General Mathematics Vol. 32, No. 2 (2024), 84–97

DOI: 10.2478/gm-2024-0015

sciendo

Exponential Bernstein-Durrmeyer operators [1]

Ştefan-Lucian Garoiu

Abstract

In this paper we introduce an exponential variant of Bernstein-Durrmeyer operators. Regarding this new operators we obtain some convergence results, a Voronovskaya type theorem and some quantitative estimates using the first order modulus of continuity and the second order modulus of smoothness and then the relation between them and K-functionals. Also we study their simultaneous approximation properties.

2010 Mathematics Subject Classification: 41A10, 41A25, 41A35, 41A36, 41A52, 41A81

Key words and phrases: Durrmeyer operators, exponential operators, K-functionals, moduli of smoothness

1 Introduction

Recently, a lot of studies regarding exponential variants of classical approximation operators have been obtained. Among these we mention the exponential variant of: Bernstein operators (see [5]), Szasz-Mirakjan operators (see [1], [2]) and also in ([4]) the authors introduced an exponential version of Kantorovich operators. Namely, they introduced the operators:

$$(1)\qquad K_n f(x) := \sum_{k=0}^{n} e^{\mu x} p_{n,k}(a_{n+1}(x))(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)e^{-\mu t}dt,\ x\in[0,1],\ n\in\mathbb{N},$$

[1] *Received 3 December, 2024*

Accepted for publication (in revised form) 31 December, 2024

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ 0 \le k \le n, \tag{2}$$

and f is a continuous function on $[0,1]$, $a_n(x) := \frac{e^{\frac{\mu x}{n}}-1}{e^{\frac{\mu}{n}}-1}$ and $\mu > 0$. The authors proved that these operators preserve exponential function $exp_\mu(x) = e^{\mu x}$ and a lot more properties of these operators. Also another version of exponential Kantorovich operators was introduced in [6] and more results concerning approximation by exponential type operators can be seen in [3] and [12].

Other important operators in Approximation Theory are Bernstein-Durrmeyer operators (which were introduced by Durrmeyer in [11] and independently by A. Lupaș in [14] with their properties later studied in [7], [8], [9], [10], [16] and many others), which are of the form:

$$M_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}\,(x) \int_0^1 p_{n,k}\,(t)\, f\,(t)\, dt,\ f \in L^\infty[0,1]. \tag{3}$$

The aim of our paper will be to introduce an exponential variant of operators in (3) as follows:

$$M_n^* f(x) = (n+1) e^{\mu x} \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) e^{-\mu t} dt, \tag{4}$$

with $f \in C([0,1]),\ x \in [0,1],\ n \in \mathbb{N}$, and where $\mu > 0$ is a real parameter.

First, we have the following simple remark, which states that our operators preserve exponential functions $\exp_\mu(x) = e^{\mu x}$, $x \in [0,1]$.

Remark 1 *For $x \in [0,1]$ and $n \in \mathbb{N}$ we have that:*

$$M_n^* exp_\mu(x) = (n+1) e^{\mu x} \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) dt = e^{\mu x} \sum_{k=0}^{n} p_{n,k}(x) = e^{\mu x}.$$

Next, the following identity holds.

Remark 2 *For $f \in C([0,1]),\ n \in \mathbb{N},\ x \in [0,1]$ we have*

$$M_n^\star f(x) = exp_\mu(x) M_n(f \exp_{-\mu})(x). \tag{5}$$

Note that our operators are constructed in a similar fashion as the operators in (1), however here we won't need the modification of the argument of the Bernstein basis polynomials $p_{n,k}$.

Let

$$\omega_1(f,\delta) = \sup\{|f(t) - f(x)|,\ |t-x| < \delta\ x,t \in [0,1],\ \delta > 0\}, \tag{6}$$

be the first order modulus of continuity. For $f \in C^1([0,1])$ by the second order modulus of smoothness we mean

$$\omega_2(f,\delta) = \sup_{h\in[0,\delta]} \sup_{x\in[0,1-\frac{h}{2}]} \left|\Delta_h^2(f,x)\right|, \tag{7}$$

where $\Delta_h^2(f,x) = f(x) - 2f(x+h) + f(x+2h)$ is the second order finite difference of f.

Using the above moduli we will provide some quantitative estimates of the degree of convergence of our operators. Another estimation of the degree of approximation by our operators will be given using the well-known Peetre's K - functionals

$$\mathcal{K}_j(f,\delta) = \inf_{g\in C^j[0,1]} \{\|f-g\|_\infty + \delta^j \|g^{(j)}\|_\infty\},\ f\in C([0,1]),\ \delta>0,\ j=1,2. \tag{8}$$

and the following relation between $\mathcal{K}_j$ and ω_j:

$$\mathcal{K}_j(f,\delta) \le C\omega_j(f,\delta),\ j=1,\ 2$$

where C is a constant depending only on j (for more details see [13]). Here by $C^j([0,1])$ we mean the space of functions having continuous j^{th} derivative on $[0,1]$ for $j=1,\ 2$.

Our paper will be divided in five sections. In Section 2 we will prove that our operators satisfy the famous Korovkin's theorem conditions and also that they converge to functions belonging to a weighted version of L^p spaces. Next, in Section 3 we will obtain a Voronovskaya type theorem. In Section 4 we will provide some quantitative estimates of the degree of approximation by our operators and finally in Section 5 we will prove a simultaneous approximation result.

2 Convergence results

Let $\gamma_i(x) = x^i \exp_\mu(x)$, $i=0,\ 1,\ 2$, $x\in[0,1]$. Then in view of the following Lemma we can consider functions γ_i, $i=0,\ 1,\ 2$, as the test functions which will be used when verifying the conditions of Korovkin theorem for operators M_n^*.

Lemma 1 *The set $\{\gamma_i(x)|\ i=0,\ 1,\ 2,\dots,\ x\in[0,1]\}$ is a Chebyshev system.*

Proof. Let us take $x_0,\ x_1,\ x_2 \in [0,1]$ with $x_0 \ne x_1 \ne x_2$. Since the determinant of $\{\gamma_0(x_0), \gamma_1(x_1),\ \gamma_2(x_2)\}$ is a Vandermonde determinant multiplied by $\exp_\mu(x_0 + x_1 + x_2)$ which is not zero, then this set is a Chebyshev set.

In what follows, by $C([0,1])$ we mean the space of continuous functions on $[0,1]$. For test functions $\gamma_0,\ \gamma_1,\ \gamma_2$ the following result holds.

Lemma 2 *The following identities are true:*

$$M_n^*\gamma_0(x) = \gamma_0(x) = exp_\mu(x), \tag{9}$$

$$M_n^*\gamma_1(x) = \frac{n}{n+2}\gamma_1(x) + \frac{1}{n+2}\gamma_0(x) \tag{10}$$

$$M_n^*\gamma_2(x) = \frac{n(n-1)}{(n+2)(n+3)}\gamma_2(x) + \frac{n}{(n+2)(n+3)}\gamma_1(x) + \frac{2}{(n+2)(n+3)}\gamma_0(x) \tag{11}$$

Proof. We have that $M_n^*\gamma_i(x) = exp_\mu(x)M_n e_i(x)$, $i = 0,\ 1,\ 2$ where $e_i(x) = x^i$, $i = 0,\ 1,\ 2$ so from the expression of $M_n e_i(x)$, $i = 0,\ 1,\ 2$ (see [9]) our results hold.

Now we can prove our convergence result.

Theorem 1 *Let $f \in C([0,1])$. Then $M_n^* f$ converges uniformly to f on $[0,1]$.*

Proof. From (9), (10) and (11) we have that $M_n^*\gamma_i(x) \to \gamma_i(x)$, $i = 0,\ 1,\ 2$ as $n \to \infty$. Also, since γ_0, γ_1 and γ_2 form a Chebyshev set, by Korovkin theorem we have that $M_n^* f$ converges uniformly to f on $[0,1]$.

Further, we will prove that our operators approximate functions belonging to L^p spaces. However, since our operators are defined as in (4) it will be in hand to work with a weighted version of L^p spaces, namely we will prove the convergence of operators M_n^* in $L^p_\mu([0,1])$. Having this in mind, we say that a function $f : [0,1] \to \mathbb{R}$ belongs to $L^p_\mu([0,1])$ if $f \in L^p([0,1])$ and

$$\|f\|_{p,\mu} = \left\{ \int_0^1 \left|e^{-\mu t} f(t)\right|^p dt \right\}^{\frac{1}{p}} < \infty. \tag{12}$$

Even though, if $f \in L^p_\mu([0,1])$ then $f \in L^p([0,1])$ (and conversely), our choice of the space is motivated by the definition of operators M_n^*.

Theorem 2 *Let $f \in L^p_\mu([0,1])$. Then:*

$$\|M_n^* f\|_{p,\mu} \le \|f\|_{p,\mu}. \tag{13}$$

Moreover,

$$\lim_{n\to\infty} \|M_n^* f - f\|_{p,\mu} = 0. \tag{14}$$

Proof. Let $f \in L^p_\mu([0,1])$. From Jensen inequality we have that:

$$
\begin{aligned}
\|M_n^* f\|_{p,\mu}^p &= \int_0^1 \left| (n+1) \sum_{k=0}^n p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) e^{-\mu t} dt \right|^p dx \\
&\le \int_0^1 \sum_{k=0}^n p_{n,k}(x)(n+1) \left(\int_0^1 p_{n,k}(t) \left| f(t) \right|^p e^{-p\mu t} dt \right) dx \\
&= \sum_{k=0}^n \int_0^1 p_{n,k}(t) \left| f(t) \right|^p e^{-p\mu t} dt \int_0^1 p_{n,k}(x)(n+1) dx \\
&= \sum_{k=0}^n \int_0^1 p_{n,k}(t) \left| f(t) \right|^p e^{-p\mu t} dt \\
&= \int_0^1 \sum_{k=0}^n p_{n,k}(t) \left| f(t) \right|^p e^{-p\mu t} dt \\
&= \|f\|_{p,\mu}^p,
\end{aligned} \tag{15}
$$

which proves (13).

Now, in order to prove (14) since $C([0,1])$ is dense in $L^p([0,1])$ then if $f \in L^p_\mu([0,1])$, for $\varepsilon > 0$, there is $g \in C([0,1])$ such that $\|f - g\|_p \le \frac{\varepsilon}{4}$. So,

$$
\begin{aligned}
\|f - g\|_{p,\mu} &= \left\{ \int_0^1 \left| e^{-\mu t} \left[f(t) - g(t) \right] \right|^p dt \right\}^{\frac{1}{p}} \\
&\le \left\{ \int_0^1 \left| f(t) - g(t) \right|^p dt \right\}^{\frac{1}{p}} = \|f - g\|_p \le \frac{\varepsilon}{4}.
\end{aligned} \tag{16}
$$

Next, we have that

$$
\|M_n^* f - f\|_{p,\mu} \le \|M_n^* f - M_n^* g\|_{p,\mu} + \|M_n^* g - g\|_{p,\mu} + \|g - f\|_{p,\mu}. \tag{17}
$$

Since:

$$
\|M_n^* g - g\|_{p,\mu} \le \|M_n^* g - g\|_p \le \|M_n^* g - g\|_\infty,
$$

and from Theorem 1 there exists $n_\varepsilon \in \mathbb{N}$ such that $n \ge n_\varepsilon \in \mathbb{N}$, $\forall\ n \in \mathbb{N}$, so, it follows that:

$$
\|M_n^* g - g\|_{p,\mu} \le \frac{\varepsilon}{2}.
$$

Furthermore, from (13) we have that:

$$
\|M_n^* f - M_n^* g\|_{p,\mu} \le \|M_n^*\|_{p,\mu} \cdot \|g - f\|_{p,\mu} \le \frac{\varepsilon}{4},
$$

hence:

$$
\|M_n^* f - f\|_{p,\mu} \le 2\|g - f\|_{p,\mu} + \|M_n^* g - g\|_{p,\mu} < \varepsilon,
$$

which proves relation (14).

3 Voronovskaya theorem

In this section we will give a Voronovskaya result regarding our operators M_n^*. In what follows by $C^2([0,1])$ we mean the space of continuous functions on $[0,1]$ which admit a second derivative at $x \in (0,1)$.

Theorem 3 *If $f \in C^2([0,1])$ then, for $x \in [0,1]$:*

$$\lim_{n\to\infty} n(M_n^* f - f)(x) = [\mu^2 x(1-x) - \mu(1-2x)]f(x) + [1-2x-2\mu x(1-x)]f'(x) + x(1-x)f''(x). \tag{18}$$

Proof. Let $g \in C^2([0,1])$ be a function such that $g(t) = e^{-\mu t}f(t)$. Using Taylor's formula with integral remainder for function g we have that:

$$g(t) = g(x) + g'(x)(t-x) + \frac{1}{2}g''(x)(t-x)^2 + R(t,x) \tag{19}$$

where $R(t,x) = \int_x^t (t-u)[g''(u) - g''(x)]du$.

Now, applying operators M_n to (19) we get:

$$M_n g(x) = g(x)m_0(x) + g'(x)m_1(x) + \frac{1}{2}g''(x)m_2(x) + M_n(R(\cdot,x))(x), \tag{20}$$

where $m_j(x) = M_n((e_1 - xe_0)^j)(x),\ j = 0,1,2,\dots$ are the moments of Bernstein-Durrmeyer operators.

Since,

$$g'(x) = e^{-\mu x}f'(x) - \mu e^{-\mu x}f(x),$$

and

$$g''(x) = f''(x)e^{-\mu x} - 2\mu f'(x)e^{-\mu x} + \mu^2 f(x)e^{-\mu x},$$

multiplying (20) by $e^{\mu x}$ and using the fact that $(M_n^* f)(x) = e^{\mu x}M_n(f\exp_{-\mu})(x)$ we get:

$$\begin{aligned} M_n^* f(x) &= f(x)m_0(x) + m_1(x)(f'(x) - \mu f(x)) \\ &\quad + \frac{1}{2}m_2(x)(f''(x) - 2\mu f'(x) + \mu^2 f(x)) + e^{\mu x}M_n(R(\cdot,x))(x). \end{aligned} \tag{21}$$

Let us denote $\psi(t) = t(1-t)$. It is well-known that ([9]):

$$m_0(t) = 1,$$

$$m_1(t) = \frac{1}{n+2}\psi'(t),$$

$$m_2(t) = \frac{2n-6}{(n+2)(n+3)}\psi(t) + \frac{2}{(n+2)(n+3)},$$

$$m_3(t) = \frac{12(n-1)}{(n+2)(n+3)(n+4)}\psi'(t)\psi(t) + \frac{6}{(n+2)(n+3)(n+4)}\psi'(t),$$

$$\begin{aligned} m_4(t) &= \frac{12\left(n^2 - 21n + 10\right)}{(n+2)(n+3)(n+4)(n+5)}\psi^2(t) \\ &+ \frac{24(3n-5)}{(n+2)(n+3)(n+4)(n+5)}\psi(t) + \frac{24}{(n+2)(n+3)(n+4)(n+5)}, \end{aligned}$$

so, after multiplying (21) by n , and passing to the limit as $n \to \infty$, we obtain:

$$\lim_{n\to\infty} n(M_n^* f - f)(x) = [\mu^2 x(1-x) - \mu(1-2x)]f(x) + [1 - 2x - 2\mu x(1-x)]f'(x)$$
$$+x(1-x)f''(x) + e^{\mu x} \lim_{n\to\infty} nM_n(R(\cdot,x))(x).$$

Lastly, we will prove that $\lim_{n\to\infty} nM_n(R(\cdot,x))(x) = 0$. In this sense we have:

$$|R(t,x)| \le \int_x^t (t-u)|g''(u) - g''(x)|du \le \int_x^t (t-u)\omega_1(g'', |u-x|)du$$
$$\le \omega_1(g'',\delta) \int_x^t (t-u)\left(1 + \frac{(u-x)^2}{\delta^2}\right) du,$$

where we choose $\delta = \frac{1}{\sqrt{n}}$. Using the identity:

$$(u-x)^2 = (u-t)^2 + 2(u-t)(t-x) + (t-x)^2,$$

in the inequality above we obtain:

$$|R(t,x)| \le \omega_1(g'',\delta)$$
$$\times \left(\int_x^t (t-u)du + \frac{1}{\delta^2}\int_x^t (t-u)[(u-t)^2 + 2(u-t)(t-x) + (t-x)^2]du\right)$$
$$= \omega_1(g'',\delta)\left[\frac{(t-x)^2}{2} + n\frac{(t-x)^4}{12}\right],$$

therefore, we will obtain that:

$$nM_n(|R(\cdot,x)|)(x) \le \omega_1\left(g'', \frac{1}{\sqrt{n}}\right)\left(\frac{1}{2}nm_2(x) + \frac{n^2}{12}m_4(x)\right),$$

but, since the parenthesis above is bounded we have $\lim_{n\to\infty} nM_n(|R(\cdot,x)|)(x) = 0$, hence we get our Voronovskaya type result.

4 Quantitative estimates

In this section we will provide some quantitative estimates of the approximation by operators M_n^*. Namely, we have the following result.

Theorem 4 *For $f \in C([0,1])$, $n \in \mathbb{N}$, $x \in [0,1]$ and $\delta > 0$, we have*

$$|f(x) - M_n^\star f(x)| \le e^{\mu x}\left[1 + \frac{(2n-6)x(1-x)+2}{\delta^2(n+2)(n+3)}\right]\omega_1\left(f \cdot exp_{-\mu},\delta\right). \tag{22}$$

and

$$|f(x) - M_n^\star f(x)| \le e^{\mu x}\left[\frac{|1-2x|}{\delta(n+2)}\omega_1\left(f \cdot exp_{-\mu},\delta\right)\right.$$
$$\left. + \left(1 + \frac{(2n-6)x(1-x)+2}{2\delta^2(n+2)(n+3)}\right)\omega_2\left(f \cdot exp_{-\mu},\delta\right)\right] \tag{23}$$

Consequently, we have

$$\|f - M_n^\star f\| \le \frac{3}{2}e^{\mu} \cdot \omega_1\left(f \cdot exp_{-\mu}, \frac{1}{\sqrt{n}}\right). \tag{24}$$

and

$$\|f - M_n^\star f\| \le e^{\mu} \cdot \left[\frac{1}{\sqrt{n}} \cdot \omega_1\left(f \cdot exp_{-\mu}, \frac{1}{\sqrt{n}}\right) + \frac{5}{4}\omega_2\left(f \cdot exp_{-\mu}, \frac{1}{\sqrt{n}}\right)\right]. \tag{25}$$

Proof. We have

$$|f(x) - M^\star f(x)| = e^{\mu x}|g(x) - M_n g(x)|,$$

where we denoted $g = f\exp_{-\mu}$. Then, for $\delta > 0$, applying the estimate given in [15], since $M_n e_0(x) = 1$, we have

$$|g(x) - M_n g(x)| \le \Big(1 + \delta^{-2}M_n(e_1 - xe_0)^2(x)\Big)\omega_1(g, \delta).$$

Also, using the estimate given in [17] we have

$$\begin{aligned} |g(x) - M_n g(x)| &\le \frac{1}{\delta}|M_n(e_1 - xe_0)(x)|\omega_1(g, \delta) \\ &+\Big(1 + \frac{1}{2\delta^2}M_n(e_1 - xe_0)^2(x)\Big)\omega_2(g, \delta). \end{aligned}$$

Then, using the known relations we get

$$M_n(e_1 - xe_0)(x) = \frac{|1-2x|}{n+2}, \quad M_n(e_1 - xe_0)^2(x) = \frac{(2n-6)x(1-x)+2}{2\delta^2(n+2)(n+3)}.$$

Relations (24) and (25) are obtained from relations (22) and (23) by taking $\delta = \frac{1}{\sqrt{n}}$ and by simple majorizations.

Further, another estimate using ω_1 can be given in the following theorem.

Theorem 5 *Let $f \in C([0,1])$. For $x \in [0,1]$ and $n \in \mathbb{N}$, we have that:*

$$|f(x) - M_n^* f(x)| \le \left(e^{2\mu x} + \frac{1}{2}e^{\mu x}\right)\omega_1\left(f \cdot e^{-\mu \cdot}, \frac{1}{\sqrt{n}}\right). \tag{26}$$

Proof. Let $x \in [0,1]$ and $\delta > 0$. We have that:

$$\begin{aligned} |f(x) - M_n^* f(x)| &\le \left|f(x) - \frac{M_n^*\gamma_0(x)}{\gamma_0(x)}f(x)\right| + \left|\frac{M_n^*\gamma_0(x)}{\gamma_0(x)}f(x) - M_n^* f(x)\right| \\ = \frac{1}{\gamma_0(x)}|f(x)| \cdot |\gamma_0(x) - M_n^*\gamma_0(x)| &+ \frac{1}{\gamma_0(x)}|M_n^*\gamma_0(x)f(x) - \gamma_0(x)M_n^* f(x)| \\ &= \frac{1}{\gamma_0(x)}M_n^*|\gamma_0 f(x) - \gamma_0(x)f|(x). \end{aligned}$$

But, for $t \in [0,1]$ we have

$$
\begin{aligned}
|\gamma_0(t)f(x) - \gamma_0(x)f(t)| = |e^{\mu t}f(x) - e^{\mu x}f(t)| &\leq e^{\mu(t+x)}\left|\frac{f(x)}{e^{\mu x}} - \frac{f(t)}{e^{\mu t}}\right| \\
&\leq exp_\mu(\cdot + x)\omega_1(f\cdot exp_{-\mu}, |t-x|) \\
&\leq exp_\mu(\cdot + x)\left(1 + \frac{(t-x)^2}{\delta^2}\right)\omega_1(f\cdot exp_{-\mu}, \delta),
\end{aligned}
$$

hence,

$$
\begin{aligned}
M_n^*|\gamma_0 f(x) - \gamma_0(x)f|(x) &\leq M_n^*\left(\left(1 + \frac{(e_1 - xe_0)^2}{\delta^2}\right)exp_\mu(\cdot + x)\right)\omega_1(f\cdot exp_{-\mu}, \delta) \\
&= \left(e^{2\mu x} + \frac{1}{\delta^2}e^{\mu x}M_n((e_1 - xe_0)^2)\right)\omega_1(f\cdot exp_{-\mu}, \delta).
\end{aligned}
$$

However, we have that:

$$
\begin{aligned}
M_n(e_1 - xe_0)^2(x) &= \frac{2n-6}{(n+2)\,(n+3)}x(1-x) + \frac{2}{(n+2)\,(n+3)} \\
&\leq \frac{n+1}{2}(n+2)(n+3) < \frac{1}{2n}.
\end{aligned} \tag{27}
$$

Now, setting $\delta = \frac{1}{\sqrt{n}}$, we have:

$$
|f(x) - M_n^* f(x)| \leq (e^{2\mu x} + \frac{1}{2}e^{\mu x})\omega_1\left(f\cdot e^{-\mu\cdot}, \frac{1}{\sqrt{n}}\right),
$$

which is our result.

In what follows, we will proceed with giving an estimate of the degree of approximation, of functions belonging to $C([0,1])$ in terms of the moduli in (6) and (7) using Peetre's K-functionals from (8) and the relation between them and the smoothness moduli.

Theorem 6 *If $f \in C([0,1])$ and $n \in \mathbb{N}$ we have that:*

$$
\begin{aligned}
\|M_n^* f - f\|_\infty &\leq \frac{\mu^2 e^\mu + 4\mu}{4n}\|f\|_\infty + K_{n,\mu}^1\omega_1\left(f, \frac{e^\mu(2\mu+1)+4}{e^\mu(8n+\mu^2)+4\mu}\right) \\
&\quad + K_{n,\mu}^2\omega_2\left(f, \sqrt{\frac{e^\mu(2\mu+1)+4}{e^\mu(8n+\mu^2)+4\mu}}\right),
\end{aligned} \tag{28}
$$

where $K_{n,\mu}^1 = C_1\frac{\left[e^\mu(8n+\mu^2)+4\mu\right](\mu e^\mu+2)}{2n[e^\mu(2\mu+1)+4]}$ and $K_{n,\mu}^2 = C_2\frac{e^{2\mu}(8n+\mu^2)+4\mu e^\mu}{4n[e^\mu(2\mu+1)+4]}$ with C_1 and C_2 constants.

Proof. Let us fix $f \in C([0,1])$ and $g \in C^2([0,1])$. Then we have:

$$\|M_n^* f - f\|_\infty \le \|M_n^* f - M_n^* g\|_\infty + \|M_n^* g - g\|_\infty + \|g - f\|_\infty.$$

Because $M_n^* f(x) = e^{\mu x} M_n(f \exp_{-\mu})(x)$, so $\|M_n^*\| \le e^{\mu}$,, which means that the relation above becomes:

$$\|M_n^* f - f\|_\infty \le 2e^{\mu}\|f - g\|_\infty + \|M_n^* g - g\|_\infty, \tag{29}$$

so, we will need to estimate $\|M_n^* g - g\|_\infty$. To do this, first, let $h \in C^2([0,1])$ such that $h(t) = e^{-\mu t} g(t)$ and let us write the Taylor polynomial with integral remainder $R(t,x) = \int_x^t h''(u)(t-u)^2 du$ associated to h :

$$h(t) = h(x) + h'(x)(t-x) + R(t,x),$$

which after multiplying by $e^{\mu t}$ becomes:

$$g(t) = e^{\mu t} e^{-\mu x} g(x) + e^{\mu t} e^{-\mu x}[g'(x) - \mu g(x)](t-x) + e^{\mu t} R(t,x),$$

therefore, applying M_n^* to the relation above and taking into consideration that $M_n^*(exp_\mu(e_1 - xe_0)^i)(x) = e^{\mu x} M_n((e_1 - xe_0)^i)(x)$, $i = 0,\ 1, \dots$ we get:

$$(M_n^* g - g)(x) =: (g'(x) - \mu g(x))\frac{1-2x}{n+2} + M_n^*\left(e^{\mu \cdot} R(\cdot, x)\right)(x).$$

But,

$$\left| e^{\mu t} \int_x^t h''(u)(t-u)du \right| \le e^{\mu t} \int_x^t |h''(u)| \cdot (t-u)du \le e^{\mu t} \frac{(t-x)^2}{2} \|h''\|_\infty.$$

Then

$$|(M_n^* g - g)(x)| \le |g'(x) - \mu g(x)| \cdot \left| \frac{1-2x}{n+2} \right| + \frac{1}{2} e^{\mu} M_n((e_1 - xe_0)^2)(x) \|h''\|_\infty. \tag{30}$$

We have:

$$\begin{aligned} \|h''\|_\infty &= \max_{x \in [0,1]} |g''(x)e^{-\mu x} - 2\mu g'(x) e^{-\mu x} + \mu^2 g(x)| \\ &\le \|g''\|_\infty + 2\mu \|g'\|_\infty + \mu^2 \|g\|_\infty, \end{aligned}$$

therefore returning to (30), using the inequality in (27) and the fact that $\|g\|_\infty \le \|f - g\|_\infty + \|f\|_\infty$ we get:

$$\begin{aligned} |(M_n^* g - g)(x)| &\le \frac{1}{n}\big(|g'(x)| + \mu |g(x)|\big) + \frac{e^{\mu}}{4n}(\|g''\|_\infty + 2\mu\|g'\|_\infty + \mu^2\|g\|_\infty) \\ &\le \frac{e^{\mu}\mu^2 + 4\mu}{4n}\|g\|_\infty + \frac{\mu e^{\mu} + 2}{2n}\|g'\|_\infty + \frac{e^{\mu}}{4n}\|g''\|_\infty \\ &\le \frac{\mu^2 e^{\mu} + 4\mu}{4n}(\|f - g\|_\infty + \|f\|_\infty) + \frac{\mu e^{\mu} + 2}{2n}\|g'\|_\infty + \frac{e^{\mu}}{4n}\|g''\|_\infty. \end{aligned}$$

Now, going back to (29) we have that:

$$\|M_n^* f - f\|_\infty \le \frac{\mu^2 e^\mu + 4\mu}{4n}\|f\|_\infty + \frac{\left[e^\mu(8n+\mu^2)+4\mu\right]}{4n}\|f-g\|_\infty + \frac{\mu e^\mu + 2}{2n}\|g'\|_\infty + \frac{e^\mu}{4n}\|g''\|_\infty$$
$$= \frac{\mu^2 e^\mu + 4\mu}{4n}\|f\|_\infty + \frac{\left[e^\mu(8n+\mu^2)+4\mu\right](\mu e^\mu+2)}{2n[e^\mu(2\mu+1)+4]}\left(\|f-g\|_\infty + \frac{e^\mu(2\mu+1)+4}{e^\mu(8n+\mu^2)+4\mu}\|g'\|\right)$$
$$+\frac{e^{2\mu}(8n+\mu^2)+4\mu e^\mu}{4n[e^\mu(2\mu+1)+4]}\left(\|f-g\|_\infty + \frac{e^\mu(2\mu+1)+4}{e^\mu(8n+\mu^2)+4\mu}\|g''\|_\infty\right)$$

so, after passing to the infimum in the left hand side of the inequality above we obtain

$$\|M_n^* f - f\|_\infty \le \frac{\mu^2 e^\mu + 4\mu}{4n}\|f\|_\infty + \frac{\left[e^\mu(8n+\mu^2)+4\mu\right](\mu e^\mu+2)}{2n[e^\mu(2\mu+1)+4]}\mathcal{K}_1\left(f, \frac{e^\mu(2\mu+1)+4}{e^\mu(8n+\mu^2)+4\mu}\right)$$
$$+\frac{e^{2\mu}(8n+\mu^2)+4\mu e^\mu}{4n[e^\mu(2\mu+1)+4]}\mathcal{K}_2\left(f, \sqrt{\frac{e^\mu(2\mu+1)+4}{e^\mu(8n+\mu^2)+4\mu}}\right),$$

which after using the fact that $\mathcal{K}_j(f,\delta) \le C_j\omega_j(f,\delta)$, $j = 1,\ 2$, where C_j is a constant depending only on j, we get our result.

5 Simultaneous approximation

In what follows we will provide a simultaneous approximation result concerning our operators. In order to do this we will need the r^{th} order derivative of M_n^*. First let us recall a result which is due to Derriennic (see [9]):

Lemma 3 *Let $f \in C^r([0,1])$. Then, for $x \in [0,1]$, $n \in \mathbb{N}$:*

$$\text{(31)} \qquad \frac{d^r}{dx^r}M_n f(x) = \frac{(n+1)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r} p_{n-r,k}(x)\int_0^1 p_{n+r,k+r}(t)f^{(r)}(t)dt.$$

Next, we will make the following notation

$$\text{(32)} \qquad Q_{n,j}f(x) = \frac{(n+1)!n!}{(n-j)!(n+j)!}\sum_{k=0}^{n-j} p_{n-j,k}(x)\int_0^1 p_{n+j,k+j}(t)f(t)dt$$

Remark 3 *We have that:*

$$\text{(33)} \qquad \frac{d^j}{dx^j}(M_n f)(x) = Q_{n,j}f^{(j)}(x).$$

From (32) we can see that operators $Q_{n,j}$ are linear and positive and the following Lemma, due to Derriennic (see [9]), holds.

Lemma 4 *Let $f \in C([0,1])$ be a function. Then $Q_{n,j}f$ converges uniformly to f.*

Now, we can proceed with the main result of this section.

Theorem 7 *Let $f \in C^r([0,1])$ and $s = 0, 1, \ldots, r$. Then we have that:*

$$\lim_{n\to\infty} \frac{d^s}{dx^s}(M_n^* f)(x) = f^{(s)}(x), \tag{34}$$

uniformly with respect to $x \in [0,1]$.

Proof. First we will find the r^{th} derivative of our operators. From Leibniz rule follows:

$$\frac{d^r}{dx^r}(M_n^* f)(x) = \sum_{j=0}^{r} \binom{r}{j} \mu^{r-j} \frac{d^r}{dx^r}(M_n f e^{-\mu\cdot})(x) e^{\mu x}, \tag{35}$$

Now, (33) yields

$$\frac{d^r}{dx^r}(M_n^* f)(x) = e^{\mu x} \sum_{j=0}^{r} \binom{r}{j} \mu^{r-j} Q_{n,j}((f e^{-\mu\cdot})^{(j)})(x) \tag{36}$$

Again, using Leibniz rule above we have:

$$\frac{d^r}{dx^r}(M_n^* f)(x) = e^{\mu x} \sum_{j=0}^{r} \binom{r}{j} \mu^{r-j} Q_{n,j} \left(\sum_{s=0}^{j} \binom{j}{s} (-\mu)^{j-s} f^{(s)} e^{-\mu\cdot} \right)(x), \tag{37}$$

and after a change in summation order we get:

$$\frac{d^r}{dx^r}(M_n^* f)(x) = e^{\mu x} \sum_{s=0}^{r} \sum_{j=s}^{r} \mu^{r-s} (-1)^{j-s} \binom{r}{j} \binom{j}{s} Q_{n,j}(f^{(s)} e^{-\mu\cdot})(x). \tag{38}$$

Next, from Lemma 4 we have that:

$$Q_{n,j}(f^{(s)} e^{(-\mu\cdot)})(x) = (f^{(s)} e^{-\mu\cdot})(x) + o\left(1\right), \ (n \to \infty). \tag{39}$$

Now, after passing to the limit and replacing (39) in (38) we get

$$\begin{aligned} \lim_{n\to\infty} \frac{d^r}{dx^r}(M_n^* f)(x) &= \lim_{n\to\infty} e^{\mu x} \sum_{s=0}^{r} \sum_{j=s}^{r} \left(f^{(s)}(x) e^{-\mu x} + o\left(1\right) \right) \mu^{r-s} (-1)^{j-s} \binom{r}{j} \binom{j}{s} \\ &= \sum_{s=0}^{r} \mu^{r-s} f^{(s)}(x) \sum_{j=s}^{r} (-1)^{j-s} \binom{r}{j} \binom{j}{s} \end{aligned}$$

which after using the well known identity:

$$\sum_{j=s}^{r} (-1)^{j-s} \binom{r}{j} \binom{j}{s} = \begin{cases} 1, & s = r \\ 0, & s < r \end{cases}$$

yields

$$\lim_{n\to\infty} \frac{d^r}{dx^r}(M_n^* f)(x) = f^{(r)}(x),$$

which is our result.

References

[1] T. Acar, A. Aral, D. Cardenas-Morales, P. Garrancho, *Szász-Mirakyan type operators which fix exponentials*, Results Math., vol. 72, no. 3, 2017, 1341-1358.

[2] T. Acar, A. Aral, H. Gonska, *On Szász-Mirakyan operators preserving* e^{2ax}*,* $a > 0$*,* Mediterr. J. Math., vol. 14, no. 6, 2017.

[3] A. M. Acu, A. Aral, I. Rașa, *New properties of operators preserving exponentials*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., vol. 117, no. 1, 2023.

[4] L. Angeloni, D. Costarelli, *Approximation by exponential - type polynomials*, J. Math. Anal. Appl., vol. 532, no. 1, 2024.

[5] A. Aral, D. Cardenas-Morales, P. Garrancho, *Bernstein-type operators that reproduce exponential functions*, J. Math. Inequal., vol. 12, no. 3, 2018, 861-872.

[6] A. Aral, D. Otrocol, I. Rașa, *On approximation by some Bernstein-Kantorovich exponential-type polynomial*, Period. Math. Hung., vol. 79, 2019, 236-254.

[7] E. Berdysheva, K. Jetter, J. Stöckier, *Durrmeyer operators and their natural quasi-interpolants*, Stud. Comput. Math., vol. 12, 2006, 1-21.

[8] H. Berens, T. Xu, *On Bernstein-Durrmeyer polynomials with Jacobi weights*, In Approximation Theory and Functional Analysis (C. K. Chui ed.), 25-43, 1991.

[9] M.-M. Derriennic, *Sur l'approximation de functions integrable sur [0; 1] par des polynomes de Bernstein modifies*, J. Approx. Theory, vol. 31, 1981, 323-343.

[10] M. M. Derriennic, *On multivariate approximation by Bernstein-type polynomials*, J. Approx. Theory, vol. 45, no. 2, 1985, 155-166.

[11] J. L. Durrmeyer, *Une formule d'inversion de la Transformee de Laplace, Applications a la Theorie des Moments*, These de 3e Cycle, Faculte des Sciences de l'Universite de Paris, 1967.

[12] V. Gupta, *Approximation with certain exponential operators*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., vol. 114, no. 2, 2022, 51.

[13] H. Johnen, *Inequalities connected with the moduli of smootness*, Mat. Vesn., N. Ser., vol. 9, no. 24, 1972, 289-303.

[14] A. Lupas, *Die Folge der Betaoperatoren. Dissertation*, Universität Stuttgart, 1972.

[15] B. Mond, *Note on the degree of approximation by linear positive operators*, J. Approx. Theory, vol. 18, 1976, 304-306.

[16] R. Păltănea, *Sur un opérateur polynomial défini sur l'ensemble des fonctions intégrables*, Babeș Bolyai Univ., Fac. Math., Res. Semin., vol. 2, 1983, 101-106.

[17] R. Păltănea, *Optimal estimates with moduli of continuity*, Result. Math., vol. 32, 1997, 318-331.

Ștefan-Lucian Garoiu
Transilvania University of Brașov
Faculty of Mathematics and Computer Science
Department of Mathematics and Computer Science
Address: Iuliu Maniu, 50, Brașov, Brașov, 500091
e-mail: stefan.garoiu@unitbv.ro