligned}
d(\mathbf{S}\chi_1, \mathbf{T}\chi_2) &\leq \delta_1 d(\mathbf{A}\chi_1, \mathbf{B}\chi_2) + \delta_2 d(\mathbf{A}\chi_1, \mathbf{S}\chi_1) + \delta_3 d(\mathbf{B}\chi_2, \mathbf{T}\chi_2) \\
&\quad + \delta_4 \max\{d(\mathbf{S}\chi_1, \mathbf{B}\chi_2), d(\mathbf{A}\chi_1, \mathbf{T}\chi_2)\}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
d(\chi_1, \chi_2) &\leq \delta_1 d(\chi_1, \chi_2) + \delta_4 d(\chi_1, \chi_2) \\
&= [\delta_1 + \delta_4] d(\chi_1, \chi_2) \\
&< d(\chi_1, \chi_2)
\end{aligned}$$

a contradiction, hence, the common fixed point is unique.

Example 2 *As in the first example, let* $\mathbf{X} = \{0, 1, 2\}$. *Define*

$$\begin{aligned} d: \quad & \mathbf{X} \times \mathbf{X} \to [0, +\infty) \\ & (x, y) \to (x-y)^2. \end{aligned}$$

Now, Consider four mappings as follows:

$$\mathbf{A}x = \begin{cases} 0 \text{ when } x = 0, \\ 2 \text{ when } x \in \{1, 2\}, \end{cases} \quad \mathbf{B}x = \begin{cases} 0 \text{ when } x = 0, \\ 1 \text{ when } x \in \{1, 2\}, \end{cases}$$

$$\mathbf{S}x = \mathbf{T}x = 0 \text{ when } x \in \{0, 1, 2\}.$$

We already proved that $(X, d, b = 3)$ *is a complete strong b-metric space. Also, it is very easy to see that mappings* **A** *and* **S** *as well as* **B** *and* **T** *are occasionally weakly* **A**-*biased (respectively* **B**-*biased) of type* $(\mathbb{A})$. *Taking* $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \frac{1}{11}$, *then,* $\delta_1 + (b+1)(\delta_2 + \delta_3) + \delta_4 = \frac{10}{11} < 1$.

Let the constructed table

♠	$x = y = 0$	$x,\ y \in \{1, 2\}$	$x = 0$ *and* $y \in \{1, 2\}$	$y = 0$ *and* $x \in \{1, 2\}$
$d(\mathbf{S}x, \mathbf{T}y)$	0	0	0	0
$d(\mathbf{A}x, \mathbf{B}y)$	0	1	1	4
$d(\mathbf{A}x, \mathbf{S}x)$	0	4	0	4
$d(\mathbf{B}y, \mathbf{T}y)$	0	1	1	0
$d(\mathbf{S}x, \mathbf{B}y)$	0	1	1	0
$d(\mathbf{A}x, \mathbf{T}y)$	0	4	0	4

Discussion: *We have four cases:*

First case: *If* $x = y = 0$, *then,* $\mathbf{A}x = \mathbf{S}x = \mathbf{B}y = \mathbf{T}y = 0$, *therefore*

$$\begin{aligned} 0 = d(\mathbf{S}x, \mathbf{T}y) \quad \leq \quad & \delta_1 d(\mathbf{A}x, \mathbf{B}y) + \delta_2 d(\mathbf{A}x, \mathbf{S}x) + \delta_3 d(\mathbf{B}y, \mathbf{T}y) \\ & + \delta_4 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\} = 0. \end{aligned}$$

Second case: *If* $x,\ y \in \{1, 2\}$, $\mathbf{A}x = 2$, $\mathbf{S}x = 0$, $\mathbf{B}y = 1$, $\mathbf{T}y = 0$, *hence*

$$\begin{aligned} 0 = d(\mathbf{S}x, \mathbf{T}y) \quad & \leq \quad \frac{1}{11} \times 1 + \frac{1}{11} \times 4 + \frac{1}{11} \times 1 + \frac{1}{11} \max\{1, 4\} \\ & = \quad \frac{10}{11} \\ & = \quad \delta_1 d(\mathbf{A}x, \mathbf{B}y) + \delta_2 d(\mathbf{A}x, \mathbf{S}x) + \delta_3 d(\mathbf{B}y, \mathbf{T}y) \\ & \quad\quad + \delta_4 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}. \end{aligned}$$

Third case: *If* $x = 0$, $y \in \{1, 2\}$, $\mathbf{A}x = 0$, $\mathbf{S}x = 0$, $\mathbf{B}y = 1$, $\mathbf{T}y = 0$, *thus*

$$\begin{aligned}
0 = d(\mathbf{S}x, \mathbf{T}y) &\leq \frac{1}{11} \times 1 + \frac{1}{11} \times 0 + \frac{1}{11} \times 1 + \frac{1}{11} \max\{1, 0\} \\
&= \frac{3}{11} \\
&= \delta_1 d(\mathbf{A}x, \mathbf{B}y) + \delta_2 d(\mathbf{A}x, \mathbf{S}x) + \delta_3 d(\mathbf{B}y, \mathbf{T}y) \\
&\quad + \delta_4 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\}.
\end{aligned}$$

Fourth case: *If* $y = 0$, $x \in \{1, 2\}$, $\mathbf{A}x = 2$, $\mathbf{S}x = 0$, $\mathbf{B}y = 0$, $\mathbf{T}y = 0$, *and*

$$\begin{aligned}
0 = d(\mathbf{S}x, \mathbf{T}y) &\leq \frac{1}{11} \times 4 + \frac{1}{11} \times 4 + \frac{1}{11} \times 0 + \frac{1}{11} \max\{0, 4\} \\
&= \frac{12}{11} \\
&= \delta_1 d(\mathbf{A}x, \mathbf{B}y) + \delta_2 d(\mathbf{A}x, \mathbf{S}x) + \delta_3 d(\mathbf{B}y, \mathbf{T}y) \\
&\quad + \delta_4 \max\{d(\mathbf{S}x, \mathbf{B}y), d(\mathbf{A}x, \mathbf{T}y)\},
\end{aligned}$$

consequently, only "zero" satisfies the equalities $A0 = S0 = B0 = T0 = 0$.

4 Conclusion

In this contribution, we proved the existence and uniqueness of common fixed points for two pairs of occasionally weakly biased mappings of type ($\mathbb{A}$) in complete strong b-metric spaces. Our theorems improve some existing results in complete metric as well as complete strong b-metric spaces.

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