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Some Geometric Properties of Subclasses of p-valent Functions with negative coefficients defined by Opoola Differential Operator [1]

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Abstract

In this work, the geometric properties of new subclasses $A_g^m(p,t,b,\alpha)$ and $M_g^m(p,t,b,\alpha;\Lambda)$ were studied. In particular the characterisation property and coefficient bound for functions in $A_g^m(p,t,b,\alpha)$, distortion and growth property for functions in $M_g^m(p,t,b,\alpha;\Lambda)$ and inclusion relations for functions in the two subclasses were established.

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1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions that are analytic in the unit disk,

$$\mathfrak{U} = \{z \in \mathbf{C} : |z| < 1\}.$$

Also, we let $\mathcal{S}$ denote a subclass of $\mathcal{A}$ consisting of functions that are univalent in the unit disk $\mathfrak{U}$ and have the following normalization (see Duren [8]).

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

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Definition 1 *Let* $T(m,p)$ *denote the class of functions* $f(z)$ *of the form:*

$$f(z) = z^p - \sum_{k=m+p}^{\infty} a_k z^k \quad (a_k \geq 0; m, p \in N), \tag{2}$$

which are analytic and p-valent functions in the unit disk $\mathfrak{U}$ *(see [4] and [17]).*

Definition 2 *Let*

$$f(z) = z^p - \sum_{k=m+p}^{\infty} a_k z^k, \quad a_k \geq 0$$

and

$$g(z) = z^p + \sum_{k=m+p}^{\infty} b_k z^k$$

, then convolution(Hadamard product) of f *and* g *is defined as*

$$(f * g)(z) = z^p - \sum_{k=m+p}^{\infty} a_k b_k z^k, \quad z \in \mathfrak{U}$$

(see [3] and [11])

Definition 3 *The set of points* z *where*
i) $|z - z_0| < \rho, \quad \rho > 0$ *is* $\rho-$*neighbourhood of points* $z_0 \in \mathcal{C}$ *and*
ii) $0 < |z - z_0| < \rho, \quad \rho > 0$ *is deleted* $\rho-$*neighbourhood of points* $z_0 \in \mathcal{C}$*,(see Duren [8]).*

Definition 4 *Let* $f \in T(m,p)$*, then* $(m,\delta) -$ *neighbourhood of functions* $f \in T(m,p)$ *is define as*

$$N^p_{m,\delta}(f;q) = \{q : q(z) \in T(m,p), q(z) = z^p - \textstyle\sum_{k=m+p}^{\infty} c_k z^k \quad \text{and} \quad \sum_{k=m+p}^{\infty} k|a_k - c_k| \leq \delta\}. \tag{3}$$

In particular, if
$h(z) = z^p \quad (p \in \mathbf{N}),$ *then*

$$N^p_{m,\delta}(h;q) = \{q : q(z) \in T(m,p), q(z) = z^p - \sum_{k=m+p}^{\infty} c_k z^k \quad \text{and} \quad \sum_{k=m+p}^{\infty} k|c_k| \leq \delta\}. \tag{4}$$

For concept on (m,δ)-neighbourhood (see [7], [10], [13], [17] and [18]).

Definition 5 *Let $f \in A$, then the Opoola differential operator $D^n_{t,\beta,\mu}f(z) : A \to A$ is defined by*

$$\begin{aligned} &D^0_{t,\beta,\mu}f(z) = f(z) \\ &D^1_{t,\beta,\mu}f(z) = (1+(\beta-\mu-1)t)f(z) - z(\beta-\mu)t + ztf'(z) = zD_tf(z) \\ &D^2_{t,\beta,\mu}f(z) = zD_t(D^1_{t,\beta,\mu}f(z)) \\ &\vdots \\ &D^n_{t,\beta,\mu}f(z) = zD_t(D^{n-1}_{t,\beta,\mu}f(z)), \end{aligned} \tag{5}$$

where

$$D_tf(z) = 1+\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n a_k z^{k-1}, \quad n \in N_0, \quad t \geq 0, \quad z \in \mathfrak{U} \quad \text{and} \quad 0 \leq \mu \leq \beta$$

(see [5], [6], [9], [12] and [15]) for few authors who has used this operator.

Theorem 1 *Let f be analytic inside and on simple closed curve C such that f is not a constant, then the maximum value of f, $|f(z)|$ is attained on C and not inside C. (see [1] and [11]).*

Definition 6 *Let the function $f \in T(m,p)$ and $g \in \mathcal{S}$, then $f \in A^m_g(p,t,b,\alpha)$. If it satisfies:*

$$\left|\frac{1}{b}\left[\frac{z((F_{(t,\beta,\mu)} * g)(z))'}{(F_{(t,\beta,\mu)} * g)(z)} - p\right]\right| < \alpha, \tag{6}$$

$(z \in U; \quad m,p \in N; \quad b \in C\backslash\{0\}; \quad t \geq 0; \quad 0 < \alpha \leq 1; \quad m,p \in N)$
and

$$F_{(t,\beta,\mu)}(z) = (1-(\beta-\mu-1)t)f(z) + ztf'(z) - z(\beta-\mu), \quad (t \geq 0, \quad 0 \leq \mu \leq \beta). \tag{7}$$

Definition 7 *Let $f \in T(m,p)$ and $q \in A^m_g(p,t,b,\alpha)$, then $f \in M^m_g(p,t,b,\alpha;\Lambda)$. If it satisfies the following non-homogeneous Cauchy-Euler differential equation:*

$$z^2\frac{d^2w}{dz^2}+2(1+\Lambda)\frac{dw}{dz}+\Lambda(1+\Lambda)w = (p+\Lambda)(p+\Lambda+1)q(z), \quad (w = f(z); \Lambda > -p). \tag{8}$$

It is noteworthy that class $A^m_g(p,t,b,\alpha)$ generalises some well-known subclasses of p-valent functions with negative coefficients and a complex order. For example,

Remark 1 *if $t = 0$, $\alpha = 1$, $b = p(1-\alpha)$, ($p \in \mathbf{N}$; $0 \leq \alpha < 1$ and $m+p = n$), then $A^m_g(p,t,b,\alpha)$ becomes $TS^*_g(p,m,\alpha)$ studied by (see Ali, Khan, Ravichandran and Subramanian [3]).*

Remark 2 *if* $t = 0$ *and* $\alpha = 1$, *then* $A_g^m(p, t, b, \alpha)$ *gives class* $S_g(p, n, b, m)$ *studied by (see Prajapat, Raina and Srivastava [16]).*

Remark 3 *if* $\alpha = \beta$ *and* $F_{t,\beta,\mu}(z) = F_\lambda(z)$ *in equation* (7), *then* $A_g^m(p, t, b, \alpha)$ *becomes* $S_g^n(p, \lambda, b, \beta)$ *studied by (see Mostafa [13]). Just to mention few.*

2 Main Results

Theorem 2 *Let the function* $f \in T(m, p)$. *Then* $f \in A_g^m(p, t, b, \alpha)$ *if and only if*

(9)
$$\sum_{k=m+p}^{\infty}(k+\alpha|b|-p)[1+(k+\beta-\mu-1)t]a_kb_k \le \alpha|b|\,[1+(p+\beta-\mu-1)t]-(1+\alpha|b|-p)(\beta-\mu)t,$$

$(b \in C\backslash\{0\};\quad t \ge 0;\quad 0 < \alpha \le 1)$.

Proof. Let $f \in A_g^m(p, t, b, \alpha)$, then by definition

$$\left|\frac{1}{b}\left\{\frac{z((F_{(t,\beta,\mu)} * g)(z))'}{(F_{(t,\beta,\mu)} * g)(z)} - p\right\}\right| < \alpha$$

$$\text{i.e}\quad \frac{1}{|b|}\left|\frac{z((F_{(t,\beta,\mu)} * g)(z))'}{(F_{(t,\beta,\mu)} * g)(z)} - p\right| < \alpha.$$

Since $|\Re(z)| \le |z|$
it implies that,

$$\frac{1}{|b|}\left|\Re\left\{\frac{z((F_{(t,\beta,\mu)} * g)(z))'}{(F_{(t,\beta,\mu)} * g)(z)} - p\right\}\right| < \alpha$$

$$i.e\quad \left|\Re\left\{\frac{z((F_{(t,\beta,\mu)} * g)(z))'}{(F_{(t,\beta,\mu)} * g)(z)} - p\right\}\right| < \alpha|b|$$

$$\Rightarrow p - \alpha|b| < \Re\left\{\frac{z[(F_{(t,\beta,\mu)} * g)(z)]'}{(F_{(t,\beta,\mu)} * g)(z)}\right\} < p + \alpha|b|$$

$$\Rightarrow \Re\left\{\frac{z[(F_{(t,\beta,\mu)} * g)(z)]'}{(F_{(t,\beta,\mu)} * g)(z)}\right\} > p - \alpha|b| \tag{10}$$

$$F_{(t,\beta,\mu)}(z) = ztf'(z) - zt(\beta-\mu) + (1 + (\beta-\mu-1)t)f(z)$$

$$f(z) = z^p - \sum_{k=m+p}^{\infty} a_kz^k, \quad a_k \ge 0$$

$$f'(z) = pz^{p-1} - \sum_{k=m+p}^{\infty} ka_kz^{k-1}$$

$$ztf'(z) = ptz^p - \sum_{k=m+p}^{\infty} kta_k z^k.$$

Therefore

$$F_{(t,\beta,\mu)}(z) = ptz^p - \sum_{k=m+p}^{\infty} kta_k z^k - zt(\beta-\mu) + (1+(\beta-\mu-1)t)(z^p - \sum_{k=m+p}^{\infty} a_k z^k)$$

$$F_{t,\beta,\mu}(z) = [1+(p+\beta-\mu-1)t]z^p - z(\beta-\mu)t - [1+(k+\beta-\mu-1)t]\sum_{k=m+p}^{\infty} a_k z^k$$

Taking $\beta - \mu = \tau$ in the above equation, to have

$$F_{(t,\beta,\mu)}(z) = [1+(p+\tau-1)t]z^p - z\tau t - [1+(k+\tau-1)t]\sum_{k=m+p}^{\infty} a_k z^k$$

and

$$(11)\quad (F_{(t,\beta,\mu)} * g)(z) = [1+(p+\tau-1)t - \frac{\tau t}{z^{p-1}}]z^p - [1+(k+\tau-1)t]\sum_{k=m+p}^{\infty} a_k b_k z^k.$$

Differentiating (11), to have

$$(12)\quad [(F_{(t,\beta,\mu)}*g)(z))]' = p[1+(p+\tau-1)t]z^{p-1} - \tau t - \sum_{k=m+p}^{\infty} k[1+(k+\tau-1)t]a_k b_k z^{k-1}$$

and

$$(13)\quad z[(F_{(t,\beta,\mu)}*g)(z)]' = p[1+(p+\tau-1)t]z^p - \tau tz - \sum_{k=m+p}^{\infty} k[1+(k+\tau-1)t]a_k b_k z^k,$$

substituting (11) and (12) into (10), to have

$$\Re\left\{\frac{p[1+(p+\tau-1)t]z^p - \tau tz - \sum_{k=m+p}^{\infty} k[1+(k+\tau-1)t]a_k b_k z^k}{[1+(p+\tau-1)t]z^p - \tau tz - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k z^k}\right\} > p - \alpha\,|b|$$

$$\Re\left\{\frac{p[1+(p+\tau-1)t] - \tau tz^{1-p} - \sum_{k=m+p}^{\infty} k[1+(k+\tau-1)t]a_k b_k z^{k-p}}{[1+(p+\tau-1)t] - \tau tz^{1-p} - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k z^{k-p}}\right\} > p - \alpha\,|b|\,.$$

Choosing the value of z on the real axis and $|z| \to 1$, gives

$$\frac{p[1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty} k[1+(k+\tau-1)t]a_k b_k}{[1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k} \geq p - \alpha\,|b|$$

i.e

$$(14)\quad p[1+(p+\tau-1)t]-\tau t-\sum_{k=m+p}^{\infty}k[1+(k+\tau-1)t]a_kb_k >$$
$$(p-\alpha|b|)\left[[1+(p+\tau-1)t]-\tau t-\sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_kb_k\right]$$

i.e $\alpha|b|[1+(p+\tau-1)t]-[(1-p)+\alpha|b|]\tau t > \sum_{k=m+p}^{\infty}(k-p+\alpha|b|)[1+(k+\tau-1)t]a_kb_k$

$$\Rightarrow \sum_{k=m+p}^{\infty}(k-p+\alpha|b|)[1+(k+\tau-1)t]a_kb_k < \alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t.$$

Thus, by the maximum modulus theorem,

(15)
$$\sum_{k=m+p}^{\infty}(k-p+\alpha|b|)[1+(k+\beta-\mu-1)t]a_kb_k \le \alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t.$$

Conversely,
assume that (9) holds and letting $\beta-\mu=\tau$, to obtain that,

$$\sum_{k=m+p}^{\infty}(k-p+\alpha|b|)[1+(k+\tau-1)t]a_kb_k \le \alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t$$

$$(16)\quad \Longrightarrow \sum_{k=m+p}^{\infty}(m+\alpha|b|)[1+(k+\tau-1)t]a_kb_k \le \alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t$$

$$(17)\quad \text{i.e}\quad \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_kb_k \le \frac{\alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t}{m+\alpha|b|}.$$

Now,

$$(18)\quad \left|\frac{z((F_{(t,\beta,\mu)}*g)(z))'}{((F_{(t,\beta,\mu)}*g)(z))'}-p\right| =$$
$$\left|\frac{p[1+(p+\tau-1)t]z^p-\tau tz-\sum_{k=m+p}^{\infty}k[1+(k+\tau-1)t]a_kb_kz^k}{[1+(p+\tau-1)t]z^p-\tau tz-\sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_kb_kz^k}-p\right|$$

$$= \left| \frac{(p-1)\tau t z^{1-p} - \sum_{k=m+p}^{\infty}(k-p)[1+(k+\tau-1)t]a_k b_k z^{k-p}}{[1+(p+\tau-1)t] - \tau t z^{1-p} - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k z^{k-p}} \right| \tag{19}$$

Choosing the value of z along real axis and $r \longrightarrow 1$, to have

$$\left| \frac{(p-1)\tau t - \sum_{k=m+p}^{\infty}(k-p)[1+(k+\tau-1)t]a_k b_k}{[1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k} \right| \tag{20}$$

$$= \frac{\left|(p-1)\tau t - \sum_{k=m+p}^{\infty}(k-p)[1+(k+\tau-1)t]a_k b_k\right|}{\left|[1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k\right|}. \tag{21}$$

Considering the denominator in (21),

$$\left|[1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k\right|$$

$$\geq |[1+(p+\tau-1)t] - \tau t| - \left|\sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k\right|$$

$$[1+(p+\tau-1)t] - \tau t - \left|\sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k\right|$$

$$\geq [1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty}|[1+(k+\tau-1)t]a_k b_k|$$

$$\geq [1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k, \quad a_k b_k \geq 0.$$

Substituting the above inequality into (21), gives

$$\left|\frac{z[(F_{(t,\beta,\mu)} * g)(z)]'}{[(F_{(t,\beta,\mu)} * g)]} - p\right| \leq \frac{\left|(p-1)\tau t - \sum_{k=m+p}^{\infty}(k-p)[1+(k+\tau-1)t]a_k b_k\right|}{[1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k} \tag{22}$$

$$\frac{\left|(p-1)\tau t - \sum_{k=m+p}^{\infty}(k-p)[1+(k+\tau-1)t]a_k b_k\right|}{[1+(p+\tau-1)t] - \tau t - \sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_k b_k} \leq \alpha|b| \tag{23}$$

(24)
$$\alpha|b|\sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_kb_k \le$$
$$\alpha|b|\left[(1+(p+\tau-1)t)-\tau t\right]-\left|(p-1)\tau t-\sum_{k=m+p}^{\infty}(k-p)[1+(k+\tau-1)t]a_kb_k\right|.$$

Using triangle inequality and simplifying (24), gives

$$\sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_kb_k \le \frac{\alpha|b|(1+(p+\tau-1)t)-(p-1+\alpha|b|)}{m+\alpha|b|}. \tag{25}$$

Hence, $f \in A_g^m(p,t,b,\alpha)$.

Theorem 3 *Let the function* $f \in A_g^m(p,t,b,\alpha)$. *Then for* $b_k \ge b_{m+p}$

$$\sum_{k=m+p}^{\infty} a_k \le \frac{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}} \tag{26}$$

and

$$\sum_{k=m+p}^{\infty} ka_k \le \frac{(m+p)[\alpha|b|(1+(p+\beta-\mu-1)t)-(1-p+\alpha|b|)(\beta-\mu)t]}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}}, \tag{27}$$

$(p > |b|)$

Proof. Let $f \in A_g^m(p,t,b,\alpha)$ and $\beta-\mu=\tau$. Then from (17),

$$\sum_{k=m+p}^{\infty}(m+\alpha|b|)[1+(k+\tau-1)t]a_kb_k \le \alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t$$

$$\Rightarrow \sum_{k=m+p}^{\infty}(m+\alpha|b|)[1+(m+p+\tau-1)t]a_kb_{m+p} \le \alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t$$

$$\sum_{k=m+p}^{\infty} a_k \le \frac{\alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t}{(m+\alpha|b|)[1+(m+p+\tau-1)t]b_{m+p}}.$$

From (15), to have

$$p[1+(p+\tau-1)t]-\tau t-\sum_{k=m+p}^{\infty}k[1+(k+\tau-1)t]a_kb_k$$

$$>(p-\alpha|b|)\left[[1+(p+\tau-1)t]-\tau t-\sum_{k=m+p}^{\infty}[1+(k+\tau-1)t]a_kb_k\right].$$

Simplifying and substituting (9) into the above inequality, gives

$$\sum_{k=m+p}^{\infty} k[1+(m+p+\tau-1)t]a_k b_{m+p}$$

$$\leq \alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t+(p-\alpha|b|)\frac{\alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t}{m+\alpha|b|}$$

$$\text{i.e} \quad \sum_{k=m+p}^{\infty} k a_k \leq \frac{(m+p)\left[\alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t\right]}{(m+\alpha|b|)[1+(m+p+\tau-1)t]b_{m+p}},$$

which gives the desired results.

Theorem 4 *Let $f \in M_g^m(p,t,b,\alpha;\Lambda)$ and $z \in \mathfrak{U}$, then*

(28)

$$|z|^p - \frac{\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\}\,(p+\Lambda)(p+\Lambda+1)}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t](m+p+\Lambda)b_{m+p}}|z|^{m+p} \leq |f(z)|$$

and

(29)

$$|f(z)| \leq |z|^p - \frac{\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\}\,(p+\Lambda)(p+\Lambda+1)}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t](m+p+\Lambda)b_{m+p}}|z|^{m+p}$$

Proof. Let $f \in T(m,p)$, $q \in A_g^m(p,t,b,\alpha)$, occurring in the non-homogeneous Cauchy-Euler differential equation (8) with $c_k \geq 0 \quad (k \geq m+p)$ and $\beta - \mu = \tau$ i.e $q \in T(m,p)$, and $q(z) = z^p - \sum_{k=m+p}^{\infty} c_k z^k$, $c_k \geq 0$,
so,

$$z^2\frac{d^2w}{dz^2} + 2(1+\Lambda)z\frac{dw}{dz} + \Lambda(1+\Lambda)w = (p+\Lambda)(p+\Lambda+1)q(z)$$

becomes,

$$[p(p-1)+2p(1+\Lambda)+\Lambda(1+\Lambda)]\,z^p - \sum_{k=m+p}^{\infty}[k(k-1)+2k(1+\Lambda)+\Lambda(1+\Lambda)]\,a_k z^k$$

$$= (p+\Lambda)(p+\Lambda+1)[z^p - \sum_{k=m+p}^{\infty} c_k z^k],$$

comparing the coefficient of the above equation, gives

$$[k(k-1)+2k(1+\Lambda)+\Lambda(1+\Lambda)]\,a_k = (p+\Lambda)(p+\Lambda+1)c_k$$

$$\Rightarrow a_k = \frac{(p+\Lambda)(p+\Lambda+1)c_k}{k^2+2k\Lambda+\Lambda^2+k+\Lambda}$$

$$= \frac{(p+\Lambda)(p+\Lambda+1)c_k}{(k+\Lambda)^2+(k+\Lambda)}.$$

Therefore,

$$a_k = \frac{(p+\Lambda)(p+\Lambda+1)}{(k+\Lambda)(k+\Lambda+1)}c_k \quad (k \geq m+p).$$

So that

$$f(z) = z^p - \sum_{k=m+p}^{\infty} a_k z^k$$

(30) becomes

$$f(z) = z^p - \sum_{k=m+p}^{\infty} \frac{(p+\Lambda)(p+\Lambda+1)}{(k+\Lambda)(k+\Lambda+1)} c_k z^k \tag{30}$$

and

$$|f(z)| \leq |z|^p + \sum_{k=m+p}^{\infty} \frac{(p+\Lambda)(p+\Lambda+1)}{(k+\Lambda)(k+\Lambda+1)} c_k |z|^{m+p}, \quad c_k \geq 0. \tag{31}$$

Since $q \in A_g^m(p,t,b,\alpha)$, then from Theorem (2), to have

$$c_k \leq \frac{\alpha|b|[1+(p+\tau-1)t] - (1-p+\alpha|b|)\tau t}{(m+\alpha|b|)[1+(m+p+\tau-1)t]b_{m+p}}, \quad k \geq m+p. \tag{32}$$

Substituting (32) into (31) gives,

$$|f(z)| \leq |z|^p + \left[\frac{\{\alpha|b|[1+(p+\tau-1)t] - (1-p+\alpha|b|)\tau t\}(p+\Lambda)(p+\Lambda+1)}{(m+\alpha|b|)[1+(m+p+\tau-1)t]b_{m+p}}|z|^{m+p}\right] \left[\sum_{k=m+p}^{\infty} \frac{1}{(k+\Lambda)(k+\Lambda+1)}\right].$$

(31a)

Finally,

$$\sum_{k=m+p}^{\infty} \frac{1}{(k+\Lambda)(k+\Lambda+1)} = \sum_{k=m+p}^{\infty} \left[\frac{1}{k+\Lambda} - \frac{1}{k+\Lambda+1}\right]$$

$$= \frac{1}{m+p+\Lambda} \quad (\Lambda \in R \backslash [-m-p, -m-p-1, ...]). \tag{33}$$

Now, (31a) becomes

$$|f(z)| \leq |z|^p + \frac{\{\alpha|b|[1+(p+\tau-1)t]+(1-p+\alpha|b|)\tau t\}\,(p+\Lambda)(p+\Lambda+1)}{(m+\alpha|b|)[1+(m+p+\tau-1)t]b_{m+p}(m+p+\Lambda)}|z|^{m+p}.$$

Applying triangle inequality on (31), to obtain

$$|f(z)| \geq |z|^p - \frac{\{\alpha|b|[1+(p+\tau-1)t]-(1-p+\alpha|b|)\tau t\}\,(p+\Lambda)(p+\Lambda+1)}{(m+\alpha|b|)[1+(m+p+\tau-1)t]b_{m+p}(m+p+\Lambda)}|z|^{m+p}$$

Theorem 5 *Let $f \in M_g^m(p,t,b,\alpha;\Lambda)$ and $z \in \mathfrak{U}$, then*

$$\psi(p,n)|z|^{p-n} + \frac{\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\}\,(p+\Lambda)(p+\Lambda+1)\psi(m+p,n)}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}(m+p+\Lambda)}|z|^{m+p-n} \leq \Big|$$

and

$$\left|f^{(n)}(z)\right| \leq \psi(p,n)|z|^{p-n} + \frac{\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\}\,(p+\Lambda)(p+\Lambda+1)\psi(m+p,n)}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}(m+p+\Lambda)}|z$$

where

$$\psi(p,n) = \frac{p!}{(p-n)!} = \begin{cases} 1, \ n=0 \\ p(p-1)\cdots(p-n+1), \ n \neq 0. \end{cases}$$

Proof. Let $f \in T(m,p)$ and $q \in A_g^m(p,t,b,\alpha)$ occurring in the non-homogeneous Cauchy-Euler differential equation (8), with $c_k \geq 0 \quad k \geq m+p$. From (24),

$$|f(z)| \leq |z|^p + \frac{\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\}\,(p+\lambda)(p+\lambda+1)}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}(m+p+\lambda)}|z|^{m+p}$$

$$|f'(z)| \leq p|z|^{p-1} + \frac{\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\}\,(p+\Lambda)(p+\Lambda+1)(m+p)}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}(m+p+\Lambda)}|z|^{m+p-1}$$

$$\left|f^{(n)}(z)\right| \leq \psi(p,n)|z|^{p-n} + \frac{\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\}\,(p+\Lambda)(p+\Lambda+1)\Psi(m+p,n)}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}(m+p+\Lambda)}|z$$

where

$$\psi(p,n) = \frac{p!}{(p-n)!} = \begin{cases} 1, \ n=0 \\ p(p-1)\cdots(p-n+1), \ n \neq 0 \end{cases}$$

Theorem 6 *Let* $f \in A_g^m(p,t,b,\alpha)$, *then*

$$A_g^m(p,t,b,\alpha) \subset N_{m,\delta}^p(h;q), \tag{34}$$

where

(35)
$$h(z) = z^p \quad and \quad \delta = \frac{(m+p)\left\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\right\}}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}}$$

Proof. Let $f \in A_g^m(p,t,b,\alpha)$, then from (23)

$$\sum_{k=m+p}^{\infty} ka_k \le \frac{(m+p)\left\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\right\}}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}}$$

Thus,

$$\sum_{k=m+p}^{\infty} ka_k \le \delta, \quad |a_k| = a_k$$

and

$$f(z) = z^p - \sum_{k=m+p}^{\infty} a_k z^k$$

which gives $f(z) \in N_{m,\delta}^p(h,q)$ with

$$\delta = \frac{(m+p)\left\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\right\}}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}}$$

Theorem 7 *Let* $f \in M_g^m(p,t,b,\alpha;\Lambda)$, *then*

$$M_g^m(p,t,b,\alpha;\Lambda) \subset N_{m,\delta}^p(f;q) \tag{36}$$

where $q(z) = \left\{q : q(z) = z^p - \sum_{k=m+p}^{\infty} c_k z^k and \sum_{k=m+p}^{\infty} |a_k - c_k| \le \delta\right\}$ *and*

(37)
$$\delta = \frac{m+(p+\Lambda)(p+\Lambda+2)}{m+p+\Lambda}\left[\frac{(m+p)\left\{\alpha|b|[1+(p+\beta-\mu-1)t]-(1-p+\alpha|b|)(\beta-\mu)t\right\}}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}}\right].$$

Proof. Let $f \in M_g^m(p,t,b,\alpha;\Lambda)$,then

$$a_k = \frac{(p+\Lambda)(p+\Lambda+1)}{(k+\Lambda)(k+\Lambda+1)}c_k \quad k \ge m+p$$

and

$$\sum_{k=m+p}^{\infty} k|a_k - c_k| \le \sum_{k=m+p}^{\infty} kc_k + \sum_{k=m+p}^{\infty} ka_k, \quad (a_k, c_k \ge 0)$$

, to have

$$\sum_{k=m+p}^{\infty} k|a_k - c_k| \leq \sum_{k=m+p}^{\infty} \left[1 + \frac{(p+\Lambda)(p+\Lambda+1)}{(k+\Lambda)(k+\Lambda+1)}\right] kc_k. \tag{38}$$

Since $q(z) \in A_g^m(p,t,b,\alpha)$, from Theorem 1,

$$\sum_{k=m+p}^{\infty} kc_k \leq \frac{(m+p)[\alpha|b|(1+(p+\beta-\mu-1)t) - (1-p+\alpha|b|)(\beta-\mu)t]}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}} \quad (p > |b|)$$

So (35) becomes

$$\sum_{k=m+p}^{\infty} k|a_k - c_k| \leq \left[1 + \sum_{k=m+p}^{\infty} \frac{(p+\Lambda)(p+\Lambda+1)}{(k+\Lambda)(k+\Lambda+1)}\right] \left\{ \frac{(m+p)[\alpha|b|(1+(p+\beta-\mu-1)t) - (1-p+\alpha|b|)(\beta-\mu)}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}} \right.$$

$$\leq \left(1 + \frac{(p+\Lambda)(p+\Lambda+1)}{m+p+\Lambda}\right) \left[\frac{(m+p)[\alpha|b|(1+(p+\beta-\mu-1)t) - (1-p+\alpha|b|)(\beta-\mu)t]}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}}\right]$$

$$\leq \left(\frac{m+(p+\Lambda)(p+\Lambda+2)}{m+p+\Lambda}\right) \left[\frac{(m+p)[\alpha|b|(1+(p+\beta-\mu-1)t) + (1-p+\alpha|b|)(\beta-\mu)t]}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}}\right]$$

i.e $f \in N_{m,\delta}^p(f:q)$ with

$$\delta = \left(\frac{m+(p+\Lambda)(p+\Lambda+2)}{m+p+\Lambda}\right) \left[\frac{(m+p)[\alpha|b|(1+(p+\beta-\mu-1)t) - (1-p+\alpha|b|)(\beta-\mu)t]}{(m+\alpha|b|)[1+(m+p+\beta-\mu-1)t]b_{m+p}}\right].$$

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