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Generalized Kantorovich operators [1]

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Abstract

In this paper we will propose a class of generalized Kantorovich type operators constructed using a general differential operator with non-constant coefficients $D^l g(x) = \sum_{i=0}^{l} a_i(x) g^{(i)}(x)$ and its corresponding antiderivative operator I^l with respect to the composition $D^l \circ I^l = Id$. The operators studied are of the type $K_n = D^l \circ L_n \circ I^l$ where L_n are positive linear operators. For these operators we will prove an approximation result and a Voronovskaja type theorem. Also, a simultaneous approximation result will be provided for a particular case. The operators studied in this paper are linear but not positive operators.

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1 Introduction

It is well known that classical approximation operators are positive linear operators. For this kind of operators, the approximation of continuous functions can be proved by Korovkin type theorems.

In this paper we will introduce a new class of operators which approximate all continuous functions on $[0, 1]$ but are not positive operators.

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We will propose a generalization of some positive linear operators in Kantorovich sense and we will prove that the approximation of all continuous function on $[0,1]$ holds. Also, a general Voronovskaja type theorem will be provided.

For a particular choice of coefficients of differential operator D^l, we will see that these operators posses the property of simultaneous approximation as well.

Finally, we will provide an approximation theorem and some Voronovskaja type theorems for the Kantorovich generalization of the Bernstein operators, Durrmeyer operators, Kantorovich operators, Stancu operators and U_n^ρ operators.

This topic was studied in particular cases in some papers such as [7, 12, 15, 16].

2 Construction of the operators

We will prove a uniform approximation result and a Voronovskaja type theorem for some Kantorovich type operators constructed using a more general differential operator: let $l \in \{1, 2, \dots\}$,

$$D^l g(x) = \sum_{i=0}^{l} a_i(x) g^{(i)}(x), \tag{1}$$

with $a_i(x) \in C(\mathbb{R}),\ i \in \{0, 1, \dots, l\}$. By an antiderivative operator corresponding to D^l we mean an operator I^l which satisfies $D^l \circ I^l = Id$. This condition leads to:

$$\left(D^l \circ I^l\right)(f) = f,$$

which is equivalent with:

$$a_l(x)\left(I^l f\right)^{(l)} + a_{l-1}(x)\left(I^l f\right)^{(l-1)} + \dots + a_1(x)\left(I^l f\right)' + a_0(x)\left(I^l f\right) = f. \tag{2}$$

Equation (2) is a linear differential equation of order l with non-constant coefficients. For this kind of differential equation we recall the following existence and unicity theorem:

Theorem 1 *[2] Suppose that $a_i \in C(I),\ i \in \{0, 1, \dots, l\}$, where $I \subset \mathbb{R}$ is an open interval with $a \in I$. Having $b_0, b_1, \dots, b_l \in \mathbb{R}$ such that the initial conditions $y^{(i)}(a) = b_i$ holds, $i \in \{0, 1, \dots, l\}$, the equation*

$$a_l(x) y^{(l)} + a_{l-1}(x) y^{(l-1)} + \dots + a_1(x) y' + a_0(x) y = f(x),\ f \in C(I), \tag{3}$$

has an unique solution $y \in C^l(I)$.

From the above theorem we have that for equation (2) there are solutions, and by imposing some initial conditions we will find a unique one. We mention that for the construction of the operators studied in this paper, the exact expression of the antiderivative operators does not play an important role since choosing a different antiderivative operator, our operator will be different, but the approximation processes we study do not depend on the choice of the antiderivative operator.

Remark 1 *From Theorem 1 we have that the solution of* (2) *exists on an open interval* I. *In order to prove our results, let us take* I *such that* $[0,1] \subset I$.

Definition 1 *Let* $(L_n)_n$ *be a sequence of linear and positive operators on* $[0,1]$. *We introduce the following Kantorovich type operators*

$$K_n = D^l \circ L_n \circ I^l, \tag{4}$$

which have the expression

$$\begin{aligned} K_n(f,x) &= \left(D^l \circ L_n \circ I^l\right)(f,x) \\ &= D^l\left(L_n(F,x)\right) \\ &= \sum_{i=0}^{l} a_i(x)\, L_n^{(i)}(F,x),\ x \in [0,1],\ f \in C[0,1], \end{aligned} \tag{5}$$

where $F(x) = I^l f(x)$.

In order to prove our results we will need the following.

Let I be a compact interval and $M_{n,k}(x) = L_n\left(\frac{(t-x)^k}{k!}, x\right),\ x \in I.$

In paper [8] it was proved the following result.

Theorem 2 *Let* $(L_n)_n$ *be a sequence of positive and linear operators such that* L_n *is convex of order* $r-1$, *with* $r \geq 0$ *an integer. Suppose that* $M_{n,4}(x) = o(M_{n,2}(x))$ *uniformly for* $x \in I$, *and* $M_{n,0}(x) = 1$, *and that there exists two functions* $a, b : I \to \mathbb{R}$ *such that* $\lim_{n\to\infty} nM_{n,1}(x) = a(x)$ *and* $\lim_{n\to\infty} nM_{n,2}(x) = b(x)$. *Then, for* $f \in C^{r+2}(I)$ *the following limit holds uniformly:*

$$\lim_{n\to\infty} n\left[L_n^{(r)}(f,x) - f^{(r)}(x)\right] = \left[a(x) f'(x) + b(x) f''(x)\right]^{(r)},\ x \in I. \tag{6}$$

3 Approximation properties

For the operators K_n introduced above we can state the following approximation results.

Theorem 3 *Let* $(L_n)_n$ *be a sequence of linear and positive operators on* $[0,1]$, *which has the property of simultaneous approximation* $\left\|L_n^{(k)} g - g^{(k)}\right\| \to 0$ *uniformly on* $[0,1]$, $g \in C^k[0,1]$, $k = \overline{0,l}$, $l \in \{0,1,2\ldots\}$. *Then the following convergence holds*

$$\|K_n f - f\| \to 0,\ uniformly\ on\ [0,1], \tag{7}$$

with $f \in C[0,1]$.

Proof. Let $f \in C[0,1]$.

$$\begin{aligned}
|K_n(f,x) - f(x)| &= \left|\sum_{i=0}^{l} a_i(x) L_n^{(i)}(F,x) - f(x)\right| \\
&= \left|\sum_{i=0}^{l} a_i(x) L_n^{(i)}(F,x) - \sum_{i=0}^{l} a_i(x) F^{(i)}(x)\right| \\
&= \left|\sum_{i=0}^{l} a_i(x)\left(L_n^{(i)}(F,x) - F^{(i)}(x)\right)\right| \\
&\leq \sum_{i=0}^{l} |a_i(x)| \left|L_n^{(i)}(F,x) - F^{(i)}(x)\right|, \ \forall\, x \in [0,1].
\end{aligned}$$

Now, because $a_i \in C[0,1]$ are continous, therefore functions a_i are bounded, and because L_n has the property of simultaneous approximation we can state the following: $\forall \varepsilon > 0$

$$|K_n(f,x) - f(x)| < \varepsilon, \ \forall x \in [0,1] \ \textit{and} \ f \in C[0,1],$$

hence the unform convergence

$$\|K_n f - f\| \to 0$$

holds on $[0,1]$.

Using Theorem 2 we can prove the following Voronovskaja type theorem for our operators K_n.

Theorem 4 *Let $(L_n)_n$ be a sequence of positive and linear operators such that L_n is convex of order $r-1$, with $r \geq 0$ an integer. Suppose that $M_{n,4}(x) = o(M_{n,2}(x))$ uniformly for $x \in [0,1]$, and $M_{n,0}(x) = 1$, and that there exists two functions $a, b : [0,1] \to \mathbb{R}$ such that $\lim_{n\to\infty} nM_{n,1}(x) = a(x)$ and $\lim_{n\to\infty} nM_{n,2}(x) = b(x)$. Let $K_n = D^l \circ L_{n+l} \circ I^l$. Then, for $f \in C^{r+2}[0,1]$ the following limit holds uniformly:*

$$\begin{aligned}
\lim_{n\to\infty} n[K_n(f,x) - f(x)] &= \sum_{i=0}^{l} a_i(x)\left[a(x)F'(x) + b(x)F''(x)\right]^{(i)} \\
&= D^l\left(a(x)F'(x) + b(x)F''(x)\right), \ x \in [0,1]. \qquad (8)
\end{aligned}$$

Proof. Let us compute the following

$$\begin{aligned}
n[K_n(f,x) - f(x)] &= n\left[\sum_{i=0}^{l} a_i(x) L_n^{(i)}(F,x) - f(x)\right] \\
&= n\left[\sum_{i=0}^{l} a_i(x) L_n^{(i)}(F,x) - \sum_{i=0}^{l} a_i(x) F^{(i)}(x)\right] \\
&= \sum_{i=0}^{l} a_i(x)\left\{n\left[L_n^{(i)}(F,x) - F^{(i)}(x)\right]\right\}.
\end{aligned}$$

Now, passing to the limit when $n \to \infty$ we obtain

$$\begin{aligned}\lim_{n\to\infty} n\left[K_n\left(f,x\right)-f\left(x\right)\right] &= \sum_{i=0}^{l} a_i\left(x\right)\left\{\lim_{n\to\infty} n\left[L_n^{(i)}\left(F,x\right)-F^{(i)}\left(x\right)\right]\right\}\\ &= \sum_{i=0}^{l} a_i\left(x\right)\left[a\left(x\right)F'\left(x\right)+b\left(x\right)F''\left(x\right)\right]^{(i)}\\ &= D^l\left(a\left(x\right)F'\left(x\right)+b\left(x\right)F''\left(x\right)\right),\ x\in\left[0,1\right].\end{aligned}$$

4 Simultaneous approximation result

In this case we consider $a_i\left(x\right)=a_i\in\mathbb{R},\ i\in\{0,1,\ldots,l\},\ l\geq 0$, real constants and the differential operator

$$D^{*l}f\left(x\right)=\sum_{i=0}^{l}a_if^{(i)}\left(x\right), \tag{9}$$

and I^{*l} a corresponding antiderivative operator with respect to $D^{*l}\circ I^{*l}g=g$, which leads to a linear differential equation of order l with constant coefficients $a_i\in\mathbb{R}$. For this kind of equation it is well known that solutions exist but they are not unique until we impose some initial conditions which are not needed for the following result. Denote $I^{*l}f:=F^*$.

Let L_n be a sequence of operators and define K_n^* as:

$$K_n^*=D^{*l}\circ L_n\circ I^{*l}. \tag{10}$$

For operators K_n^* we can state the following simultaneous approximation result:

Theorem 5 *Let $\left(L_n\right)_n$ be a sequence of operators having the property that $\left(L_ng\right)^{(j)}\to g^{(j)}$ uniformly for $x\in[0,1]$ and $g\in C^j\left[0,1\right],\ 0\leq j\leq r+l,\ r\geq 0.$ Then*

$$\left(K_n^*f\right)^{(r)}\to f^{(r)},\ f\in C^r\left[0,1\right], \tag{11}$$

and $F^*\in C^{r+l}\left[0,1\right].$

Proof. Let us consider the derivative of K_n^*f :

$$\begin{aligned}\left(K_n^*f\right)^{(r)} &= \left[\left(D^{*l}\circ L_n\circ I^{*l}f\right)\left(x\right)\right]^{(r)}\\ &= \left[\sum_{i=0}^{l}a_iL_n^{(i)}\left(F,x\right)\right]^{(r)}\\ &= \sum_{i=0}^{l}a_iL_n^{(i+r)}\left(F,x\right).\end{aligned} \tag{12}$$

Now, because of the simultaneous approximation property of L_n, we have that for all $\varepsilon_i > 0$, $i \in \{0, 1, \dots, l\}$, the inequality $\left| L_n^{(i+r)}(F, x) - F^{(i+r)}(x) \right| < \varepsilon_i$ holds, therefore we get:

$$\begin{aligned}
(13) \qquad \left| (K_n^* f)^{(r)} - f^{(r)}(x) \right| &= \left| \sum_{i=0}^{l} a_i L_n^{(i+r)}(F, x) - f^{(r)}(x) \right| \\
&= \left| \sum_{i=0}^{l} a_i L_n^{(i+r)}(F, x) - \left[\sum_{i=0}^{l} a_i F^{(i)}(x) \right]^{(r)} \right| \\
&= \left| \sum_{i=0}^{l} a_i \left(L_n^{(i+r)}(F, x) - F^{(i+r)}(x) \right) \right| \\
&\leq \sum_{i=0}^{l} |a_i| \left| L_n^{(i+r)}(F, x) - F^{(i+r)}(x) \right| \\
&< \sum_{i=0}^{l} |a_i| \varepsilon_i < \varepsilon,
\end{aligned}$$

therefore $\left| (K_n^* f)^{(r)} - f^{(r)}(x) \right| < \varepsilon$ uniformly for $x \in [0, 1]$.

5 Particular cases

In this section we will study the generalized Kantorovich modification for some well known operators.

In order to simplify the notations we introduce the following differential operator

$$(14) \qquad D_y^l g(x) = \sum_{i=1}^{l} y^i a_i(x) g^{(i-1)}(x).$$

5.1 Bernstein operators

Let B_n be Bernstein operators,

$$(15) \qquad B_n(f, x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \ x \in [0, 1], \ f \in C[0, 1],$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}$, for $0 \leq k \leq n$, and $p_{n,k}(x) = 0$ for $k > n$. It is well known that Bernstein operators play an important role in approximation theory. Some of the results obtained for Bernstein operators will be useful in our paper. We will recall some of them.

Theorem 6 *[3]Let $f \in C[0,1]$, then the following uniform convergence holds*

$$\|B_n f - f\| \to 0. \tag{16}$$

We will also need the simultaneous approximation result:

Theorem 7 *[5]If $f \in C^k[0,1]$, then*

$$\left\| (B_n f)^{(k)} - f^{(k)} \right\| \to 0. \tag{17}$$

Let $L_n = B_{n+l}$. Therefore,

$$\begin{aligned} K_n^B(f,x) &= \left(D^l \circ B_{n+l} \circ I^l \right)(f,x) \\ &= \sum_{i=0}^{l} a_i(x)\, B_{n+l}^{(i)}(F,x), \ x \in [0,1], \ f \in C[0,1]. \end{aligned} \tag{18}$$

Because Bernstein operators have the simultaneous approximation property, we can state the following result obtained by applying Theorem 3:

Theorem 8 *The following convergence*

$$K_n^B(f,x) \to f(x) \tag{19}$$

holds uniformly for $x \in [0,1]$ and $f \in C[0,1]$.

Now, in order to obtain a Voronovskaja type result for $K_n^B(f,x)$ we will need the following

$$\begin{aligned} M_{n,0}(x) &= 1; \ M_{n,1}(x) = 0; \ M_{n,2}(x) = \frac{x(1-x)}{2n}; \\ M_{n,4}(x) &= \frac{x(1-x)}{n^2}\left[\left(3 - \frac{6}{n}\right) x(1-x) + \frac{1}{n} \right]. \end{aligned} \tag{20}$$

We can see that $M_{n,4}(x) = o(M_{n,2}(x))$, hence Theorem 4 can be applied with

$$\begin{aligned} a(x) &= \lim_{n\to\infty} nM_{n,1}(x) = 0 \ \textit{and} \\ b(x) &= \lim_{n\to\infty} nM_{n,2}(x) = \frac{x(1-x)}{2}. \end{aligned} \tag{21}$$

Let us denote:

$$A(x) = \sum_{i=0}^{l} \left(a_i''(x)\, F^{(i)}(x) + 2a_i'(x)\, F^{(i+1)}(x) \right), \ x \in [0,1], \ a_i \in C^2[0,1]. \tag{22}$$

We have:

Theorem 9 *Let $a_i \in C^2[0,1]$ and $F \in C^{l+2}[0,1]$. The following limit holds*

$$\lim_{n\to\infty} n\left[K_n^B(f,x) - f(x)\right] \tag{23}$$
$$= \frac{1}{2}x(1-x)f''(x) + \xi(x), \ x \in [0,1], \ f \in C^2[0,1].$$

where

$$\xi(x) = \frac{1}{2}(1-2x)\frac{\partial D_y^l F''(x)}{\partial y}|_{y=1} - \frac{1}{2}\frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1} - \frac{1}{2}x(1-x)A(x).$$

Proof. We apply Theorem 4 for K_n^B with $a(x) = 0$ and $b(x) = \frac{x(1-x)}{2}$. We get by using Leibniz rule:

$$\begin{aligned}
\lim_{n\to\infty}\left[K_n^B(f,x) - f(x)\right] &= D^l\left(\frac{x(1-x)}{2}F''(x)\right) \\
&= \sum_{i=0}^{l} a_i(x)\left(\frac{x(1-x)}{2}F''(x)\right)^{(i)} \\
&= \sum_{i=0}^{l} a_i(x)\sum_{k=0}^{i}\binom{i}{k}\left(\frac{x(1-x)}{2}\right)^{(k)}\left(F''(x)\right)^{(i-k)} \\
&= \sum_{i=0}^{l} a_i(x)\left[\frac{x(1-x)}{2}F^{(i+2)}(x) + i\frac{1-2x}{2}F^{(i+1)}(x) - \frac{i(i-1)}{2}F^{(i)}(x)\right] \\
&= \frac{x(1-x)}{2}D^lF''(x) + \frac{1-2x}{2}\frac{\partial D_y^l F''(x)}{\partial y}|_{y=1} - \frac{1}{2}\frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1}.
\end{aligned}$$

We now treat separately the term $D^lF''(x)$. In order to get the desired result we shall compute the second derivative of the expression $D^lF(x) = f(x)$. We get

$$f''(x) = \sum_{i=0}^{l} a_i''(x)F^{(i)}(x) + 2\sum_{i=0}^{l} a_i'(x)F^{(i+1)}(x) + \sum_{i=0}^{l} a_i(x)F^{(i+2)}(x),$$

which leads to

$$D^lF''(x) = f''(x) - \sum_{i=0}^{l}\left(a_i''(x)F^{(i)}(x) + 2a_i'(x)F^{(i+1)(x)}\right) = f''(x) - A(x), \tag{24}$$

hence our result is proved.

5.2 Durrmeyer operators

Let D_n be Durrmeyer operators defined as

$$D_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)\,dt, \ x \in [0,1], \ f \in C[0,1],$$

with $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, for $0 \le k \le n$, and $p_{n,k}(x) = 0$ for $k > n$.

Durrmeyer operators are useful in approximation of functions $f \in L_1[0,1]$. It was also proved that Durrmeyer operators can be used in simultaneous approximation:

Theorem 10 *[4]If $f \in C^k[0,1]$, then*

$$\left\| (D_n f)^{(k)} - f^{(k)} \right\| \to 0.$$

We choose $L_n = D_{n+l}$ in our construction, therefore

$$K_n^D(f,x) = \left(D^l \circ D_{n+l} \circ I^l\right)(f,x) \tag{25}$$
$$= \sum_{i=0}^{l} a_i(x) D_{n+l}^{(i)}(F,x), \; x \in [0,1], \; f \in C[0,1].$$

Now, because the simultaneous approximation result holds, we can state the following uniform approximation result for operators K_n^D by applying Theorem 3:

Theorem 11 *The following convergence*

$$K_n^D(f,x) \to f(x) \tag{26}$$

holds uniformly for $x \in [0,1]$ and $f \in C[0,1]$.

It is also known that D_n are convex operators of order $r-1$, $r \ge 0$ (see [7]) and that

$$\begin{aligned} M_{n,0}(x) &= 1, \; M_{n,1}(x) = \frac{1-2x}{n+2}, \\ M_{n,2}(x) &= \frac{(2n-6)x(1-x)+2}{2(n+2)(n+3)}, \end{aligned}$$

and that $M_{n,4}(x) = o(M_{n,2}(x))$ uniformly for $x \in [0,1]$.

Now, we can apply Theorem 4 with the following

$$\begin{aligned} a(x) &= \lim_{n\to\infty} n M_{n,1}(x) = 1 - 2x \textit{ and} \\ b(x) &= \lim_{n\to\infty} n M_{n,2}(x) = x(1-x), \end{aligned} \tag{27}$$

and we get:

Theorem 12 *Let $a_i \in C^2[0,1]$ and $F \in C^{l+2}[0,1]$. The following limit holds*

$$\lim_{n\to\infty} n\left[K_n^D(f,x) - f(x)\right] \tag{28}$$
$$= x(1-x) f''(x) + (1-2x) f'(x) + \eta(x), \; x \in [0,1], \; f \in C^2[0,1],$$

where

$$\eta(x) = -x(1-x)\sum_{i=0}^{l} a_i'(x)F^{(i)}(x) - (1-2x)\left[A(x) + \frac{\partial D_y^l F''(x)}{\partial y}|_{y=1}\right]$$
$$-\frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1} - 2\frac{\partial D_y^l F'(x)}{\partial y}|_{y=1}.$$

Proof. The proof is similar to the one of the Theorem 9, taking into account the term $D^l F'(x)$ which can expressed as

$$D^l F'(x) = f'(x) - \sum_{i=0}^{l} a_i'(x) F^{(i)}(x). \tag{29}$$

5.3 Kantrovich operators

Let $\tilde{K}_n$ be Kantorovich operators defined as

$$\tilde{K}_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ \ x\in[0,1],\ \ f\in C[0,1],$$

with $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$, for $0 \le k \le n$, and $p_{n,k}(x) = 0$ for $k > n$.

For these operators we have that the simultaneous approximation result holds:

Theorem 13 *[7] If $f \in C^k[0,1]$, then*

$$\left\|\left(\tilde{K}_n f\right)^{(k)} - f^{(k)}\right\| \to 0.$$

We choose $L_n = \tilde{K}_{n+l}$ in our construction, therefore

$$\begin{aligned} K_n^{\tilde{K}}(f,x) &= \left(D^l \circ \tilde{K}_{n+l} \circ I^l\right)(f,x) \\ &= \sum_{i=0}^{l} a_i(x)\tilde{K}_{n+l}^{(i)}(F,x),\ \ x\in[0,1],\ \ f\in C[0,1]. \end{aligned} \tag{30}$$

Now, because the simultaneous approximation result holds for $K_n^{\tilde{K}}$, we can apply Theorem 4 and we get the following uniform approximation result:

Theorem 14 *The following convergence*

$$K_n^{\tilde{K}}(f,x) \to f(x) \tag{31}$$

holds uniformly for $x \in [0,1]$ and $f \in C[0,1]$.

It is also known that Kantorovich operators $\tilde{K}_n$ are convex operators of order $r-1,\ r\geq 0$ (see [7]) and that

$$\begin{aligned} M_{n,0}(x) &= 1,\ M_{n,1}(x)=\frac{1-2x}{2(n+1)}, \\ M_{n,2}(x) &= \frac{3x(1-x)(n-1)+1}{6(n+1)^2}, \end{aligned}$$

and that $M_{n,4}(x)=o(M_{n,2}(x))$ uniformly for $x\in[0,1]$.

Now, we can apply Theorem 4 with the following

(32)
$$\begin{aligned} a(x) &= \lim_{n\to\infty} nM_{n,1}(x)=\frac{1-2x}{2} \text{ and} \\ b(x) &= \lim_{n\to\infty} nM_{n,2}(x)=\frac{x(1-x)}{2}, \end{aligned}$$

and we get:

Theorem 15 *Let $a_i\in C^2[0,1]$ and $F\in C^{l+2}[0,1]$. The following limit holds*

(33)
$$\begin{aligned} &\lim_{n\to\infty} n\left[K_n^{\tilde{K}}(f,x)-f(x)\right] \\ &=\frac{x(1-x)}{2}f''(x)+\frac{(1-2x)}{2}f'(x)+\frac{\eta(x)}{2},\ x\in[0,1],\ f\in C^2[0,1], \end{aligned}$$

where

$$\begin{aligned} \eta(x) = -x(1-x)\sum_{i=0}^{l} a_i'(x)F^{(i)}(x)-(1-2x)\left[A(x)+\frac{\partial D_y^l F''(x)}{\partial y}|_{y=1}\right] \\ -\frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1}-2\frac{\partial D_y^l F'(x)}{\partial y}|_{y=1}, \end{aligned}$$

as in Theorem 12.

Proof. The result is obtained by similar computation as in Theorems 9 and 12.

5.4 Stancu operators

Let $S_n^{\alpha,\beta}$ be Bernstein-Stancu operators defined as

$$S_n^{\alpha,\beta}(f,x)=\sum_{k=0}^{n} p_{n,k}(x)f\left(\frac{k+\alpha}{n+\beta}\right),\ x\in[0,1],\ f\in C[0,1],$$

with $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$, for $0\leq k\leq n$, and $p_{n,k}(x)=0$ for $k>n$, and $\frac{k+\alpha}{n+\beta}\in[0,1],\ 0\leq k\leq n$.

For these operators we have that the simultaneous approximation result holds:

Theorem 16 *[1]If $f \in C^k[0,1]$, then*

$$\left\| \left(S_n^{\alpha,\beta} f \right)^{(k)} - f^{(k)} \right\| \to 0.$$

We choose $L_n = S_{n+l}^{\alpha,\beta}$ in our construction, therefore

$$K_n^S(f,x) = \left(D^l \circ S_{n+l}^{\alpha,\beta} \circ I^l \right)(f,x) \tag{34}$$
$$= \sum_{i=0}^{l} a_i(x) \left(S_{n+l}^{\alpha,\beta} \right)^{(i)} (F,x), \ x \in [0,1], \ f \in C[0,1].$$

Because the simultaneous approximation result holds for K_n^S, we can state the following uniform approximation result using Theorem 3:

Theorem 17 *The following convergence*

$$K_n^S(f,x) \to f(x) \tag{35}$$

holds uniformly for $x \in [0,1]$ and $f \in C[0,1]$.

It is also known that the Stancu operators $S_n^{\alpha,\beta}$ are convex operators of order $r-1, \ r \geq 0$ (see [8]) and that

$$\begin{aligned} M_{n,0}^{\alpha,\beta}(x) &= 1, \ M_{n,1}^{\alpha,\beta}(x) = \frac{\alpha - \beta x}{n+\beta}, \\ M_{n,2}^{\alpha,\beta}(x) &= \frac{1}{2}\left[\left(\frac{\alpha - \beta x}{n+\beta} \right)^2 + \left(\frac{n}{n+\beta} \right)^2 \frac{x(1-x)}{n} \right], \end{aligned}$$

and that $M_{n,4}^{\alpha,\beta}(x) = o\left(M_{n,2}^{\alpha,\beta}(x) \right)$ uniformly for $x \in [0,1]$.

Now, we can apply Theorem 4 with the following

$$\begin{aligned} a(x) &= \lim_{n\to\infty} n M_{n,1}^{\alpha,\beta}(x) = \alpha - \beta x, \ \textit{and} \\ b(x) &= \lim_{n\to\infty} n M_{n,2}(x) = \frac{x(1-x)}{2}, \end{aligned} \tag{36}$$

and we get:

Theorem 18 *Let $a_i \in C^2[0,1]$ and $F \in C^{l+2}[0,1]$. The following limit holds*

$$\lim_{n\to\infty} n\left[K_n^S(f,x) - f(x) \right] \tag{37}$$
$$= (\alpha - \beta x) f'(x) + \frac{x(1-x)}{2} f''(x) - \theta(x), \ x \in [0,1], \ f \in C^2[0,1],$$

where

$$\theta(x) = (\alpha - \beta x)\sum_{i=0}^{l} a_i'(x)\, F^{(i)}(x) + \frac{x(1-x)}{2} A(x) + \beta \frac{\partial D_y^l F'(x)}{\partial y}|_{y=1}$$
$$+\frac{1-2x}{2}\frac{\partial D_y^l F''(x)}{\partial y}|_{y=1} + \frac{1}{2}\frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1}.$$

Proof. By direct computations, similar to Theorem 9.

5.5 U_n^ρ operators

U_n^ρ operators are defined as

$$U_n^\rho(f,x) = \sum_{k=0}^{n} F_{n,k}^\rho(f)\, p_{n,k}(x), \ f \in C[0,1], \ x \in [0,1], \tag{38}$$

with

$$F_{n,k}^\rho(f) = \int_0^1 f(t)\,\mu_{n,k}^\rho(t)\,dt, \ \text{and} \tag{39}$$

$$\mu_{n,k}^\rho(t)\,dt = \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)}, \tag{40}$$

where $B(\cdot,\cdot)$ is Euler's Beta function and $\rho > 0$.

For the U_n^ρ operators we have the following results proved in paper [9].

Theorem 19 *[9] U_n^ρ are convex operators of order $r-1$, $0 \le r \le n$, $\rho > 0$, $n \ge 1$.*

Theorem 20 *[9] The following simultaneous approximation holds:*

$$\lim_{n\to\infty} \left\| (U_n^\rho f)^{(r)} - f^{(r)} \right\| = 0, \ f \in C^r[0,1], \tag{41}$$

$\rho \in (0,\infty]$.

Also, in this paper they computed the following

$$\begin{aligned} M_{n,0}(x) &= 1; \ M_{n,1}(x) = 0, \\ M_{n,2}(x) &= \frac{(\rho+1)\,x(1-x)}{2(n\rho+1)}. \end{aligned} \tag{42}$$

Now, we take $L_n = U_{n+l}^\rho$ and we get

$$UK_n^\rho(f,x) = \left(D^l \circ U_{n+l}^\rho \circ I^l\right)(f,x), \ f \in C[0,1], \ x \in [0,1]. \tag{43}$$

Since U_n^ρ have the simultaneous approximation property, we can state the following approximation result concerning UK_n^ρ by applying Theorem 3:

Theorem 21 *The following convergence*

$$UK_n^{\rho}(f,x) \to f(x) \tag{44}$$

holds uniformly for $x \in [0,1]$ and $f \in C[0,1]$ and $\rho \in (0,\infty]$.

And also, because U_n^{ρ} is convex of order $r-1$, $M_{n,4}(x) = o(M_{n,2}(x))$ uniformly for $x \in [0,1]$, and $M_{n,0}(x) = 1$, we can state the following Voronovskaja type result using Theorem 4 with $a(x) = 0$ and $b(x) = \frac{\rho+1}{2\rho}x(1-x)$:

Theorem 22 *Let $a_i \in C^2[0,1]$ and $F \in C^{l+2}[0,1]$. The following limit holds*

$$\begin{aligned} &\lim_{n\to\infty} n\left[UK_n^{\rho}(f,x) - f(x)\right] \\ &= \frac{(\rho+1)\,x\,(1-x)}{2\rho} f''(x) + \omega(x), \ x \in [0,1], \ f \in C^2[0,1], \end{aligned} \tag{45}$$

where $\omega(x) = -\frac{(\rho+1)x(1-x)}{2} A(x) + \frac{\rho+1}{2\rho}(1-2x)\frac{\partial D_y^l F''(x)}{\partial y}|_{y=1} - \frac{\rho+1}{2}\frac{\partial^2 D_y^l F'(x)}{\partial y^2}|_{y=1}$.

6 Nonpositivity of operators

Proposition 1 *Operators K_n^B, K_n^D, K_n^S, $K_n^{\tilde{K}}$ and UK_n^{ρ} are linear but not positive operators.*

Proof. The linearity of operators is obvious from the definition and from the fact that Bernstein operators, Durrmeyer operators, Stancu operators, Kantorovich operators and U_n^{ρ} operators are linear. For the positivity we will give some counterexample:

Example 1 *Let $D^* f(x) = f'(x) - xf(x)$. For this example we will need the exact expression of the antiderivative operator I^* associated to D^*, therefore we choose I^* such that $D^* \circ I^* = Id$ and it satisfies the initial condition $I^* f(0) = 0$. For simplicity, we make the notation $I^* f := F$. The condition $D^* \circ I^* f = D^* F = f$ leads to the following differential equation with initial condition:*

$$\begin{cases} F' - xF = f \\ F(0) = 0 \end{cases}, \tag{46}$$

for which we have the solution

$$F(x) = e^{\frac{x^2}{2}} \int_0^x f(t)\, e^{-\frac{t^2}{2}} dt. \tag{47}$$

Now, take $f^*(t)=e^{\frac{t^2}{2}-5t}>0$ and let us compute, for $n=1$ the expression of the operator K_1^B:

$$
\begin{aligned}
K_1^B(f^*,x) &= (D^*\circ B_2\circ I^*)\,f^*(x)=D^*(B_2(F,x)) \\
&= B_2'(F,x)-xB_2(F,x) \\
&= 2F\left(\frac{1}{2}\right)\left(x^3-x^2-2x+1\right)+F(1)\left(2x-x^2\right)
\end{aligned}
\tag{48}
$$

If we take $x=1$, we obtain

$$
\begin{aligned}
K_1^B(f^*,1) &= -2e^{\frac{1}{8}}\int_0^{\frac{1}{2}} e^{-5t}e^{\frac{t^2}{2}}e^{-\frac{t^2}{2}}dt+e^{\frac{1}{2}}\int_0^1 e^{-5t}e^{\frac{t^2}{2}}e^{-\frac{t^2}{2}}dt \\
&= -8.8531\times10^{-2}<0,
\end{aligned}
\tag{49}
$$

therefore the operators K_n^B are not positive.

Example 2 *If we take the same differential operator $D^*f(x)=f'(x)-xf(x)$ with its corresponding antiderivative operator $F(x)=e^{\frac{x^2}{2}}\int_0^x f(t)\,e^{-\frac{t^2}{2}}dt$, we get*

$$
\begin{aligned}
K_1^S(f^*,x) &= \left(D^*\circ S_2^{\alpha,\beta}\circ I^*\right)f^*(x)=D^*(S_2^{\alpha,\beta}(F,x)) \\
&= S_2^{\alpha,\beta\prime}(F,x)-xS_2^{\alpha,\beta}(F,x) \\
&= 2F\left(\frac{\alpha}{2+\beta}\right)\left(-x^3+2x^2+x-2\right)+F\left(\frac{1+\alpha}{2+\beta}\right)\left(2x^3-2x^2-4x+2\right) \\
&\quad +F\left(\frac{2+\alpha}{2+\beta}\right)\left(2x-x^3\right).
\end{aligned}
\tag{50}
$$

If we chose $f^*(t)=e^{\frac{t^2}{2}}>0$, $\alpha=1,\ \beta=2$, and we compute for $x=1$ we get

$$
K_1^S(f^*,1)=-1.8393<0,
\tag{51}
$$

hence K_n^S are not positive operators.

Example 3 *In order to prove that K_n^D, $K_n^{\tilde{K}}$ and UK_n^{ρ} are not positive, we will consider the following differential operator $\tilde{D}f=f'-f$, which has the corresponding antiderivative operator $\tilde{I}f(x):=\tilde{F}(x)=e^x\int_0^x f(t)\,e^{-t}dt$, obtained after imposing the initial condition $\tilde{I}f(0)=0$. We choose $n=1$ and $f(t)=e^t>0$. For the Kantorovich modification of Durrmeyer operators we get:*

$$
\begin{aligned}
K_1^D(f,x) &= D_2'\left(\tilde{F},x\right)-D_2\left(\tilde{F},x\right) \\
&= 3\left(-3+4x-x^2\right)\int_0^1(1-t)^2\tilde{F}(t)\,dt+3\left(4-10x+x^2\right)\int_0^1 t(1-t)\,\tilde{F}(t)\,dt \\
&\quad +3\left(2x-x^2\right)\int_0^1 t^2\tilde{F}(t)\,dt.
\end{aligned}
\tag{52}
$$

Computing in $x = 1$ we obtain:

$$K_1^D(f, 1) = -0.16784 < 0.$$

For the Kantorovich modification of Kantorovich operators we have

$$K_1^{\tilde{K}}(f, x) = \tilde{K}_2'\left(\tilde{F}, x\right) - \tilde{K}_2\left(\tilde{F}, x\right) \tag{53}$$
$$= 3\left(-3 + 4x - x^2\right)\int_0^{\frac{1}{3}} \tilde{F}(t)\,dt + 3\left(2 - 6x + 2x^3\right)\int_{\frac{1}{3}}^{\frac{2}{3}} \tilde{F}(t)\,dt$$
$$+3\left(2x - x^2\right)\int_{\frac{2}{3}}^{1} \tilde{F}(t)\,dt,$$

and by taking $x = 1$ we get:

$$K_1^{\tilde{K}}(f, 1) = -1.0011 < 0,$$

therefore the Kantorovich modifications of Durrmeyer operators and Kantorovich operators are not positive operators.

For UK_n^ρ we consider $f(t) = e^t > 0$ and $\rho = 5$. The value of the operators in $x = 1$ is

$$UK_1^5(f, 1) = U_2^{5\prime}\left(\tilde{F}, 1\right) - U_2^5\left(\tilde{F}, 1\right) \tag{54}$$
$$= -2F_{2,1}^5 + \tilde{F}(1)$$
$$= -14.714 < 0,$$

hence the Kantorovich modification of U_n^ρ is not a positive operator.

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