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Combining chaotic dynamical systems using the fuzzy logic XOR operator [1]

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Abstract

In this paper we explore whatever combining two chaotic dynamical systems using the fuzzy logic operator XOR can maintain or not the chaotic properties of the resulting dynamical system. This study is motivated by techniques used in applications to secure communications and cryptography.

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1 Introduction

Chaotic dynamical systems are commonly used as models in a wide range of applications, cryptography [1, 6, 12] image and speech encryption and retrieval [2, 7, 8, 10] and achievement of associative memory properties [3, 4, 5].

Extreme sensitivity to initial conditions is an interesting property of chaotic systems. This property makes chaotic systems a worthy choice for constructing cryptosystems.

Another idea is to mix two or more chaotic dynamical systems to gain more unpredictably in order to enhance encryption process [7, 8, 9].

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We ask ourselves this question, whatever combining two chaotic dynamical systems permits to maintain chaotic property of the resulting one? In particular whatever combining two chaotic dynamical systems by fuzzy logic operators, mainly the XOR operator gives rise to a chaotic dynamical system.
In this paper we restrict our study to the XOR operator this operator is widely suggested in literature although other operators have been considered and studied such as the exponential function [8].
In other words consider two chaotic dynamical systems (I, F) and (I, G) is the dynamical system $(I, F\ XOR\ G)$ still chaotic?

We studied some combination of known chaotic dynamical systems and checked whatever the combination using XOR operator is still chaotic or not. This gave us some preliminary remarks about how to combine chaotic dynamical systems in order to maintain the chaotic properties of the resulting dynamical system.

We provide a sufficient condition to check if the combination has chances to succeed. Rather than the popular Lyapunov exponents method used almost systematically in the applied mathematics literature we use a tool provided by ergodic theory [11].

2 Dynamical systems

2.1 Topological dynamics

A dynamical system (X, F) consists of a compact metric space X and a continuous self–map F .
A point x is said periodic if there exists $p > 0$ with $F^p(x) = x$. The least p with this property is called the period of x. A point x is eventually or ultimately periodic if $T^m(x)$ is periodic for some $m \geq 0$.
In the same way a dynamical system is said periodic if there exists $p > 0$ with $F^p(x) = x$ for every $x \in X$ and eventually periodic if F^m is periodic for some $m \geq 0$.
A point $x \in X$ is said to be an equicontinuity point, or to be Lyapunov stable, if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$ one has $d(F^n(y), F^n(x)) < \epsilon$ for any integer $n \geq 0$.
We say that (X, F) is sensitive if for any $x \in X$ we have :

$$\exists \epsilon > 0, \forall \delta > 0, \exists y \in B_\delta(x), \exists n \geq 0 \text{ such that } d(F^n(y), F^n(x)) \geq \epsilon.$$

We say that (X, T) is expansive if we have :

$$\exists \epsilon > 0, \forall x \neq y \in X, \exists n \geq 0, d(F^n(x), F^n(y)) \geq \epsilon.$$

A dynamical system (X, T) is transitive if for any nonempty open sets $U, V \subset A^{\mathbb{Z}}$ there exists $n > 0$ with $U \cap F^{-n}(V) \neq \emptyset$. This is equivalent to the existence of a

point with a dense orbit.

A dynamical system is said topologically mixing if for any nonempty open sets $U, V \subset A^{\mathbb{Z}}, U \cap F^{-n}(V) \neq \emptyset$ for all sufficiently large n.
Let (X, F) be a dynamical system, endow the set X with a sigma algebra $\mathbb{B}$. The function F preservers some measure μ on the sigma algebra $\mathbb{B}$ iff for every $B \in \mathbb{B}$ we have $\mu(F^{-1}(B)) = \mu(B)$. We say then that $(X, \mathbb{B}, F, \mu)$ is a measurable dynamical system.

The topological support of a measure is defined as the set of all points $x \in X$ for which every open neighborhood of x has positive measure.
A dynamical system $(X, \mathbb{B}, F, \mu)$ is ergodic if every invariant subset of X is either of measure 0 or of measure 1.
Equivalently, if for any measurable $U, V \subset X$, there exists some $n \in \mathbb{N}$ such that $\mu(U \cap F^{-n}(V)) > 0$.
If a dynamical system is ergodic then it is transitive on the topological support of the measure.

For dynamical systems defined on the interval a common used measure is the Lebesgue measure, the topological support of the Lebesgue measure is $\mathbb{R}$.

2.2 Chaotic dynamical systems on the interval

Dynamical systems defined on the interval have a particular behavior, a rich literature is devoted to the subject, we will recall here some results about their properties.

In matter of chaos The Devaney's chaos is seen as a combination of unpredictably (sensitivity) and regular behaviors (periodic points), transitivity ensuring that the system is undecomposable.

Definition 1. *A topological dynamical system (X, f) is chaotic in the sense of Devaney if :*
(1) Is transitive.
(2) The set of periodic points is dense in X.
(3) Is sensitive to initial conditions.

It is know that for every dynamical system the conditions (1) and (2) implies the condition (3) which lead to the so called modified Devaney definition of chaos which rather uses only conditions (1) and (2).
For interval maps transitivity is enough to imply the other two conditions.

Proposition 1. *An interval map is chaotic in the sense of Devaney if and only if it is transitive.*

For the proof of this result you can look at [13].

3 Results

3.1 Preliminary remarks

One issue when using the fuzzy logic XOR operator is preserving the invariance of the resulting dynamical system if we want to combine two dynamical systems they must be defined on the same interval but this is not enough to ensure invariance of the resulting combined dynamical system.
If the two dynamical systems are defined on the interval $[0, 1]$ it is easy to show that the result will be invariant on the interval $[0, 1]$. One solution to overcome the problem of the invariance is to rescale every dynamical system defined on a given interval to $[0, 1]$.
Below we give two examples the first one of an XOR combination of two chaotic dynamical systems which is not chaotic and the second one where the result is a chaotic dynamical system.

Example 1. *Let us consider the two dynamical systems* $([0, 1], f_r)$ *and* $([0, 1], T)$ *where* f_r *and* T *are the logistic map and the tent map, respectively, defined as follows:*

$$f_r(x) = r.x.(1 - x)$$

$$T(x) = \begin{cases} -2x + 1, & 0 \leq x \leq 0.5; \\ 2x - 1, & 0.5 \leq x \leq 1 \end{cases}$$

In our work, r *is greater than* 3.57*, as the logistic map is chaotic for* $r > 3.57$ *and non-chaotic for* $r \leq 3.57$*.*

These two dynamical systems are well known chaotic systems, let us consider their fuzzy XOR *combination* H *defined by*

$$H(x) = \max(f_r(x), T(x)) - \min(f_r(x), T(x))$$

It possesses two fixed points, the fixed point 0 is unstable while the fixed point 0.23 is asymptotically stable. The basin of attraction of the point 0 contains 4 isolated points while the basin of attraction of 0.23 contain the hole interval except the basin of attraction of 0.

In figure 1, the Lyapunov exponents of the logistic map, the tent map, and H *are all positive, which confirms their chaotic nature. A positive Lyapunov exponent allows us to infer sensitivity to initial conditions and highlight the chaotic character.*

However, for a value lower than 3.57*, such as* $r = 3$*, the logistic map is not chaotic (as shown by the negative Lyapunov exponent in Figure 2). Similarly, for this value, the map* H *also exhibits no chaotic behavior (negative Lyapunov exponent, see Figure 2).*

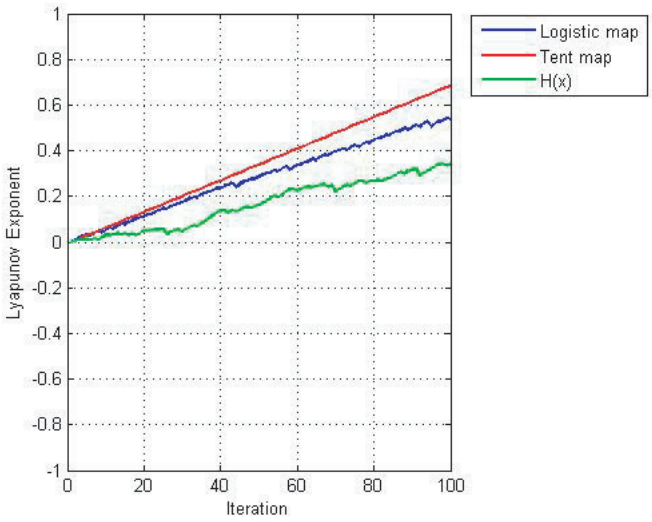


Figure 1: Lyapunov exponents of the chaotic logistic map ($r = 3.9$), chaotic tent map and chaotic H.

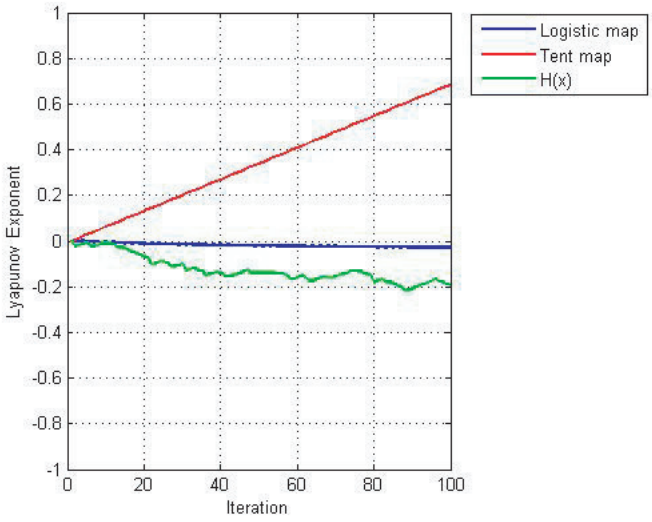


Figure 2: Lyapunov exponents of the non-chaotic logistic map ($r = 3$), chaotic tent map and non-chaotic H.

Example 2. *Consider the two following dynamical systems the two dynamical systems* $([0,1], T)$ *and* $([0,1], ST)$*, where* T *is the tent map and* ST *the map defined by:*

$$ST(x) = \begin{cases} -2x+1 : 0 \leq x \leq \frac{1}{2} \\ 2x-1 : \frac{1}{2} \leq x \leq 1 \end{cases}$$

The graph of the function ST is as an inverted Tent map graph. The map ST is chaotic as the point $\frac{\pi}{3.5}$ *has a dense orbit. The map T XOR ST is chaotic.*

3.2 Initial numerical investigation summary

Along with the two examples shown before we have tested some other dynamical systems, part of the results is shown in the following table :

XOR	Cubic map C	Logistic map f_r	Tent map T	Inverted Tent map ST
Doubling map B	Non chaotic	Non chaotic	Non chaotic	Non chaotic
Cubic map C	.....	Chaotic	Non chaotic	Non chaotic
Logistic map f_r	.....	.....	Non chaotic	Chaotic
Tent map T	.....	.....	.....	Chaotic
Inverted Tent map ST	.....	.....	.....	.....

We recall below the definitions of the classical and less classical maps used throughout this paper :

Map name	Doubling map B	Cubic map C	Logistic map $f_r(x)$
Expression	$2xmod1 = \begin{cases} 2x, 0 \leq x < 0.5 \\ 2x-1, 0.5 \leq x < 1 \end{cases}$	$16x^3 - 24x^2 + 9x$	$r.x.(1-x)$
Map name	Tent map T	Inverted Tent map ST	
Expression	$\begin{cases} -2x+1, 0 \leq x \leq 0.5 \\ 2x-1, 0.5 \leq x \leq 1 \end{cases}$	$\begin{cases} -2x+1 : 0 \leq x \leq \frac{1}{2} \\ 2x-1 : \frac{1}{2} \leq x \leq 1 \end{cases}$	

The observation of the results of the table suggests that combining two dynamical systems using the XOR operator leads to the resulting dynamical system to be chaotic if their graphs have some form of symmetry to the horizontal line $y = \frac{1}{2}$.

3.3 Mirror effect and number of full branches

The aim of this section is to come with some criterion choice. The optimal situation is that the two dynamical systems have to be symmetrical to the horizontal line $y = \frac{1}{2}$ this is what we will call a mirror effect. In this situation we can show that the combination is a chaotic dynamical using tools from ergodic theory.

Definition 2. *Let* $I \subset \mathbb{R}$ *be an interval. A map* $f : I \to I$ *is a full branch map if there exists a finite or countable partition* $\mathcal{P}$ *of* I *into subintervals such that for each* $w \in \mathcal{P}$ *the map* $f_{int(w)} : int(w) \to int(I)$ *is a bijection.*

A map f is a piecewise continuous (resp C^1, C^2, affine) full branch map if for each $w \in \mathcal{P}$ the map $f_{int(w)} : int(w) \to int(I)$ is a homeomorphism (respectively, C^1 diffeomorphism, C^2 diffeomorphism, affine).

Definition 3. *A full branch map has bounded distortion if*

$$\sup_{n\in\{1,2\}} \sup_{w^{(n)}\in\mathcal{P}^{(n)}} \sup_{x,y\in w^{(n)}} \log \left| Df^{(n)}(x) / Df^{(n)}(y) \right| < \infty$$

here (n) stands for the nth derivative.

If the function is piecewise affine then the distortion is 0.

Theorem 1. *Let $f : I \to I$ be a full branch map with bounded distortion. Then Lebesgue measure is ergodic.*

For the proof of this result you may look at [11].

Example 3. *The tent map, the doubling map and the logistic map have two full branches with bounded distortion, the cubic map have three full branches with bounded distortion.*

Proposition 2. *Consider $([0,1], f)$ and $([0,1], g)$ two dynamical systems, suppose that the graphs of f and g are symmetrical according to the horizontal line $y = \frac{1}{2}$ and that the number of full branches of f and g are equal to k.*
The dynamical system $([0,1], f\ XOR\ g)$ has $2k$ full branches hence is chaotic.

Proof. Suppose that we have a partition $\mathcal{P}$ of $[0,1]$ into subintervals such that for each $w \in \mathcal{P}$ the map $f_{int(w)} : int(w) \to]0,1[$ is a bijection.
As g is a mirror of f we obtain by symmetry :

$$(f\ XOR\ g)(x) = \begin{cases} 2\left|f(x) - 0.5\right|, & \text{if } f(x) \le 0.5; \\ 2\left|f(x) - 0.5\right|, & \text{if } f(x) \ge 0.5. \end{cases}$$

As f is a bijection there is a partition $w = w_1 \cup w_2$ such that :

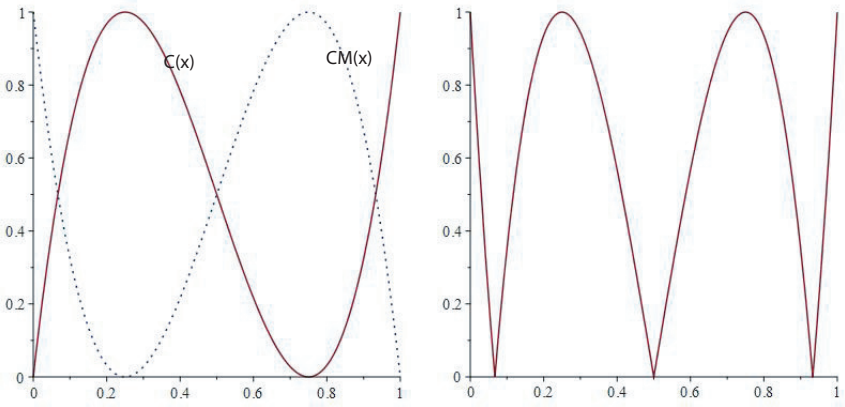
$$\begin{aligned} (f\ XOR\ g)(x) &= \begin{cases} 2\left|f(x) - 0.5\right|, & \text{if } x \in w_1; \\ 2\left|f(x) - 0.5\right|, & \text{if } x \in w_2. \end{cases} \\ &\Rightarrow \begin{cases} (f\ XOR\ g)(w_1) =]0,1] \\ (f\ XOR\ g)(w_1) =]0,1[\end{cases} \end{aligned}$$

Thus $f\ XOR\ g$ has two full branches on w.

Example 4. *Let us consider the cubic map $([0,1], C)$,*
where

$$C(x) = 16x^3 - 24x^2 + 9x.$$

The partition $\left\{\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]\right\}$ satisfies the conditions of the precedent definition, hence the cubic map is of three full branches. Its mirror CM has also three



(a) (b)

Figure 3: (a) The curves of the cubic map C (solid line) and its mirror CM (points). (b) The curve of the dynamical system $C\ XOR\ CM$.

full branches (see Figure 3(a)).

The curve in figure 3(b) represents the function $C\ XOR\ CM$, featuring six fully developed branches.

Remark 1. *The cubic function $C(x) = 16x^3 - 24x^2 + 9x$ is chaotic and $C\ XOR\ f_r$ is chaotic for all r.*

1. *The cubic function has a complex nonlinear structure that generates unstable behaviors, sensitive to initialization, with sometimes a positive Lyapunov exponent that increases for an initial value chosen outside the interval $[0, 1]$ (see the Figure 4). This figure highlights the positive Lyapunov exponent associated with the cubic function within the interval $[0, 1]$, providing confirmation of its chaotic nature in this range.*

2. *The variation of the Lyapunov exponent for the function H, which defines $C\ XOR\ f_r$, is positive (see Figure 5), which allows us to conclude that $C\ XOR\ f_r$ is chaotic. For $r = 3$, the logistic map is not chaotic (the Lyapunov exponent is negative, see Figure 5(a)), while for $r = 3.9$, the logistic map exhibits chaotic behavior (the Lyapunov exponent is positive, see Figure 5(b)). In both cases, the Lyapunov exponent of H remains positive, confirming the chaotic nature of $C\ XOR\ f_r$ for both chosen values of r.*

Proposition 3. *Consider $([0, 1], f)$ and $([0, 1], g)$ two dynamical systems, suppose that the graphs of f and g are symmetrical according to the horizontal line $y = \frac{1}{2}$ and that f and g have the full branch property then $f\ XOR\ g$ is chaotic .*

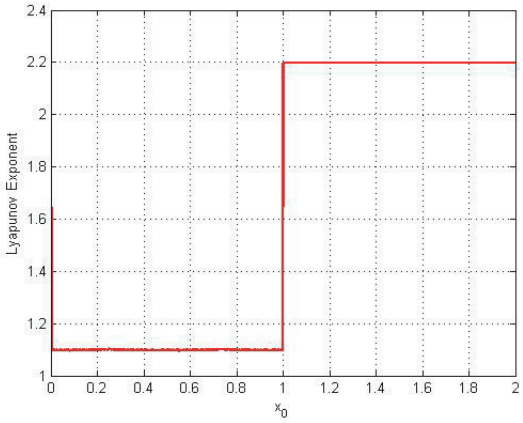
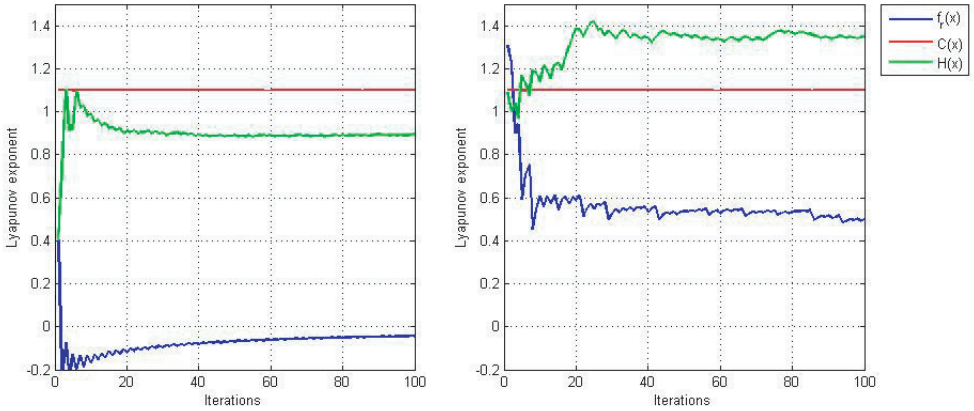


Figure 4: Lyapunov exponent of $C(x)$ according to the initial condition x_0.



(a) $r = 3$ (b) $r = 3.9$

Figure 5: Evolution of the Lyapunov exponents for $f_r(x)$, $C(x)$ and $H(x)$.

Proof. Suppose that f has k branches then f XOR g has $2k$ branches and a relevant partition w.
As the graph of g is symmetrical to the graph of f then they have the same distortion.

On each branch of the partition we have

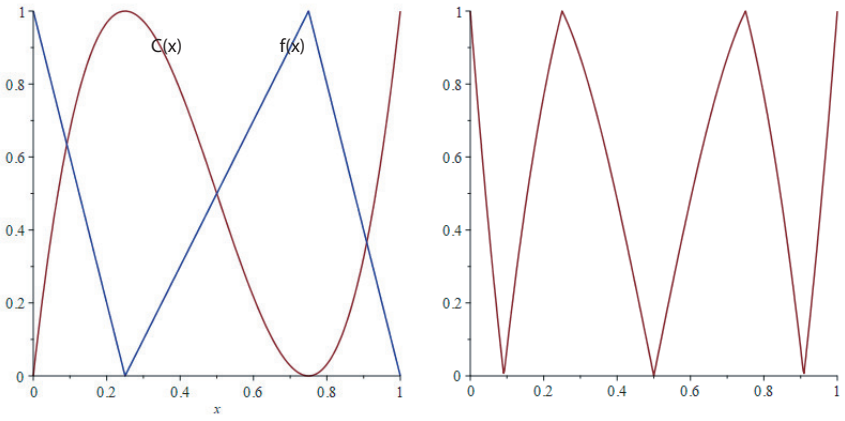
$$\sup_{n\geq 1}\sup_{w^{(n)}\in\mathcal{P}^{(n)}}\sup_{x,y\in w^{(n)}}\log\left|D\left(f\ XOR\ g\right)^{n}(x)/D\left(f\ XOR\ g\right)^{n}(y)\right|\leq$$
$$\max\left(\begin{array}{l}\sup_{n\geq 1}\sup_{w^{(n)}\in\mathcal{P}^{(n)}}\sup_{x,y\in w^{(n)}}\log\left|Df^{(n)}(x)/D\left(f\right)^{n}(y)\right|\\ ,\sup_{n\geq 1}\sup_{w^{(n)}\in\mathcal{P}^{(n)}}\sup_{x,y\in w^{(n)}}\log\left|D\left(g\right)^{n}(x)/D\left(g\right)^{n}(y)\right|\end{array}\right)$$

Therefore $f\ XOR\ g$ is a full branch map with bounded distortion. Then the Lebesgue measure is ergodic [11].
As the Lebesgue measure is ergodic then $f\ XOR\ g$ is transitive on the topological support of the Lebesgue measure which is the whole interval. Hence it is chaotic.

Remark 2. *The two maps do not need to be perfectly symmetrical to each other. Consider the cubic map and the map defined by*

$$f(x)=\begin{cases}-4x+1, 0\leq x\leq 0.25;\\ 2x-0.5, 0.25\leq x\leq 0.75;\\ 4x-4, 0\leq x\leq 1.\end{cases}$$

This map has been defined just by mirroring maximum and minimum points of the Cubic map to the line $y=\frac{1}{2}$. The result give us a six full branches map as shown in the figure 6(a) and figure 6(b).



(a) (b)

Figure 6: (a) Curve of the cubic map C and the function f. (b) Curve of $C\ XOR\ f$ with six full branches.

In figure 7, the Lyapunov exponents of the two chaotic functions, $C(x)$ and $f(x)$, are positive, indicating that both functions exhibit chaotic behavior. Furthermore, the variation of the Lyapunov exponent associated with $H(x)$, corresponding

to C XOR f, is also positive. This positive variation reinforces the conclusion that C XOR f exhibits chaotic dynamics. Consequently, the overall behavior of C XOR f can be definitively classified as chaotic, as clearly demonstrated by the positive Lyapunov exponents shown in figure 7.

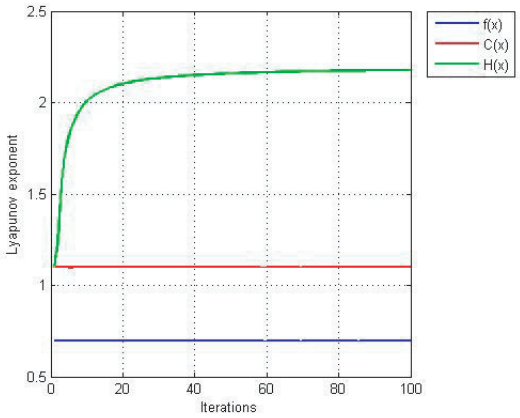


Figure 7: Evolution of the Lyapunov exponents for $f(x)$, $C(x)$ and $H(x)$ with $x_0 = 0.4$.

4 Conclusion

We investigated an ideal situation to combine chaotic one dimensional maps using the fuzzy logic XOR operator.
Using tools from ergodic theory we were able to establish a result under the slightly strong condition than Devaney's chaos, it is worth noting that this condition is satisfied by a majority of classical chaotic maps on the interval used in applications.

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