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Modified Szász-Mirakyan operators [1]

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Abstract

For a modified sequence of Szász-Mirakyan operators we establish rates of convergence using the Ditzian– Totik modulus of continuity. Moreover, we characterize functions satisfying a Lipschitz type condition.

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1 Introduction

Throughout the paper we set $I = [0, \infty)$, $C(I)$ is the family of all continuous functions $f : I \to \mathbb{R}$, and $e_k(x) = x^k$, $k \in \mathbb{N}_0$.

As a modification of Szász-Mirakyan operators, for $n \in \mathbb{N}$ and $x \in I$, in [4] Duman and Özarslan introduced the operators

$$E_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_{n,k}(x), \qquad p_{n,k}(x) = e^{-nu_n(x)} \frac{(nu_n(x))^k}{k!},$$

where $\{u_n(x)\}$ is a sequence of continuously differentiable functions, $u_n : I \to I$. These operators are interesting only if there exists $x_0 > 0$ such that $u_n(x_0) \neq x_0$, otherwise, one recovers the classical Szász-Mirakyan operators

$$S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} \frac{n^k}{k!} f\left(\frac{k}{n}\right) x^k.$$

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In [5] Duman and Özarslan investigated the global approximation behavior of the modified Szász-Mirakyan operators E_n in some weighted spaces.

For $m \in \mathbb{N}_0$ and $x \in I$, set $\mu_m(x) = 1/(1+x^m)$ and consider the weighted space $C^*_m(I) = \{f \in C(I) : \mu_m f$ is uniformly continuous and bounded on $I\}$ endowed with the norm $\|f\|^*_m = \sup_{x\in I} \mu_m(x) \mid f(x) \mid$.

The following result was proved in [5].

Theorem 1.1 *Let D_n be given by (4) and assume*

$$u_n(0) = 0 \qquad 0 < u_n(x) \le x, x > 0, \quad n \in \mathbb{N}, \tag{1}$$

and $u'_n(x) \neq 0$, for $x \in I$. Then, for every $m \in \mathbb{N}_0$ there exists an absolute constant $M_m > 0$ such that, if $f \in C^*_m(I)$ and $x \in I$,

$$\mu_m(x) \mid E_n(f;x) - f(x) \mid \le M_m\, \omega^2_m\Big(f, \sqrt{v^2_n(x) + \frac{u_n(x)}{n}}\Big) + \omega^1_m(f, v_n(x)),$$

where $v_n(x) = x - u_n(x)$, $\omega^1_m(f,\delta) = \sup_{0\le h\le\delta,\, x\in I} \mu_m(x) \mid f(x+h) - f(x) \mid$ and

$$\omega^2_m(f,\delta) = \sup_{0\le h\le\delta} \sup_{x\in I} \frac{\mid f(x+2h) - 2f(x+h) + f(x) \mid}{1+x^m}.$$

Notice that (without additional conditions) we can not use Theorem 1.1 to obtain an estimate of $\|D_n(f) - f\|^*_m$ in the norm of the space $C^*_m(I)$. In [5, pag. 77] Duman and Özarslan asserted that, if

$$\lim_{n\to\infty} \sup_{x\in I} \mid u_n(x) - x \mid = 0 \tag{2}$$

and $f \in C^*_m(I)$, then $\lim_{n\to\infty} \|D_n(f) - f\|^*_m = 0$.

In this work we study a variation of the operators E_n in other functional spaces with other conditions concerning $u_n(x)$.

For $m \in \mathbb{N}_0$ set

$$\varrho_m(x) = \frac{1}{(1+x)^m}, \qquad x \in I,$$

and consider the weighted space $C_m(I)$ of all functions $f \in C(I)$ such that

$$\|f\|_m = \sup_{x\in I} \varrho_m(x) \mid f(x) \mid < \infty,$$

and there exists $A_m(f) \in \mathbb{R}$ such that

$$A_m(f) = \lim_{x\to\infty} \varrho_m(x) f(x). \tag{3}$$

Throughout this work we assume that $m \in \mathbb{N}_0$ is fixed, $C_m(I)$ is defined as above and

$$D_n(f,x) = \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) p_{n,k}(x), \tag{4}$$

But

$$\sup_{n\in\mathbb{N}}\sup_{x\in I} \mid u_n'(x) \mid < \infty.$$

and there exist constants $K_1 > 1$ and $K_2 > 1$ such that, for each $n \in \mathbb{N}$ and every $x > 0$,

$$0 < x \le u_n(x) \le K_1 x, \qquad \text{and} \quad \mid u_n(x) - x \mid \le \frac{K_2}{n} \tag{5}$$

It follows from (5) that $u_n(0) = 0$. Notice that, if $u_n(x) \neq x$, then the condition $x \le u_n(x)$ was not considered by Duman and Özarslan. In [5] there are not examples of function u_n satisfying (1), different from $e_1(x)$. It is easy to see that the functions

$$u_n(x) = x\Big(1 + \frac{1}{n(1+x)}\Big)$$

satisfy the assumptions assumed by us.

Let $\mathcal{D}(I)$ be the family of all $f \in C(I)$ such that, for each $n \in \mathbb{N}$, the series $D_n(f)$ converges absolutely.

For $f \in C_m(I)$, $\varphi(x) = \sqrt{x}$, and $t \ge 0$ define

$$\omega^{\varphi}(f,t)_m = \sup_{0\le h\le t} \sup_{x\in I,\, h\varphi(x)/2\le x} \frac{\mid f(x+h\varphi(x)/2) - f(x-h\varphi(x)/2) \mid}{(1+x)^m}. \tag{6}$$

In this work we proved the following assertions.

Theorem 1.2 *For each $m \in \mathbb{N}_0$, there exists a constant C_m such that, for every $f \in C_m(I)$ and $n \in \mathbb{N}$*

$$\|D_n(f) - f\|_m \le C_m\, \omega^{\varphi}\Big(f, \frac{1}{\sqrt{n}}\Big)_m,$$

where the modulus is defined in (6)*.*

Theorem 1.3 *For each $0 < \alpha < 1$ and $f \in C_m(I)$ the following assertions are equivalent:*

(i) *There exists a constant C_1 such that $\|D_n(f) - f\|_m \le C_1 n^{-\alpha/2}$, for each $n \in \mathbb{N}$.*

(ii) *There exists a constant C_2 such that $\omega^{\varphi}(f,t)_m \le C_1 t^{\alpha}$, $0 < t \le 1$.*

2 Auxiliary results

For $n \in \mathbb{N}$ and $x > 0$ consider the following notations: set $P_{n,1}(x) = x$ and, for $j \in \mathbb{N}$,

$$P_{n,j+1}(x) = x\Big(x - \frac{1}{n}\Big)\cdots\Big(x - \frac{j}{n}\Big). \tag{7}$$

Proposition 2.1 *Assume that D_n is given by (4).*
(i) *If $f \in \mathcal{D}(I)$, then $D_n(f,0) = f(0)$.*
(ii) *For each $j \in \mathbb{N}$, one has*

$$D_n(P_{n,j+1}, x) = u_n^{j+1}(x).$$

In particular

$$D_n(e_1, x) = u_n(x) \qquad D_n(e_2, x) = u_n^2(x) + \frac{u_n(x)}{n},$$

and

$$D_n((e_1 - x)^2, x) = (u_n(x) - x)^2 + \frac{u_n(x)}{n}.$$

Proof. (i) Since $u_n(0) = 0$, $D_n(f,0) = f(0)$, for each $f \in \mathcal{D}(I)$.
(ii) It is clear that $D_n(e_0, x) = 1$
Notice that que $nP_{n,1}(k/n) = k$ and, for $j \in \mathbb{N}$,

$$n^{j+1} P_{n,j+1}\Big(\frac{k}{n}\Big) = k(k-1)\cdots(k-j).$$

Therefore, for each fixed $x > 0$,

$$e^{nu_n(x)} n^{j+1} D_n(P_{n,j+1}, x) = \sum_{k=j+1}^{\infty} \frac{(nu_n(x))^k}{(k-1-j)!} = (nu_n(x))^{j+1} e^{nu_n(x)}$$

and $D_n(P_{n,j+1}, x) = u_n^{j+1}(x)$.
In particular, if $j = 0$, $D_n(P_{n,1}, x) = D_n(e_1, x) = u_n(x)$.
On the other hand, since $P_{n,2}(x) = x^2 - x/n$, then

$$D_n(e_2, x) = D_n(P_{n,2}, x) + \frac{1}{n} D_n(e_1, x) = u_n^2(x) + \frac{u_n(x)}{n}.$$

Finally

$$\begin{aligned} D_n((e_1 - x)^2, x) &= D_n(e_2, x) - 2x D_n(e_1, x) + x^2 \\ &= u_n^2(x) - 2x u_n(x) + x^2 + \frac{u_n(x)}{n} = (u_n(x) - x)^2 + \frac{u_n(x)}{n}. \end{aligned}$$ □

Lemma 2.1 *There exists a constant C such that, for each $x \in I$ and $n \in \mathbb{N}$,*

$$D_n((e_1 - x)^2, x) \le C \frac{x}{n}.$$

Proof. From Proposition 2.1 and conditions (5) one has

$$D_n((e_1-x)^2,x) \leq (u_n(x)+x) \mid u_n(x)-x \mid + \frac{u_n(x)}{n} \leq (K_1+1)\frac{K_2}{n}x+\frac{K_1}{n}x \leq \frac{C}{n}x. \quad \square$$

In Proposition 2.1 we found a representation for $D_n(e_1,x)$ and $D_n(e_2,x)$. In Proposition 2.2 we present a recurrence relation to compute $D_n(e_i,x)$ for all $i \geq 2$.

The falling factorial $(x)_n$ is defined as $(x)_n = x(x-1)(x-n+1)$. It is known that $(x)_n$ is a polynomial in x with expansion

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k \tag{8}$$

where the coefficients $s(n,k)$ are the Stirling number of first type [2, p. 48]. Moreover, the numbers $s(n,k)$ satisfy the recurrence relation [2, p. 49]

$$s(n,0) = s(0,k) = 0, \quad s(0,0) = 1$$

and, for $n,k \geq 1$, $s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k)$.

It can be proved that $s(n,n) = 1$,

Proposition 2.2 *For each $i \in \mathbb{N}$, $i \geq 2$, one has*

$$D_n(e_i,x) = u_n^i(x) - \sum_{j=1}^{i-1} \frac{s(i,j)}{n^{i-j}} D_n(e_j,x).$$

Proof. Taking into account that, for $i \geq 2$,

$$n^i P_{n,i}\Big(\frac{x}{n}\Big) = x(x-1)\cdots(x-i+1) = (x)_i,$$

it follows from (8) that (recall that $s(n,n) = 1$)

$$P_{n,i}(x) = \frac{1}{n^i}\sum_{j=0}^{i} s(i,j)n^i x^j = x^i + \frac{1}{n^i}\sum_{j=0}^{i-1} s(i,j)n^j x^j.$$

Hence, it follows from (ii) in Proposition 2.1 that

$$D_n(e_i,x) = D_n(P_{n,i},x) - \sum_{j=1}^{i-1} \frac{s(i,j)D_n(e_j,x)}{n^{i-j}} = u_n^i(x) - \sum_{j=1}^{i-1} \frac{s(i,j)}{n^{i-j}} D_n(e_j,x). \quad \square$$

Lemma 2.2 *There exists a constant C_m such that, for each $n \in \mathbb{N}$ and $x \in I$, one has*

$$0 \leq u_n^m(x) \leq C_m(1+x)^m,$$

Moreover

$$\lim_{x\to\infty} \frac{u_n^m(x)}{(1+x)^m} = 1.$$

Proof. For the inequality we should consider only the case $x > 0$.

If $m = 1$,

$$0 < u_n(x) \leq K_1 \, x \leq K_1(1+x)$$

and

$$\Big| \frac{u_n(x) - x}{1+x} \Big| \leq \frac{K_2}{n(1+x)}.$$

Therefore

$$\lim_{x\to\infty} \frac{u_n(x)}{1+x} = \lim_{x\to\infty} \frac{u_n(x) - x + x}{1+x} = 1.$$

If $m > 1$, then

$$0 \leq u_n^m(x) \leq (u_n(x) - x + x)^m = x^m + \sum_{i=0}^{m-1} \binom{m}{i} x^i (u_n(x) - x)^{m-i}$$

$$\leq x^m + C_1 \sum_{i=0}^{m-1} \binom{m}{i} \frac{x^i}{n^{m-i}} \leq C_2(1+x)^m.$$

On the other hand

$$\lim_{x\to\infty} \frac{u_n^m(x)}{(1+x)^m} = \lim_{x\to\infty} \Big(\frac{u_n(x)}{1+x}\Big)^m = 1.$$

□

Lemma 2.3 *For each $i, m \in \mathbb{N}_0$, there exists a constant M_i such that, for each $n \in \mathbb{N}$ and $x \in I$, one has*

$$0 \leq D_n(e_i, x) \leq M_i(1+x)^i \quad \textit{and} \quad D_n((1+e_1)^m, x) \leq M_m(1+x)^m.$$

Proof. If $i = 0$ the inequality is simple because $D_n(e_0, x) = 1$. Hence, we assume that $i \in \mathbb{N}$.

(i) Since $D_n(e_i) = 0$ (see Proposition 2.1) we consider the case $x > 0$.

If $i = 1$, the assertion follows from Lemma 2.2, because

$$D_n(e_1, x) = u_n(x) \leq K_1 x \leq K_1(1+x).$$

Suppose that, for some $i \in \mathbb{N}$ and each j, $1 \leq j \leq i$, one has

$$D_n(e_j, x) \leq C_i(1+x)^j.$$

Since $(1+x)^j \leq (1+x)^{i+1}$ and the sum

$$\sum_{j=1}^{i} \frac{\mid s(i+1, j) \mid}{n^{i+1-i}} \leq \sum_{i=1}^{i} \mid s(i+1, i) \mid$$

is bounded, from Proposition 2.2 and Lemma 2.2 we obtain

$$0 \leq D_n(e_{i+1}, x) = u_n^{i+1}(x) - \sum_{j=1}^{i} \frac{s(i+1, j)}{n^{i+1-i}} D_n(e_j, x) \leq C_{mi+1}(1+x)^{i+1}$$

and we have the result.

(ii) Set $\gamma_m = \max\{M_i, 0 \le i \le m\}$. Taking into account that $(1+x)^i \le (1+x)^m$, for $x \ge 0$ and $0 \le i \le m$, we obtain

$$D_n((1+e_1)^m, x) = \sum_{i=0}^{m} \binom{m}{i} D_n(e_i, x) \le \gamma_m \sum_{i=0}^{m} \binom{m}{i} (1+x)^i \le 2^m \gamma_m (1+x)^m.$$

This completes the proof. □

Corollary 2.1 (i) *If $n \in \mathbb{N}$, $m > 1$, and $1 \le i \le m-1$, then*

$$\lim_{x\to\infty} \varrho_m(x) D_n(e_i, x) = 0.$$

Moreover, for each $m \in \mathbb{N}_0$, one has

$$\lim_{x\to\infty} \varrho_m(x) D_n(e_m, x) = 1 = \lim_{x\to\infty} \varrho_m(x) D_n(1/\varrho_m, x). \tag{9}$$

Proof. (i) It follows from Lemma 2.3 because

$$\varrho_m(x) D_n(e_i, x) \le C_i \frac{(1+x)^i}{(1+x)^m} \le C_i \frac{(1+x)^{m-1}}{(1+x)^m}.$$

For $m = 0$ equation (9) is trivial.

For $m = 1$ equation (9) was verified in Lemma 2.2.

If $m > 1$, taking into account Proposition 2.2 and Lemma 2.2 we obtain

$$\lim_{x\to\infty} \varrho_m(x) D_n(e_m, x) = \lim_{x\to\infty} \varrho_m(x) \Big(u_n^m(x) - \sum_{i=1}^{m-1} \frac{s(j,i)}{n^{m-i}} D_n(e_i, x) \Big) = 1.$$

Finally

$$\lim_{x\to\infty} \varrho_m(x) D_n(1/\varrho_m, x) = \lim_{x\to\infty} \varrho_m(x) D_n((1+e_1)^m, x)$$
$$= \lim_{x\to\infty} \varrho_m(x) \sum_{j=0}^{m} \binom{m}{j} D_n(e_j, x) = \lim_{x\to\infty} \varrho_m(x) D_n(e_m, x) = 1. \quad \square$$

Corollary 2.2 *If $f \in C_m(I)$, then $D_n((e_1 - x)f, x) \in C(I)$. Moreover, there exists a constant C such that, for each $n \in \mathbb{N}$ and $f \in C_m(I)$, one has*

$$\mid D_n((e_1 - x)f, x) \mid \le C_2 \, \|f\|_m \, (1+x)^m \, \frac{\sqrt{x}}{\sqrt{n}}.$$

Proof. We will verify that the series $D_n((e_1 - x)f, x)$ converges absolutely. In fact, it follows from Lemmas 2.3 and 2.1 that

$$\mid D_n((e_1 - x)f, x) \mid \le \|f\|_m \, D_n\Big(\mid e_1 - x \mid (1+e_1)^m, x \Big)$$
$$\le \sqrt{M_{2m}} \, \|f\|_m \, (1+x)^m \, \sqrt{D_n((e_1 - x)^2, x)} \le C \|f\|_m \, (1+x)^m \, \frac{\sqrt{x}}{\sqrt{n}}. \quad \square$$

3 D_n as a bounded endomorphism

Proposition 3.1 *For each $m \in \mathbb{N}_0$, there exists a constant M_m such that, for each $f \in C_m(I)$ and $n \in \mathbb{N}$,*

$$\|D_n(f)\|_m \leq M_m \|f\|_m,$$

where M_m is the constant in Lemma 2.3. *Moreover*

$$\lim_{x\to\infty} \varrho_m(x) D_n(f,x) = A_m(f),$$

where $A_m(f)$ is given as in (3).

In particular, for each $n \in \mathbb{N}$, $D_n : C_m(I) \to C_m(I)$.

Proof. If $f \in C_m(I)$, it follows from Lemma 2.3 that, for $n \in \mathbb{N}$ and $x \in I$,

$$\varrho_m(x) \mid D_n(f,x) \mid \leq \|f\|_m \, \varrho_m(x) \, D_n((1+e_1)^m, x) \leq M_m \|f\|_m.$$

Let us verify that the limits $\lim_{x\to\infty} \varrho_m(x) D_n(f,x)$ exists.

From (9) we know that there exists $N_1 > 0$ such that, for $x > N_1$,

$$\varrho_m(x) D_n(1/\varrho_m, x) \leq 2.$$

If $f \in C_m(I)$, set

$$B = 1 + \mid A_m(f) \mid + \|f\|_m.$$

Since

$$x = x - u_n(x) + u_n(x) \leq \frac{K_2}{n} + u_n(x),$$

we know that $\lim_{x\to\infty} u_n(x) = \infty$. Hence, there exists $N_2 > N_1$ such that, for $x > N_2$, $nu_n(x) > 1$.

Given $\varepsilon > 0$, there exists $N_3 > N_2$ such that, for $x > N_3$,

$$\mid \varrho_m(x) f(x) - A_m(f) \mid < \frac{\varepsilon}{6}.$$

Since $k/n \to \infty$, as $k \to \infty$, there exists $q \in \mathbb{N}$ such that $k/n > N_3$, for every $k > q$. Hence

$$\mid \varrho_m(k/n) f(k/n) - A_m(f) \mid < \frac{\varepsilon}{6}.$$

Notice that, if $x > N_3$, then

$$e^{-nu_n(x)} \sum_{k=0}^{q} \frac{(nu_n(x))^k}{k!} \left(1 + \frac{k}{n}\right)^m \leq e^{-nu_n(x)} + \left(1 + \frac{q}{n}\right)^m e^{-nu_n(x)} (nu_n(x))^q e.$$

Taking into account that (q is fixed)

$$\lim_{x\to\infty} \frac{(nu_n(x))^q}{e^{nu_n(x)}} = \lim_{y\to\infty} \frac{y^q}{e^y} = 0,$$

there exists $N_4 > N_3$ such that, for $x > N_4$,

$$e^{-nu_n(x)} \sum_{k=0}^{q} \frac{(nu_n(x))^k}{k!} \Big(1 + \frac{k}{n}\Big)^m \le \frac{\varepsilon}{2B}.$$

Therefore, if $x > N_4$, then

$$\begin{aligned}
&\varrho_m(x) \mid D_n(f,x) - A_m(f) D_n(1/\varrho_m, x) \mid \\
&\quad = \varrho_m(x) \left| \sum_{k=0}^{\infty} p_{n,k}(x) \frac{(\varrho_m(k/n) f(k/n) - A_m(f))}{\varrho_m(k/n)} \right| \\
&\quad \le \varrho_m(x) \left| \sum_{k=0}^{q} p_{n,k}(x) \frac{(\varrho_m(k/n) f(k/n) - A_m(f))}{\varrho_m(k/n)} \right| \\
&\qquad\qquad + \varrho_m(x) \left| \sum_{k=q+1}^{\infty} p_{n.k}(x) \frac{(\varrho_m(k/n) f(k/n) - A_m(f))}{\varrho_m(k/n)} \right| \\
&\quad \le \Big(\|f\|_m + \mid A_m(f) \mid \Big) \varrho_m(x) e^{-nu_n(x)} \sum_{k=0}^{q} \frac{(nu_n(x))^k}{k!} \Big(1 + \frac{k}{n}\Big)^m \\
&\qquad\qquad + \frac{\varepsilon}{6} \, \varrho_m(x) \sum_{k=q+1}^{\infty} p_{n,k}(x) \Big(1 + \frac{k}{n}\Big)^m \\
&\quad \le \frac{\varepsilon}{2} + \frac{\varepsilon}{6} \varrho_m(x) D_n(1/\varrho_m, x) < \varepsilon.
\end{aligned}$$

We have proved that

$$\lim_{x\to\infty} \Big(\varrho_m(x) D_n(f,x) - A_m(f) D_n(1/\varrho_m, x) \Big) = 0.$$

Finally

$$\begin{aligned}
&\lim_{x\to\infty} \varrho_m(x) D_n(f,x) \\
&\quad = \lim_{x\to\infty} \varrho_m(x) \Big(D_n(f,x) - A_m(f) D_n(1/\varrho_m, x) + A_m(f) D_n(1/\varrho_m, x) \Big) \\
&\quad = A_m(f) \lim_{x\to\infty} \varrho_m(x) D_n(1/\varrho_m, x) = A_m(f). \qquad \square
\end{aligned}$$

Remark 3.1 Later we will need a modification of the first inequality in Proposition 3.1. If $f \in C_m(I)$, then

$$\begin{aligned}
\varrho_m(x) \Big| \sum_{k=0}^{\infty} p_{n,k}(x) f\Big(\frac{k+1}{n}\Big) \Big| &\le \|f\|_m \, \varrho_m(x) \sum_{k=0}^{\infty} p_{n,k}(x) \Big(1 + \frac{k+1}{n}\Big)^m \\
&\le 2^m \|f\|_m \, \varrho_m(x) \sum_{k=0}^{\infty} p_{n,k}(x) \Big(1 + \frac{k}{n}\Big)^m \le 2^m \, M_m \, \|f\|_m.
\end{aligned}$$

4 Some auxiliary inequalities

Recall that $\varphi(x) = \sqrt{x}$.

For $m \in \mathbb{N}_0$ let $C^1_m(I)$ be the family of all absolutely continuous functions $F \in C_m(I)$ such that $F' \in C(I)$ and

$$\|\varphi F'\|_m = \sup_{x\in I} \mid \varrho_m(x)\sqrt{x}\, F'(x) \mid < \infty.$$

Lemma 4.1 *Suppose that $m \in \mathbb{N}_0$ and $F \in C^1_m(I)$.*

(i) *If $x, y \in I$, and $x > 0$, then*

$$\mid F(x) - F(y) \mid \leq 2\frac{\|\varphi F'\|_m}{\sqrt{x}} \mid x - y \mid \Big((1 + x)^m + (1 + y)^m\Big).$$

(ii) *There exists a constant C such that, for each $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$, one has*

$$\Big|F\Big(\frac{k+1}{n}\Big) - F\Big(\frac{k}{n}\Big)\Big| \leq C\|\varphi F'\|_m \frac{(1 + k/n)^m}{\sqrt{n}} \frac{1}{\sqrt{k+1}}.$$

Proof. (i) If $0 < x < y$,

$$\sqrt{x} \mid F(x) - F(y) \mid \leq \sqrt{x} \int_0^{y-x} \mid F'(s + x) \mid ds \leq \int_0^{y-x} \sqrt{x+s} \mid F'(s + x) \mid ds$$
$$= \int_0^{y-x} (1 + x + s)^m \frac{\sqrt{x+s} \mid F'(s + x) \mid}{(1 + x + s)^m} ds \leq \|\varphi F'\|_m \int_0^{y-x} (1 + x + s)^m ds$$
$$\leq \|\varphi F'\|_m (1 + y)^m \,(y - x).$$

If $0 \leq y < x$, then

$$\mid F(x) - F(y)| \leq \int_y^x \frac{\sqrt{s} \mid F'(s) \mid ds}{\sqrt{s}} \leq \int_y^x (1 + s)^m \frac{\|\varphi F'\|_m ds}{\sqrt{s}}$$
$$\leq (1 + x)^m \|\varphi F'\|_m \int_y^x \frac{ds}{\sqrt{s}} = 2(1 + x)^m \|\varphi F'\|_m (\sqrt{x} - \sqrt{y})$$
$$= 2(1 + x)^m \|\varphi F'\|_m \frac{x - y}{\sqrt{x} + \sqrt{y}} \leq \frac{2}{\sqrt{x}}(1 + x)^m \|\varphi F'\|_m (x - y).$$

(ii) If $k \in \mathbb{N}_0$, then

$$\mid F\Big(\frac{k+1}{n}\Big) - F\Big(\frac{k}{n}\Big) \mid \leq \int_{k/n}^{(k+1)/n} \mid F'(s) \mid ds$$
$$= \int_{k/n}^{(k+1)/n} (1 + s)^m \frac{\mid \sqrt{s} F'(s) \mid}{(1 + s)^m} \frac{ds}{\sqrt{s}} \leq \|\varphi F'\|_m (1 + (k + 1)/n))^m \int_{k/n}^{(k+1)/n} \frac{ds}{\sqrt{s}}$$
$$\leq 2^{m+1} \|\varphi F'\|_m (1 + k/n)^m \Big(\sqrt{(k + 1)/n} - \sqrt{k/n}\Big)$$

$$\leq 2^{m+1}\|\varphi F'\|_m \frac{(1+k/n)^m}{n} \frac{1}{\sqrt{(k+1)/n}+\sqrt{k/n}}$$

$$\leq C\|\varphi F'\|_m \frac{(1+k/n)^m}{\sqrt{n}} \frac{1}{\sqrt{k+1}}. \qquad \square$$

Lemma 4.2 *For each $x > 0$ and $n \in \mathbb{N}$, one has*

$$\sum_{k=0}^{\infty} \frac{p_{n,k}(x)}{k+1} \leq \frac{1}{nu_n(x)}.$$

Proof. It follows from the relations

$$nu_n(x)\sum_{k=0}^{\infty} \frac{p_{n,k}(x)}{k+1} = e^{-nu_n(x)} \sum_{k=0}^{\infty} \frac{(nu_n(x))^{k+1}}{(k+1)!} \leq 1. \qquad \square$$

5 The upper estimate

Proposition 5.1 *Assume $m \in \mathbb{N}_0$ and $\varphi(x) = \sqrt{x}$. There exists a constant C such that, for each $n \in \mathbb{N}$ and $F \in C^1_m(I)$, one has*

$$\|D_n(F) - F\|_m \leq C\frac{\|\varphi F'\|_m}{\sqrt{n}}.$$

Proof. If $x = 0$, taking into account Proposition 2.1 one has $D_n(F, 0) = F(0)$. If $x > 0$, it follows from Lemma 4.1 and Hölder inequality that

$$\Big|\sum_{k=0}^{\infty} p_{n,k}(x)\Big(F\Big(\frac{k}{n}\Big) - F(x)\Big)\Big|$$

$$\leq 2\frac{\|\varphi F'\|_m}{\sqrt{x}} \sum_{k=0}^{\infty} p_{n,k}(x)\Big|\frac{k}{n} - x\Big|\Big((1+x)^m + \Big(1+\frac{k}{n}\Big)^m\Big)$$

$$\leq 2\frac{\|\varphi F'\|_m}{\sqrt{x}}\Big\{(1+x)^m + \sqrt{D_n((1+e_1)^{2m}, x)}\Big\}\sqrt{D_n((e_1-x)^2, x)}$$

(in the case $m = 0$ we do not need Hölder inequality).

If $m > 0$, then (see Lemma 2.3)

$$\sqrt{D_n((1+e_1)^{2m}, x)} \leq \sqrt{M_{2m}}(1+x)^m.$$

Therefore, taking into account Lemma 2.1, one has

$$| D_n(F,x) - F(x) | \leq C\frac{\|\varphi F'\|_m}{\sqrt{x}} \frac{\sqrt{x}}{\sqrt{n}}(1+x)^m. \qquad \square$$

For $f \in C_m(I)$ and $t \geq 0$ define

$$K^{\varphi}(f,t)_m = \inf_{F\in C_m^1(I)} \Big\{\|f-F\|_m + t\|\varphi F'\|_m\Big\}. \tag{10}$$

It is known that $\lim_{t\to 0+}\omega^{\varphi}(f,t)_m = 0$, for each $f \in C_m(I)$. For this result the condition $\lim_{x\to\infty} \varrho_m(x)f(x) = A_m(f)$ is needed (see [3, p. 36-37]). Moreover, there exist positive constants C_1, C_2 and t_0 such that, for every $t \in (0,t_0)$ and each $f \in C_{m,\infty}(I)$,

$$C_1\omega^{\varphi}(f,t)_m \leq K^{\varphi}(f,t)_m \leq C_2\omega^{\varphi}(f,t)_m. \tag{11}$$

the second inequality holds for every $t \in (0,1]$.

Now we prove the main result.

Proof of Theorem 1.2. It follows from standard arguments. If $f \in C_m(I)$ and $F \in C_m^1(I)$, then

$$\|D_n(f)-f\|_m \leq \|D_n(f-F)-(f-F)\|_m + \|D_n(F)-F\|_m.$$

Taking into account Proposition 3.1 and Proposition 5.1, there exists a constant C_1 such that

$$\|D_n(f)-f\|_m \leq (1+M_m)\|f-F\|_m + C_1\frac{\|\varphi F'\|_m}{\sqrt{n}}.$$

Therefore, there exists a exists a constant C_2 such that

$$\begin{aligned}\|D_n(f)-f\|_m &\leq C_2 \inf_{F\in C_m^1(I)}\Big\{\|f-F\|_m + \frac{1}{\sqrt{n}}\|\varphi F'\|_m\Big\}\\ &= C_2K^{\varphi}\Big(f,\frac{1}{\sqrt{n}}\Big)_m \leq C_3\omega^{\varphi}\Big(f,\frac{1}{\sqrt{n}}\Big)_m.\end{aligned}$$

□

6 Characterization of Lipschitz classes

Lemma 6.1 *If $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $f \in C_{m,\infty}(I)$, then*

$$D_n'(f,x) = \frac{nu_n'(x)}{e^{nu_n(x)}}\sum_{k=0}^{\infty}\frac{(nu_n(x))^k}{k!}\Big(f\Big(\frac{k+1}{n}\Big)-f\Big(\frac{k}{n}\Big)\Big) \tag{12}$$

$$= \frac{nu_n'(x)}{u_n(x)}\Big(D_n((e_1-x)f,x)+(x-u_n(x))D_n(f,x)\Big). \tag{13}$$

Proof. We can assume that $f(0)=0$.

If $k \in \mathbb{N}$, then

$$p_{n,k}'(x) = -nu_n'(x)p_{n,k} + e^{-nu_n(x)}\frac{n^k}{k!}ku_n'(x)u_n^{k-1}(x)$$

$$= nu_n'(x)\Big(- p_{n,k} + \frac{1}{u_n(x)}\frac{k}{n} p_{n,k}(x)\Big).$$

Therefore

$$D_n'(f,x) = \frac{nu_n'(x)}{e^{nu_n(x)}}\Big(- \sum_{k=1}^{\infty} \frac{(nu_n(x))^k}{k!} f\Big(\frac{k}{n}\Big) + \sum_{k=1}^{\infty} \frac{(nu_n(x))^{k-1}}{(k-1)!} f\Big(\frac{k}{n}\Big)\Big)$$

$$= \frac{nu_n'(x)}{e^{nu_n(x)}} \sum_{k=0}^{\infty} \frac{(nu_n(x))^k}{k!} \Big(f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\Big)$$

and

$$D_n'(f,x) = \frac{nu_n'(x)}{u_n(x)}\Big(D_n(e_1 f, x) - u_n(x) D_n(f,x)\Big).$$

$$= \frac{nu_n'(x)}{u_n(x)}\Big(D_n((e_1 - x) f, x) + x D_n(f,x) - u_n(x) D_n(f,x)\Big). \qquad \square$$

Proposition 6.1 *Assume $m \in \mathbb{N}_0$.*

(i) *For each $n \in \mathbb{N}$ and $f \in C_m(I)$, one has $D_n'(f) \in C_m(I)$.*

(ii) *There exists a constant C such that, for each $f \in C_m(I)$ and $n \in \mathbb{N}$,*

$$\|\varphi\, D_n'(f)\|_m \le C\sqrt{n}\, \|f\|_m.$$

En particular $D_n(f) \in C_m^1(I)$.

Proof. (i) We will verify that $D_n(f)$ is absolutely continuous, $D_n'(f) \in C(I)$, $\sup_{x\in I} \varrho_m(x) \mid D_n'(f,x) \mid < \infty$ and

$$\lim_{x\to\infty} \varrho_m(x) D_n'(f,x) = 0.$$

We know that $D_n(f) \in C_m(I)$ (see Proposition 3.1). From Corollary 2.2 and (13) we deduce that $D_n'(f) \in C(I)$.

Since the sequence $\{u_n'(x)\}$ is uniformly bounded, taking into account (12), Proposition 3.1 and Remark 3.1, we obtain

$$\varrho_m(x) \mid D_n'(f,x) \mid = \Big| nu_n'(x) \sum_{k=0}^{\infty} p_{n,k}(x) \Big(f\Big(\frac{k+1}{n}\Big) - f\Big(\frac{k}{n}\Big)\Big)\Big| \le Cn\|f\|_m.$$

Hence $D_n'(f,x) \in C_m(I)$.

aaa

$$\mid D_n((e_1 - x) f, x) \mid \le C_2 \|f\|_{m,\infty} (1+x)^m \frac{\sqrt{x}}{\sqrt{n}}.$$

On the other hand, taking into account (13), Lemma 2.2, (5), and Proposition 3.1, we know that

$$\varrho_m(x) \mid D_n'(f,x) \mid \le C_1 \frac{n \varrho_m(x)}{u_n(x)} \Big(\mid D_n((e_1 - x) f, x) \mid + \mid x - u_n(x) \mid\mid D_n(f,x) \mid \Big)$$

$$\leq C_2 \frac{n}{u_n(x)}\Big(\frac{\sqrt{x}}{\sqrt{n}}+\frac{1}{n}\Big)\|f\|_m \leq C_3\Big(\frac{\sqrt{n}}{\sqrt{u_n(x)}}+\frac{1}{u_n(x)}\Big)\|f\|_m.$$

Since $\lim_{x\to\infty} u_n(x)=\infty$, one has

$$\lim_{x\to\infty} \varrho_m(x) D'_n(f,x)=0.$$

We have proved that $D'_n(f)\in C_m(I)$.

(ii) If $nx\geq 1$, then (recall that $x\leq K_1 u_n(x)$)

$$\sqrt{x}\varrho_m(x) \mid D'_n(f,x) \mid\leq C_2 \frac{n\sqrt{x}}{u_n(x)}\Big(\frac{\sqrt{x}}{\sqrt{n}}+\frac{1}{n}\Big)\|f\|_m \leq C_3\Big(\sqrt{n}+\frac{\sqrt{x}}{u_n(x)}\Big)\|f\|_m$$

$$= C_3\Big(\sqrt{n}+\frac{1}{\sqrt{n}}\frac{\sqrt{nx}}{u_n(x)}\Big)\|f\|_m \leq C_3\Big(\sqrt{n}+\frac{1}{\sqrt{n}}\frac{nx}{u_n(x)}\Big)\|f\|_m \leq C_4\sqrt{n}\|f\|_m.$$

But, if $nx\leq 1$ we use (12) to obtain

$$\varrho_m(x)\sqrt{x} \mid D'_n(f,x) \mid\leq C\varrho_m(x)\sqrt{x}\, n\sum_{k=0}^{\infty} p_{n,k}(x)\Big|f\Big(\frac{k+1}{n}\Big)-f\Big(\frac{k}{n}\Big)\Big|$$

$$\leq C\sqrt{x}\, n\,\|f\|_m \leq C\sqrt{n}\|f\|_m.$$

□

Proposition 6.2 *If $m\in\mathbb{N}_0$, there exists a constant C_m such that, for every $F\in C^1_m(I)$, $n\in\mathbb{N}$*

$$\|\varphi D'_n(F)\|_m \leq C_m\|\varphi F'\|_m.$$

Proof. Since $F(0)\in C^1_{m,\infty}(I)$, $D'_n(F-F(0),x)=D'_n(F,x)$, and $(F(x)-F(0))'=F'(x)$, we can assume that $F(0)=0$.

We use equation (12).

If $F\in C^1_{m,\infty}(I)$, as a consequence of Lemmas 4.1 and 4.2 we obtain

$$\sqrt{x} \mid D'_n(F,x) \mid\leq C_1\sqrt{nx}\|\varphi F'\|_m \sum_{k=0}^{\infty} p_{n,k}(x)\frac{(1+k/n)^m}{\sqrt{k+1}}$$

$$\leq C_1\sqrt{nx}\,\|\varphi F'\|_m\sqrt{D_n((1+e_1)^{2m},x)}\Big(\sum_{k=0}^{\infty}\frac{p_{n,k}(x)}{k+1}\Big)^{1/2}$$

$$\leq C_2\sqrt{nx}\,\|\varphi F'\|_m\frac{(1+x)^m}{\sqrt{nu_n(x)}} = C_2\|\varphi F'\|_m\frac{\sqrt{x}}{\sqrt{u_n(x)}}(1+x)^m$$

and it is sufficient to prove the result. □

We need the following result.

Lemma 6.2 (see [1, p. 100]) *Fix $\alpha\in(0,1)$, $c>0$, and let $\Omega:[0,c]\to\mathbb{R}$ be a monotone increasing function. If there exists a constant M such that, for every $h,t\in[0,c]$,*

$$\Omega(h)\leq M\Big(t^\alpha+(h/t)\Omega(t)\Big),$$

then there exists a constant C such that $\Omega(h)\leq Ch^\alpha$.

Proof of Theorem 1.3. (i) $\Rightarrow$ (ii). For $h, t \in (0,1]$, fix $n \in \mathbb{N}$ such that

$$\frac{1}{n} \le t^2 \le \frac{2}{n}. \tag{14}$$

Recall that $D_n(f) \in C^1_{m,\infty}(I)$ (see Proposition 6.1).

For each $F \in C^1_{m,\infty}(I)$, taking into account Propositions 3.1, 6.1 and 6.2, we obtain

$$\begin{aligned} K^{\varphi}(f,h)_m &\le \|f - D_n(D_n(f))\|_m + h\|\varphi\, D_n'(D_n(f))\|_m \\ &\le \|f - D_n(f)\|_m + \|D_n(f - D_n(f))\|_m \\ &\qquad + h\|\varphi\, D_n'(D_n(f-F))\|_m + h\|\varphi\, D_n'(D_n(F))\|_m \\ &\le (1 + C_m)\|f - D_n(f)\|_m + C_2 h\Big(\sqrt{n}\|D_n(f-F)\|_m + \|\varphi\, D_n'(F)\|_m\Big) \\ &\le (1 + C_m)\|f - D_n(f)\|_m + C_3 h\Big(\sqrt{n}\|f-F\|_m + \|\varphi\, F'\|_m\Big) \\ &\le C_4\Big(\frac{1}{n^{\alpha/2}} + h\sqrt{n}\Big(\|f-F\|_m + \frac{1}{\sqrt{n}}\|\varphi\, F'\|_{m]}\Big). \end{aligned}$$

Taking into account that $F \in C^1_m(I)$ is arbitrary and (14), one has

$$\begin{aligned} K^{\varphi}(f,h)_m &\le C_4\Big(\frac{1}{n^{\alpha/2}} + h\sqrt{n}K^{\varphi}\Big(f, \frac{1}{\sqrt{n}}\Big)_m\Big) \\ &\le C_4\Big(t^{\alpha} + 2\frac{h}{t}K^{\varphi}(f,t)_m\Big). \end{aligned}$$

From Lemma 6.2 we know that $K^{\varphi}(f,h)_m \le Ch^{\alpha}$, $h \in [0,1]$ and (ii) is a consequence of (11).

(ii) $\Rightarrow$ (i) It follows from Theorem 1.2. □

7 Conclusions

In Theorem 1.2 we verified the operators D_n can be use to approximate every function $f \in C_m(I)$. However, there are still some open problems related to these operators. For example, it would be interesting to obtain a Voronovskaya-type theorem.

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