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Some New Classes of General Harmonic-like Nonlinear Equations [1]

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Abstract

Some classes of general nonlinear equations, which can be viewed as a novel important special case of nonlinear equations, are introduced and investigated. Using various techniques and dynamical systems coupled with f inite difference approach, we suggest and analyzed a number of new iterative methods s for solving harmonic-like nonlinear equations. Convergence analysis of these methods is investigated under suitable conditions. One can obtain a number of new classes of general nonlinear equations by interchanging the role of operators. Various special cases are discussed as applications of the main results. Several open problems are suggested for future research.

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1 Introduction

It is well known a wide class of complex and complicated problems, which arise in different branches of mathematical and engineering sciences such as physical, biomedical, regional, optimization, ecology, economics and engineering sciences, can be studied in the general and unified framework of nonlinear equations. In recent years, several iterative methods for finding the approximate solutions of the nonlinear equation are being developed using several different techniques including Taylor series, quadrature formulas, Trapezoidal rule, variation iterations, homotopy perturbation and decomposition techniques. For more detail,see [7, 8, 11, 20, 21, 31] and the references therein. It has been proved that the general nonlinear equations are equivalent to the fixed point problems. This equivalent formulations has been exploited to suggest and analyze several multi-step explicit

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and implicit iterative methods for solving nonlinear nonlinear equations. In particular, these multi step methods can be viewed as novel modifications and generalizations of Noor (three step) iterations, the origin can be traced back to Noor [17, 18]. These Noor(three-step) iterations include Ishikawa(two-step) iterations, Mann(one-step) iteration and Picard method as special cases. For the applications, generalizations, modifications, extensions and relative aspects of Noor (three-step) iterations, see [3, 4, 5, 8, 17, 18, 22, 23, 24, 28, 29, 30, 35] and the references therein. Several new generalizations and extensions of the convex functions and convex sets have been introduced and studied to tackle unrelated complicated and complex problems in a unified manner. Anderson et al. [2] have investigated several aspects of the harmonic convex sets and harmonic convex functions, which can viewed as important generalizations of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, ifr_1 and r_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula:
More precisely, if r_1 and r_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula

$$(\frac{1}{r_1}+\frac{1}{r_2})^{-1}=\frac{r_1r_2}{r_1+r_2},$$

which is half the harmonic means. The harmonic mean is employed in finance to determine the average multiples like the price-income ratio, etc. stock exchange and financial mathematics. The harmonic mean are being used to suggest some iterative methods for solving nonlinear equations. Noor et al. [24, 25, 26, 27] introduced the new concepts of harmonic-like convex sets and harmonic-like convex functions, which can be viewed as an important generalization of harmonic convex sets and functions. Motivated and inspired by the research activities in this fast developing field, we introduce and consider some new classes of general harmonic-like nonlinear equations. We establish the equivalence between the general harmonic-like nonlinear equations and the fixed point problem. This alternate equivalence is used to prove the existence of the solution as well as to suggest multi step methods. Convergence criteria is also analyzed under some weaker conditions. To discuss the asymptotic stability of the solution and to suggest multi-step iterative methods, we apply the dynamical system technique, which is mainly due to Dupuis et al. [10] as developed in [10, 12, 14, 19, **?**, 22, 28, 32, 33, 34]. This is entirely a new way of considering the multi-step methods for solving general nonlinear equations. We introduce first order, second order and third order initial value and boundary value problems for computing the approximate solution of the general harmonic-like nonlinear equations, which is main domain of differential equations. Making use of finite forward backward interpolation and descritizing the dynamical systems, several multi-step iterative methods for solving the general nonlinear equations analyzed along with convergence analysis. It has been shown that these new iterative methods include a wide class of known and new iterative methods as special cases.

2 Formulations and basic facts

Let $\mathcal{H}$ be a real Hilbert space . We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. We also need the some basic concepts from the harmonic convex analysis [2, 15] and Noor et al. [24, 25, 26, 27].

We now introduce some new concepts in harmonic convex analysis, which are needed in the derivation of the main results.

Definition 1 *A set $\Omega\subseteq\mathcal{H}$ is said to be harmonic-like mean, if*

$$\left(\frac{2w\mu}{w+\mu}\right),\quad \forall\mu.\nu,w\in\mathcal{H}$$

It is worth mentioing that, for $w = \nu$, then the harmonic-like means becomes

$$\left(\frac{2w\mu}{w+\mu}\right) \Longrightarrow \left(\frac{2\mu\nu}{\nu+\mu}\right) \qquad \forall \mu, \nu \in \mathcal{H}.$$

which is called harmonic mean. Note that the element $w \in \mathcal{H}$ can be considered as the weight element. For the properties and applications of harmonic means, see [1, 2, 15, 24, 25, 26, 27, 28] and the references therein.
Let $\mathcal{T}, g : \mathcal{H} \Longrightarrow \mathcal{H}$ be nonlinear operators. We consider the problem of finding the solution $\mu \in \mathcal{H}$ of the nonlinear equations

$$\mathcal{T}(\mu) = 0, \tag{1}$$

which can be written as

$$g(\mu) = g(\mu) - \rho\mathcal{T}\left(\frac{2w\mu}{w+\mu}\right), \qquad \forall w, \mu \in \mathcal{H}, \tag{2}$$

where ρ is a parameter, is called the general harmonic-like nonlinear equation. By interchanging the role of the operators $\mathcal{T}$ and g, the problem (2.2) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\mathcal{T}(\mu) = \rho g\left(\frac{2w\mu}{w+\mu}\right) - \mathcal{T}(\mu), \quad \forall w, \mu \in \mathcal{H}. \tag{3}$$

Note that the symmetry between the operators$\mathcal{T}$ and g.

Special Cases

We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2).

(I). If $w = \mu$, then the problem (2) reduces to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) = g(\mu) - \rho\mathcal{T}(\mu), \tag{4}$$

which is the general nonlinear equation.

(II). For $w = \nu$, the problem (2) collapses to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) = g(\mu) - \rho\mathcal{T}\left(\frac{2\nu\mu}{\nu+\mu}\right), \tag{5}$$

is called the harmonic nonlinear equation and appears to be a new one.

(III). For $\mathcal{T} = I$, the identity operator, the problem (2)reduces to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) = \rho\left(\frac{2w\mu}{w+\mu}\right) - g(\mu), \tag{6}$$

which is called the inverse harmonic-like nonlinear equation.

(IV). Similarly, by taking $g = I$, the identity operator, the problem (3) reduces to finding $\mu \in \mathcal{H}$, such that

$$\mathcal{T}(\mu) == \rho\left(\frac{2w\mu}{w+\mu}\right) - \mathcal{T}(\mu), \tag{7}$$

which is also called the inverse harmonic-like nonlinear equation.

We need the following definitions and results in obtaining our results.

Definition 2 *[26, 27] An operator $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$ is said to be:*

1. *Strongly harmonic-likemonotone, if there exist a constant* $\alpha > 0$*, such that*
$$\langle \mathcal{T}\bigg(\frac{2w\mu}{w+\mu}\bigg) - \mathcal{T}(\frac{2w\nu}{w+\nu}\bigg), \mu-\nu\rangle \geq \alpha\|\mu-\nu\|^2, \quad \forall w,\mu,\nu \in \mathcal{H}.$$
2. *harmonic-like Lipschitz continuous, if there exist a constant* $\beta > 0$*, such tha*
$$\|\mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}\bigg(\frac{2w\nu}{w+\nu}\bigg)\| \leq \beta\|\mu-\nu\|, \quad \forall w,\mu,\nu \in \mathcal{H}.$$
3. *harmonic-like monotone, if*
$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}\bigg(\frac{2w\nu}{w+\nu}\bigg), \mu-\nu\rangle \geq 0, \quad \forall w,\mu,\nu \in \mathcal{H}.$$
4. *harmonic-like pseudo monotone, if*
$$\langle \mathcal{T}\bigg(\frac{2w\mu}{w+\mu}\bigg), \nu-\mu\rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}\bigg(\frac{2w\nu}{w+\nu}\bigg), \nu-\mu\rangle \geq 0, \quad \forall w,\mu,\nu \in \mathcal{H}.$$

For $w = \mu$, Definition 2 reduces to the classical monotonicity of the operators.

Definition 3 *An operator* $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ *is said to be:*

1. *Strongly monotone, if there exist a constant* $\alpha > 0$*, such that*
$$\langle \mathcal{T}(\mu) - \mathcal{T}(\nu), \mu-\nu\rangle \geq \alpha\|\mu-\nu\|^2, \quad \forall \mu,\nu \in \mathcal{H}.$$
2. *Lipschitz continuous, if there exist a constant* $\beta > 0$*, such that*
$$\|\mathcal{T}(\mu) - \mathcal{T}(\nu)\| \leq \beta\|\mu-\nu\|, \quad \forall \mu,\nu \in \mathcal{H}.$$
3. *Monotone, if*
$$\langle \mathcal{T}(\mu) - \mathcal{T}(\nu), \mu-\nu\rangle \geq 0, \quad \forall \mu,\nu \in \mathcal{H}.$$
4. *Pseudo monotone, if*
$$\langle \mathcal{T}(\mu), \nu-\mu\rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}(\nu), \nu-\mu\rangle \geq 0, \quad \forall \mu,\nu \in \mathcal{H}.$$

Remark 1 *Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.*

3 Itertaive Methods and Convergence Criteria

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the harmonic-like nonlinear equations
Let $\mu \in \mathcal{H}$ be the solution of (2). Then

$$g(\mu) = g(\mu) - \rho\mathcal{T}\bigg(\frac{2w\mu}{w+\mu}\bigg), \qquad \forall w,\mu \in \mathcal{H},$$

which can be written as

$$\mu = \mu - g(\mu) + \bigg[g(\mu) - \rho\mathcal{T}\bigg(\frac{2w\mu}{w+\mu}\bigg)\bigg], \qquad \forall w,\mu \in \mathcal{H}, \tag{8}$$

This implies that the general harmonic-like nonlinear equation (2) is equivalent to the harmonic-like fixed point problem. (8). We define the function Φ associated with (8) as

$$\Phi(\mu) = \mu - g(\mu) + \bigg[g(\mu) - \rho\mathcal{T}\bigg(\frac{2w\mu}{w+\mu}\bigg)\bigg], \qquad \forall w,\mu \in \mathcal{H}, \tag{9}$$

To prove the unique existence of the solution of the problem (2), it is enough to show that the map F defined by (9) has a fixed point.

Theorem 1 *Let the operator g be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, respectively. If the operator $\mathcal{T}$ be Lipschitz continuous with constant $\beta > 0$, and there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{1-k}{\beta}, \quad k < 1, \tag{10}$$

where

$$\begin{aligned} \theta &= \rho\beta + k \quad &(11)\\ k &= \sqrt{1-2\sigma+\zeta^2}+\zeta. \quad &(12) \end{aligned}$$

then there exists a unique solution of the problem (2).

Proof. Since the general harmonic-like nonlinear equations (8) and(2) are equivalent, so it is enough to show that the map $\Phi(\mu)$, defined by (9 has a fixed point.
For all $\nu \neq \mu \in \mathcal{H}$, we have

$$\begin{aligned} \left\|\Phi(\mu)-\Phi(\nu)\right\| &\leq \left\|\mu-\nu-(g(\mu)-g(\nu))\right\| \\ &\quad + \left\|\left[g(\mu)-\rho\mathcal{T}\left(\frac{2w\mu}{w+\mu}\right)\right]-\left[g(\nu)-\rho\mathcal{T}\left(\frac{2w\nu}{w+\nu}\right)\right]\right\| \\ &\leq \left\|\mu-\nu-(g(\mu)-g(\nu))\right\| \\ &\quad + \left\|g(\nu)-g(\mu)-\rho(\mathcal{T}\left(\frac{2w\nu}{w+\nu}\right)-\mathcal{T}\left(\frac{2w\mu}{w+\mu})\right)\right\| \\ &\leq \left\|\mu-\nu-(g(\mu)-g(\nu)\right\|+\zeta\left\|\nu-\mu\right\|+\rho\beta\left\|\nu-\mu\right\|. \quad (13) \end{aligned}$$

Since the operator g is strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned} \left\|\mu-\nu-(g(\mu)-g(\nu))\right\|^2 &\leq \left\|\mu-\nu\right\|^2-2\langle g(\mu)-g(\nu),\mu-\nu\rangle+\left\|g(\mu)-g(\nu)\right\|^2 \\ &\leq (1-2\sigma+\zeta^2)\left\|\mu-\nu\right\|^2. \quad (14) \end{aligned}$$

From (13) and (14), we have

$$\begin{aligned} \left\|\Phi(\mu)-\Phi(\nu)\right\| &\leq \left\{\sqrt{(1-2\sigma+\zeta^2}+\zeta+\rho\beta\right\}\left\|\mu-\nu\right\| \\ &= \theta\left\|\mu-\nu\right\|, \end{aligned}$$

where

$$\begin{aligned} \theta &= \rho\beta+k \\ k &= \sqrt{1-2\sigma+\zeta^2}+\zeta. \end{aligned}$$

From (10), it follows that $\theta < 1$, which implies that the map $\Phi(\mu)$ defined by (9) has a fixed point, which is the unique solution of (2). □

We now suggested and analyzed the three step iterative methods for solving the general harmonic-like nonlinear equations (2)

Algorithm 1 *For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$y_n = (1-\gamma_n)\mu_n + \gamma_n\Big\{\mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho\mathcal{T}\Big(\frac{2w\mu_n}{w+\mu_n}\Big)\Big]\Big\} \tag{15}$$

$$z_n = (1-\beta_n)\mu_n + \beta_n\Big\{y_n - g(y_n) + \Big[g(y_n) - \rho\mathcal{T}\Big(\frac{2wy_n}{w+y_n}\Big)\Big]\Big\} \tag{16}$$

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\Big\{z_n - g(z_n) + \Big[g(z_n) - \rho\mathcal{T}\Big(\frac{2wz_n}{w+z_n}\Big)\Big]\Big\}. \tag{17}$$

which are known as Noor iterations.
For $\gamma_n = 0$, Algorithm 1 reduces to:

Algorithm 2 *For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned} z_n &= (1-\beta_n)\mu_n + \beta_n\Big\{\mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho\mathcal{T}\Big(\frac{2w\mu_n}{w+\mu_n}\Big)\Big]\Big\} \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Big\{z_n - g(z_n) + \Big[g(z_n) - \rho\mathcal{T}\Big(\frac{2wz_n}{w+z_n}\Big)\Big]\Big\}, \end{aligned}$$

which is known as the Ishikawa iterative scheme for the problem (2).
Note that for $\gamma_n = 0$ *and* $\beta_n = 0$, Algorithm 1 is called the Mann iterative method, that is.

Algorithm 3 *For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\mu_{n+1} = (1-\beta_n)\mu_n + \beta_n\Big\{\mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho\mathcal{T}\Big(\frac{2w\mu_n}{w+\mu_n}\Big)\Big]\Big\}.$$

We suggest new perturbed iterative schemes for solving the general harmonic-like nonlinear equations (2).

Algorithm 4 *For a given μ_0, compute the approximate solution $\{\mu_n\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n + \gamma_n\Big\{\mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho\mathcal{T}\Big(\frac{2w\mu_n}{w+\mu_n}\Big)\Big]\Big\} + \gamma_n h_n \\ z_n &= (1-\beta_n)\mu_n + \beta_n\Big\{y_n - g(y_n) + \Big[g(y_n) - \rho\mathcal{T}\Big(\frac{2wy_n}{w+y_n}\Big)\Big]\Big\} + \beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Big\{z_n - g(z_n) + \Big[g(z_n) - \rho\mathcal{T}\Big(\frac{2wz_n}{w+z_n}\Big)\Big]\Big\} + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0 \le \alpha_n, \beta_n, \gamma_n \le 1; \quad \forall n \ge 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving harmonic-like nonlinear equations (2).
We now study the convergence analysis of Algorithm 1, which is the main motivation of our next result.

Theorem 2 *Let the operators $\mathcal{T}, g$ satisfy all the assumptions of Theorem 1. If the condition (10) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 1 converges to the exact solution $\mu \in \mathcal{H}$ of the harmonic-like nonlinear equations (2) strongly in $\mathcal{H}$.*

Proof. From Theorem 1, we see that there exists a unique solution $\mu \in \mathcal{H}$ of the harmonic-like nonlinear equations (2). Let $\mu \in \mathcal{H}$ be the unique solution of (2). Then, using the fixed point problem (8), we have

$$\begin{aligned}
\mu &= (1-\alpha_n)\mu + \alpha_n\Big\{\mu - g(\mu) + \Big[g(\mu) - \rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big]\Big\} \qquad (18)\\
&= (1-\beta_n)\mu + \beta_n\Big\{\mu - g(\mu) + \Big[g(\mu) - \rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big]\Big\} \qquad (19)\\
&= (1-\gamma_n)\mu + \gamma_n\Big\{\mu - g(\mu) + \Big[g(\mu) - \rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big]\Big\}. \qquad (20)
\end{aligned}$$

From (17) and (18), we have

$$\begin{aligned}
\Big\|\mu_{n+1} - \mu\Big\| &= \Big\|(1-\alpha_n)(\mu_n - \mu) + \alpha_n(z_n - \mu - (g(z_n) - g(\mu)))\\
&+ \alpha_n\Big\{\Big[g(\mu_n) - \rho\mathcal{T}\Big(\frac{2w\mu_n}{w+\mu_n}\Big)\mu\Big] - \Big[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu}\Big)\Big]\Big\}\Big\|\\
&\leq (1-\alpha_n)\Big\|\mu_n - \mu\Big\| + \alpha_n\Big\|\mu_n - \mu - (g(\mu_n) - g(\mu))\Big\|\\
&+ \alpha_n\Big\{\Big\|g(\mu_n) - g(\mu) - \rho\Big(\mathcal{T}\Big(\frac{2w\mu_n}{w+\mu_n}\Big) - \mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big)\Big\|\Big\}\\
&\leq (1-\alpha_n)\Big\|\mu_n - \mu\Big\| + \alpha_n(k+\rho\beta)\Big\|\mu_n - \mu\Big\|\\
&= (1-\alpha_n)\Big\|u_n - \mu\Big\| + \alpha_n\theta\Big\|\mu_n - \mu\Big\|, \qquad (21)
\end{aligned}$$

where θ is defined by (11).
In a similar way, from (15) and (19), we have

$$\begin{aligned}
\|z_n - \mu\| &\leq (1-\beta_n)\|\mu_n - \mu\| + 2\beta_n\theta\|y_n - \mu - (g(y_n) - g(\mu))\|\\
&+ \beta_n\|g(y_n) - g(\mu) - \rho(y_n - \mu)\| + \beta_n\eta\|y_n - \mu\|\\
&\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n(k+\rho)\|y_n - \mu\|,\\
&\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n\theta\|y_n - \mu\|, \qquad (22)
\end{aligned}$$

where θ is defined by (11).
From (15) and (20), we obtain

$$\begin{aligned}
\|y_n - \mu\| &\leq (1-\gamma_n)\|\mu_n - \mu\| + \gamma_n\theta\|\mu_n - \mu\|\\
&\leq (1-(1-\theta)\gamma_n)\|\mu_n - \mu\|\\
&\leq ||\mu_n - \mu||. \qquad (23)
\end{aligned}$$

From (22) and (23), we obtain

$$\begin{aligned}
\|z_n - \mu\| &\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n\theta\|\mu_n - \mu\|\\
&= (1-(1-\theta)\beta_n)\|\mu_n - \mu\|\\
&\leq ||\mu_n - \mu||. \qquad (24)
\end{aligned}$$

Form the above equations, we have

$$\begin{aligned}
\|\mu_{n+1} - \mu\| &\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\theta\|\mu_n - \mu\|\\
&= [1-(1-\theta)\alpha_n]\|\mu_n - \mu\|\\
&\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|\mu_0 - \mu\|.
\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (23), and (24), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

Also, we can suggest the following iterative methods for solving the harmonic-like nonlinear equations.

Algorithm 5 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\left[g(\mu_n)-\rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right)\right], \tag{25}$$

which is an implicit iterative method and is equivalent to the following two-step method.

Algorithm 6 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n-g(\mu_n)+[g(\mu_n)-\rho\mathcal{T}(\mu_n)] \\ \mu_{n+1} &= \mu_n-g(\mu_n)+\left[g(\mu_n)-\rho\mathcal{T}\left(\frac{2wz_n}{w+z_n}\right)\right]. \end{aligned}$$

Algorithm 7 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\left[g(\mu_{n+1})-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})\right], \tag{26}$$

which is known as the modified iterative method and is equivalent to the iterative method.

Algorithm 8 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n-g(\mu_n)+\left[g(\mu_n)-\rho\mathcal{T}\left(\frac{2w\mu_n}{w+\mu_n}\right)\right] \\ \mu_{n+1} &= \mu_n-g(\mu_n)+\left[g(z_n)-\rho\mathcal{T}\left(\frac{2z_n}{w+z_n}\right)\right], \end{aligned}$$

which is two-step predictor-corrector method for solving the problem (2).

We can rewrite the equation (8) as:

$$\mu=\mu-g(\mu)+\left[g\left(\frac{\mu+\mu}{2}\right)-\rho\mathcal{T}\left(\frac{2w\mu}{w+\mu}\right)\right]. \tag{27}$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 9 *[31]. For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\left[g\left(\frac{\mu_n+\mu_{n+1}}{2}\right)-\rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right)\right]. \tag{28}$$

Applying the predictor-corrector technique, we suggest the following inertial iterative method for solving the problem (2) .

Algorithm 10 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n-g(\mu_n)+\left[g(\mu_n)-\rho\mu_n\right] \\ \mu_{n+1} &= \mu_n-g(\mu_n)+\left[g\left(\frac{z_n+\mu_n}{2}\right)-\rho\mathcal{T}\left(\frac{2wz_n}{w+z_n}\right)\right]. \end{aligned}$$

One can rewrite (8) as

$$\mu = \mu - g(\mu) + \left[g\left(\frac{\mu+\mu}{2}\right) - \rho\mathcal{T}\left(\frac{2w(\mu+\mu)}{2w+\mu+\mu}\right)\right]. \tag{29}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2).

Algorithm 11 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \left[g\left(\frac{\mu_n+\mu_{n+1}}{2}\right) - \rho\mathcal{T}\left(\frac{2w(\mu_n+\mu_{n+1})}{2w+\mu_n+\mu_{n+1}}\right)\right]. \tag{30}$$

Applying the predictor-corrector technique, we suggest the inertial iterative method for solving the general harmonic-like nonlinear equations (2)

Algorithm 12 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_n}{w+\mu_n}\right)\right] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \left[g\left(\frac{\omega_n+\mu_n}{2}\right) - \rho\mathcal{T}\left(\frac{2w(z_n+\mu_n)}{2w+z_n+\mu_n}\right)\right], \end{aligned}$$

which is a new predictor-corrector two-step method.

We now suggest multi-step inertial methods for solving the harmonic-like nonlinear equations (2).

Algorithm 13 *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} z_n &= \mu_n - \theta_n(\mu_n - \mu_{n-1}), \\ y_n &= (1-\gamma_n)z_n + \gamma_n\left\{z_n - g(\omega_n) + \left[g\left(\frac{z_n+\mu_n}{2}\right) - \rho\mathcal{T}\left(\frac{2w(z_n+\mu_n)}{2w+z_n+\mu_n}\right)\right]\right\}, \\ t_n &= (1-\beta_n)y_n \\ &\quad +\beta_n\left\{y_n - g(y_n) + \left[g\left(\frac{y_n+\omega_n+\mu_n}{3}\right) - \rho\mathcal{T}\left(\frac{2w(y_n+z_n+\mu_n)}{3w+y_n+z_n+\mu_n}\right)\right]\right\}, \\ \mu_{n+1} &= (1-\alpha_n)z_n + \alpha_n(z_n - g(z_n) \\ &\quad +\alpha_n\left\{\left[g\left(\frac{z_n+y_n+t_n+\mu_n}{4}\right) - \rho\mathcal{T}\left(\frac{2w(y_n+z_n+t_n+\mu_n)}{4w+y_n+z_n+t_n+\mu_n}\right)\right]\right\}, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1$.

Using the above technique, one can investigate the convergence analysis of these inertial projection methods. We would like to mention that Algorithm 13 can be viewed as the generalizations of Noor (three-step) iterations [17, 18] for solving the fixed point problem. Similar multi-step hybrid iterative methods can be proposed and analyzed for solving system of general fixed point problems (2), which is an interesting problem.

4 Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving the general harmonic-like nonlinear equations. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [10]. It is worth mentioning that the dynamical system is a first order initial value problem. Consequently, variational inequalities and nonlinear problems arising in various branches in pure and applied sciences can now be studied via the differential equations. It has been shown [10, 12, 14, 18, 19, **?**, 22, 32, 33, 34].that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. We introduce and consider some first order, second order and third order dynamical systems associated with general harmonic-like nonlinear equations, which are used to suggest and analyze several iterative methods.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \left\{ \left[g(\mu) - \rho \mathcal{T}\left(\frac{2w\mu}{w+\mu} \right) \right] - g(\mu) \right\}. \tag{31}$$

One can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem(2), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation One can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem(2), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{32}$$

We now consider a dynamical system associated with the harmonic-like nonlinear equations. Using the equivalent formulation (3.1), we suggest a class of dynamical systems as

$$\frac{d\mu}{dt} = \lambda \left\{ \left[g(\mu) - \rho \mathcal{T}\left(\frac{2w\mu}{w+\mu} \right) \right] - g(\mu) \right\}, \quad \mu(t_0) = \alpha, \tag{33}$$

where λ is a parameter. The system of type (33) is called the harmonic-like dynamical system associated with the problem (2). From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$.. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2) can be studied. The equilibrium point of the dynamical system (33) is defined as follows.

The equilibrium point of the dynamical system (33) is defined as follows.

Definition 4 *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (44), if,*

$$\frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like nonlinear equations (2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point. This implies that $\mu \in \mathcal{H}$ is a solution of the harmonic-like nonlinear equations (2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Definition 5 *[13] The dynamical system is said to converge to the solution set S^* of (**??**), if , irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t \to \infty} dist(\mu(t), S^*) = 0, \tag{34}$$

where

$$dist(\mu, S^*) = inf_{\nu \in S^*} \|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (34) implies that

$$\lim_{t \to \infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 6 *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \leq u_1 \|\mu(t_0) - \mu^*\| exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 1 *(Gronwall Lemma) Let* $\hat{\mu}$ *and* $\hat{\nu}$ *be real-valued nonnegative continuous functions with domain* $\{t : t \leq t_0\}$ *and let* $\alpha(t) = \alpha_0(|t - t_0|)$*, where* α_0 *is a monotone increasing function. If, for* $t \geq t_0$*,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t)exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now establish that the trajectory of the solution of the dynamical system (33) converges to the unique solution of the general harmonic-like nonlinear equation (2). The analysis is in the spirit of Noor [14] and Xia and Wang [23, 24].

Theorem 3 *Let the operator* $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ *be harmonic-like Lipschitz continuous with constants* $\beta > 0$ *and the operator* g *be Lipschitz continuous with constant* $\zeta > 0$*, respectively. If* $\lambda(2\zeta+\rho\beta) < 1$*, then, for each* $\mu \in \mathcal{H}$*, there exists a unique continuous solution* $\mu(t)$ *of the dynamical system (33) with* $\mu(t_0) = \mu_0$ *over* $[t_0, \infty)$*.*

Proof. Let

$$G(\mu) = \lambda\Big\{\Big[g(\mu) - \rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big] - g(\mu)\Big\}, \quad \forall\mu \in H.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$.
$\forall\mu, \nu \in H$, we have

$$\begin{aligned}
&\Big\|G(\mu) - G(\nu)\Big\| \\
\leq\ & \lambda\Big\{\Big\|\Big[g(\mu) - \rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big] - \Big[g(\nu) - \rho\mathcal{T}\Big(\frac{2w\nu}{w+\nu}\Big)\Big]\Big\| + \Big\|g(\mu) - g(\nu)\Big\|\Big\} \\
\leq\ & \lambda\Big\{\Big\|g(\mu) - g(\nu)\Big\| + \Big\|\Big[g(\mu) - \rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big] - \Big[g(\nu) - \rho\mathcal{T}\Big(\frac{2w\nu}{w+\nu}\Big)\Big]\Big\|\Big\} \\
\leq\ & \lambda\Big\{\Big\|g(\mu) - g(\nu)\Big\| + \Big\|g(\mu) - g(\nu) - \rho(\mathcal{T}\Big(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu})\Big)\Big\|\Big\} \\
\leq\ & \lambda\Big\{\Big\|g(\mu) - g(\nu)\Big\| + \Big\|g(\mu) - g(\nu)\Big\| + \rho\Big\|\mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}\Big)\Big\|\Big\} \\
\leq\ & \lambda\{(2\zeta + \beta\rho)\}\Big\|\mu - \nu\Big\|.
\end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant
$\lambda\{(2\zeta + \rho\beta)\} < 1$ and for each $\mu \in \mathcal{H}$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (33), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider , for any $\mu \in \mathcal{H}$.

$$\Big\|G(\mu)\Big\| = \Big\|\frac{d\mu}{dt}\Big\| \leq \lambda\Big\|\Big[g(\mu) - \rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big] - g(\mu)\Big\| \leq \lambda\rho\beta.$$

Then

$$\|\mu(t)\| \leq \|\mu_0\| + \int_{t_0}^{t} \|\mu(s)\|ds \leq (\|\mu_0\| + k_2\int_{t_0}^{t} \|\mu(s)\|ds,$$

where $k_2 = \lambda\rho\beta$.
Hence by the Gronwall Lemma 1, we have

$$\|\mu(t)\| \leq \|u_0\| e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 4 *If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\sigma > 0$ and $\zeta > 0$, then the dynamical system (33) converges globally exponentially to the unique solution of the general harmonic-like nonlinear equations (2).*

Proof. Since the operators $\mathcal{T}, g$ are Lipschitz continuous, it follows from Theorem 3 that the dynamical system (33) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in \mathcal{H}$. Let $\mu(t)$ be a solution of the initial value problem (33). For a given $\mu^* \in H$ satisfying (1), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad u(t) \in \mathcal{H}. \tag{35}$$

From (33) and (35), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda\Big\langle \mu(t) - \mu^*, \frac{d\mu}{dt}\Big\rangle \\
&= 2\lambda\Big\langle \mu(t) - \mu^*, \Big[g(\mu(t)) - \rho\mathcal{T}\Big(\frac{2w\mu(t)}{w+\mu(t)} bigg)\Big] - g(\mu(t))\Big\rangle \\
&= 2\lambda\Big\langle \mu(t) - \mu^*, \Big[g(\mu(t)) - \rho\mathcal{T}\Big(\frac{2w\mu(t)}{w+\mu(t)}\Big)\mu(t)\Big] - g(\mu^*) + g(\mu^*) - g(\mu(t))\Big\rangle \\
&= -2\lambda\Big\langle \mu(t) - \mu^*, g(\mu(t)) - g(\mu^*)\Big\rangle \\
&\quad +2\lambda\Big\langle \mu(t) - \mu^*, \Big[g(\mu(t)) - \rho\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t}))\mu(t)\Big] - g(\mu^*)\Big\rangle \\
&\leq -2\lambda\Big\langle \rho(\mathcal{T}(\frac{2w\mu(t}{w+\mu(t})) - \mathcal{T}(\frac{2w\mu^*(t)}{w+\mu^*(t)})\mu^*), g(\mu(t)) - g(\mu^*)\Big\rangle \\
&\quad +2\lambda\Big\langle \mu(t) - \mu^*(t), \Big[g(\mu(t)) - \rho\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)})\Big] - \Big[g(\mu^*(t)) - \rho\mu^*(t)\Big]\Big\rangle, \\
&\leq -2\lambda\sigma\Big\|\mu(t) - \mu^*\Big\|^2 + \lambda\Big\|g(\mu(t)) - g(\mu^*)\Big\|^2 \\
&\quad +\lambda\Big\|\Big[g(\mu(t)) - \rho\mathcal{T}\Big(\frac{2w\mu(t)}{w+\mu(t)}\Big)\Big] - \Big[g(\mu^*(t)) - \rho\mathcal{T}\Big(\frac{2w\mu^*(t)}{w+\mu^*(t)}\Big)\Big]\Big\|^2
\end{aligned} \tag{36}$$

Using the Lipschitz continuity of the operators$\mathcal{T}, g$, we have

$$\begin{aligned}
&\Big\|\Big[g(\mu(t)) - \rho\mathcal{T}(\frac{2w\mu(t)}{w+\mu(t)})\Big] - \Big[g(\mu^*(t)) - \rho\mathcal{T}\Big(\frac{2w\mu^*(t)}{w+\mu^*(t)}\Big)\Big]\Big\| \\
&\quad \leq \Big\|g(\mu(t)) - g(\mu^*(t)) - \rho(\mathcal{T}\Big(\frac{2w\mu(t)}{w+\mu(t)}\Big) - \mathcal{T}\Big(\frac{2w\mu^*(t)}{w+\mu^*(t)})\Big)\Big\| \\
&\quad \leq (\zeta + \rho\beta)\Big\|\mu(t) - \mu^*(t)\Big\|.
\end{aligned} \tag{37}$$

From (36) and (37), we have

$$\frac{d}{dt}\Big\|\mu(t) - \mu^*(t)\Big\| \leq 2\xi\lambda\Big\|\mu(t) - \mu^*(t)\Big\|,$$

where

$$\xi = ((1 + \rho\beta) - 2\sigma).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\left\|\mu(t) - \mu^*\right\| \leq \left\|\mu(t_0) - \mu^*\right\| e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (33) converges globally exponentially to the unique solution of the harmonic-like nonlinear equations (2). □

We use the dynamical system (33) to suggest some iterative for solving the harmonic-like nonlinear equations (2). These methods can be viewed in the sense of Noor [17, 18] involving the double projection.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (33) becomes

$$\frac{d\mu}{dt} + g(\mu) = \left[g(\mu) - \rho\mathcal{T}\left(\frac{2w\mu}{w+\mu}\right)\right], \quad \mu(t_0) = \alpha. \tag{38}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (38), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right)\right], \tag{39}$$

where h is the step size.
Now, we can suggest the following implicit iterative method for solving the harmonic-like nonlinear equations (2).

Algorithm 14 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right) - \frac{\mu_{n+1} - \mu_n}{h}\right],$$

This is an implicit method. Algorithm 14 is equivalent to the following two-step method.

Algorithm 15 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_n}{w+\mu_n}\right)\right] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2wy_n}{w+y_n}\right) - \frac{y_n - \mu_n}{h}\right], \end{aligned}$$

Discretizing (33), we now suggest an other implicit iterative method for solving the harmonic-like nonlinear equations (2).

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = \left[g(\mu_{n+1}) - \rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right)\right], \tag{40}$$

where h is the step size.
This formulation enables us to suggest the two-step iterative method.

Algorithm 16 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_n}{w+\mu_n})\mu_n\right)\right] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \left[g(\omega_n) - \rho\mathcal{T}\left(\frac{2wy_n}{w+y_n}\right) - \frac{y_n - \mu_n}{h}\right]. \end{aligned}$$

Discretizing (33),we have

$$\frac{\mu_{n+1}-\mu_n}{h} = \mu_n - g(\mu_n) + \left[g(\mu_{n+1}) - \rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right)\right], \tag{41}$$

where h is the step size.
This helps us to suggest the following implicit iterative method for solving the problem (2).

Algorithm 17 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_n}{w+\mu_n}\right)\right] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \left[g(y_n) - \rho\mathcal{T}\left(\frac{2wy_n}{w+y_n}\right)\right]. \end{aligned}$$

Discretizing (33), we propose another implicit iterative method.

$$\frac{\mu_{n+1}-\mu_n}{h} + g(\mu_n) = \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right)\right]$$

where h is the step size.

For $h = 1$, we can suggest an implicit iterative method for solving the problem (2).

Algorithm 18 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right)\right].$$

From (38), we have

$$\frac{d\mu}{dt} + g(\mu) = \left[g((1-\alpha)\mu + \alpha\mu) - \rho\mathcal{T}\left(\frac{2w(1-\alpha)\mu + \alpha\mu)}{w+(1-\alpha)\mu+\alpha\mu}\right)\right], \tag{42}$$

where $\alpha \in [0, 1]$ is a constant.

Discretization (42) and taking $h = 1$, we have

$$\mu_{n+1} = \mu_n - g(\mu_n) + \left[g\big((1-\alpha)\mu_n + \alpha\mu_{n-1}\big) - \rho\mathcal{T}\left(\frac{2w((1-\alpha)\mu_n + \alpha\mu_{n-1})}{w+(1-\alpha)\mu_n+\alpha\mu_{n-1}}\right)\right], \tag{43}$$

which is an inertial type iterative method for solving the harmonic-like nonlinear equations (2). Using the predictor-corrector techniques, we have

Algorithm 19 *For a given μ_0, μ_1, compute μ_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \left[g(y_n) - \rho\mathcal{T}\left(\frac{2wy_n}{w+y_n}\right)\right], \end{aligned}$$

which is known as the inertial two-step iterative method.

We now introduce the second order dynamical system associated with the harmonic-like nonlinear equations (2). To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\begin{aligned} \gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} &= \lambda\left\{\left[g(\mu) - \rho\mathcal{T}\left(\frac{2w\mu}{w+\mu}\right)\right] - g(\mu)\right\}, \\ &\quad \mu(a) = \alpha, \quad \mu(b) = \beta, \end{aligned} \tag{44}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (44) is indeed a second order boundary vale problem. In a similar way, we can define the second order initial value problem associated with the dynamical system.

The equilibrium point of the dynamical system (44) is defined as follows.

Definition 7 *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (44), if,*

$$\gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like nonlinear equations (2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.
From (44), we have

$$g(\mu) = [g(\mu) - \rho\mu].$$

Thus, we can rewrite (44) as follows:

$$g(\mu) = \Big[g(\mu) - \rho\mu + \gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\Big]. \tag{45}$$

For $\lambda = 1$, the problem (44) is equivalent to finding $\mu \in \Omega$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + g(\mu) = \Big[g(\mu) - \rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{46}$$

The problem (46) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various fields of mathematical and engineering sciences is fruitful in developing implementable numerical methods for finding the approximate solutions of the harmonic-like quasi variational inequalities. Consequently, one can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the harmonic-like nonlinear equations and related optimization problems.
We discretize the second-order dynamical systems (46) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h^2} + \frac{\mu_n - \mu_{n-1}}{h} + g(\mu_n) = \Big[\mu_n - \rho\mathcal{T}\Big(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\Big)\Big], \tag{47}$$

where h is the step size.
If $\gamma = 1, h = 1$, then, from equation(47) we have

Algorithm 20 *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n + g(\mu_n) + \Big[g(\mu_n) - \rho\mathcal{T}\Big(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\Big)\Big],$$

which is the extragradient method for solving the harmonic-like noninear equations(2). Algorithm 20 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 21 *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho\mathcal{T}\Big(\frac{2wy_n}{w+y_n}\Big)\Big], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n \in [0,1]$ is a constant.
In a similar way, we have the following two-step method.

Algorithm 22 *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Big[g(y_n) - \rho\mathcal{T}\Big(\frac{2wy_n}{w+y_n}\Big)\Big], \end{aligned}$$

which is also called the double projection method for solving the harmonic-like nonlinear equations (2).

We discretize the second-order dynamical systems (33) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}+g(\mu_{n+1})=\Big[g(\mu_n)-\rho\mathcal{T}\Big(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\Big)\Big],$$

where h is the step size.
Using this discrete form, we can suggest the following an iterative method for solving the harmonic-like nonlinear equations(2).

Algorithm 23 *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\mu_{n+1} &= \mu_n - g(\mu_{n+1})\\ &+\Big[g(\mu_{n+1})-\rho\mathcal{T}\Big(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\Big)-\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}\Big].\end{aligned}$$

Algorithm 23 is called the hybrid inertial proximal method for solving the harmonic-like nonlinear equations and related optimization problems. This is a new proposed method.
Note that, for $\gamma = 0$, Algorithm 23 reduces to the following iterative method.

Algorithm 24 *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_{n+1})+\Big[g(\mu_{n+1})-\rho\mathcal{T}\Big(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\Big)+\frac{\mu_n-\mu_{n-1}}{h}\Big],$$

which is called the inertial double projection method.

We now consider the third order dynamical systems associated with the harmonic-like nonlinear equations of the type (2). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\begin{aligned}\gamma\frac{d^3\mu}{dt^3}+\zeta\frac{d^2\mu}{dt^2}+\xi\frac{d\mu}{dt}+g(\mu) &= \Big[g(\mu)-\rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)\Big],\\ &\quad u(a)=\alpha, \dot{\mu}(a)=\beta, \dot{\mu}(b)=0\end{aligned}\tag{48}$$

where $\gamma > 0, \zeta, \xi$ and $\rho > 0$ are constants. Problem (48) is called third order dynamical system associated with harmonic-like nonlinear equations (2).
The equilibrium point of the dynamical system (48) is defined as follows.

Definition 8 *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (44), if,*

$$\gamma\frac{d^3\mu}{dt^3}+\zeta\frac{d^2\mu}{dt^2}+\xi\frac{d\mu}{dt}=0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like nonlinear equations (2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Consequently, the problem (48) can be written as

$$g(\mu)=\Big[g(\mu)-\rho\mathcal{T}\Big(\frac{2w\mu}{w+\mu}\Big)+\gamma\frac{d^3\mu}{dt^3}+\zeta\frac{d^2\mu}{dt^2}+\xi\frac{d\mu}{dt}\Big].\tag{49}$$

We discretize the third-order dynamical systems (48) using central finite difference and backward difference schemes to have

$$\begin{aligned}&\gamma\frac{u_{n+2}-2u_{n+1}+2u_{n-1}-u_{n-2}}{2h^3}+\zeta\frac{u_{n+1}-2u_n+u_{n-1}}{h^2}\\ &+\xi\frac{3\mu_n-4\mu_{n-1}+\mu_{n-2}}{2h}+g(\mu_n)=\Big[g(\mu_n)-\rho\mathcal{T}\Big(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\Big)\Big],\end{aligned}\tag{50}$$

where h is the step size.
If $\gamma = 1, h = 1, \zeta = 1, \xi = 1$, then, from equation(50) after adjustment, we have

Algorithm 25 *For a given μ_0, μ_1, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho\mathcal{T}\left(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}\right) + \frac{\mu_{n-1} - 3u_n}{2}\right], \quad n = 0, 1, 2, \dots$$

which is an inertial type hybrid iterative methods for solving the harmonic-like nonlinear equations(2).

Remark 2 *For appropriate and suitable choice of the operators $\mathcal{T}, g$, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving inverse fixed point problem and related optimization problems. Using the techniques and ideas of Noor et al.[18, 19], one can discuss the convergence analysis of the proposed methods*

Conclusion

In this paper, we have introduced and studied some new classes of general harmonic-like nonlinear equations. We have proved that the general harmonic-like nonlinear equations are equivalent to the fixed point problem. The alternative equivalence has been used to suggest and propose some new multi-step explicit and implicit iterative methods for solving the general harmonic-like nonlinear equations along with convergence criteria. Dynamical system technique for solving the general nonlinear equations is exploited to discuss the asymptotic stability of the solution in the sense of Lypnouv. Second order and third order initial dynamical systems associated with the general harmonic-like nonlinear equations are considered to propose and suggest a wide class of iterative methods for finding the approximate solution, which is entirely new and novel idea. These new methods include Noor(three step) iterations, Ishikawa(two-step) iterations and Mann (one-step) iteration as special cases. It is an open problem to compare these proposed methods with other methods. Applying the technique and ideas of Ashish et. al. [3, 4, 5], Kwuni et al. [13] and Paimsangan et al. [29], one can explore the Julia set and Mandelbrot set in Noor orbit by using Noor (multi-step) iterations in the fixed point theory, fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics signal recovery, logistic maps [35]and computer aided design. This is an open problem, which deserves further research efforts.

Data availability:

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study

Conflict interest:

Authors have no conflict of interest.

Authors contributions:

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