General Mathematics Vol. 32, No. 2 (2024), 62–66

DOI: 10.2478/gm-2024-0013

On a transformation of a sequence of real numbers [1]

Daniel Florin Sofonea, Ioan Ţincu

Abstract

In this paper, we determine finite sections of a regular transformation, using random variables which follows the geometrical law, respectively Pascal′s law and the central limit theorem.

2010 Mathematics Subject Classification: 60F05.
Key words and phrases: geometrical law, regular transformation, Pascal's law.

1 Introduction

Let $A = \|a_{n,k}\|_{n,k\in\mathbb{N}}$ be a matrix with real elements. The sequence of real numbers $(s_n)_{n\in\mathbb{N}}$ is said to be A-summable to the value $s \in \mathbb{R}$ if each of the series $\sigma_n = \sum_{k=0}^{\infty} a_{n,k} \cdot s_k$, $n = 0, 1, ...$ is convergent and if $\sigma_n \to s$ for $n \to \infty$ (or $\lim_{n\to\infty} \sigma_n = s$).

Summation method A is regular if any convergent series is A-summable to its limit.

Theorem 1. *(Toeplitz) (see[2]) Summation method A is regular if and only if:*

$$\lim_{n\to\infty} a_{n,k} = 0, \; for\, all\ k\ natural. \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1, \tag{2}$$

[1] *Received 15 October, 2024*
Accepted for publication (in revised form) 4 December, 2024

(3) $\sum_{k=0}^{n} |a_{n,k}| \leq M$, *for every* n *natural*, M *being a constant independent of* n.

Remark 1. *If the elements of matrix A are positive and $\sum_{k=0}^{\infty} a_{n,k} = 1$, for every n natural, then the conditions (2) and (3) are verified.*

Note:
- $M(f)$ represents the expectation of a random variable f,
- $D^2(f)$ represents the variance (dispersion) of a random variable f,
- $[x]$ represents the wolle part of x.

Theorem 2. *(A. Reny) (see[1], [3]) Let $f_1(t), f_2(t), ..., f_n(t), ...$ be a sequence of independent random variables which follows the same distributions. If $M = M(f_k)$ and $D = D(f_k)$, $k = 1, 2, ...$, we note by $\beta_n = f_1 + ... + f_n, \beta_n^* = \dfrac{\beta_n - M(\beta_n)}{D(\beta_n)}$ and by $F_{n,\beta^*}(x)$ the distribution function corresponding to the variable β_n^*, then*

$$\lim_{n\to\infty} F_{n,\beta^*}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt, \ (\forall) x \in \mathbb{R}. \tag{4}$$

The function $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-t^2/2} dt$, represents the standard normal distribution function.

If the random variables η_1 and η_2 verify the conditions $\eta_2 = a \cdot \eta_1 + b, a, b \in \mathbb{R}$, $a > 0$, then distribution functions of these random variables verify

$$F_{\eta_2}(x) = F_{\eta_1}\left(\frac{x-b}{a}\right). \tag{5}$$

Definition 1. *The discrete random variable X is said to follows Pascal's law if has the distribution $\begin{pmatrix} k \\ P(n,k) \end{pmatrix}_{k=0,1,2,...}$, where*

$$P(n,k) = \binom{n+k-1}{k} p^n \cdot q^k, \ p \in (0,1), \ q = 1-p. \tag{6}$$

Remark 2. *For $n = 1$, we hold that X follows the geometrical law, i.e. X has the distribution $\begin{pmatrix} k \\ p \cdot q^k \end{pmatrix}_{k=0,1,2,...}$, $p \in (0,1)$, $q = 1-p$.*

2 Principal results

When the independent random variables $f_1, ..., f_n$ follows the geometrical law $\begin{pmatrix} k \\ p \cdot q^k \end{pmatrix}_{k=0,1,2,...}$, $p \in (0,1)$, $q = 1-p$, then random variable $\beta_n = f_1 + ... + f_n$ follows Pascal′s law, i.e. it has the distribution $\begin{pmatrix} k \\ P(n,k) \end{pmatrix}_{k=0,1,...}$, $P(n,k) = \binom{n+k-1}{k} p^n \cdot q^k$.

The distribution function of the random variable β_n is $F_{n,\beta}(x) = \sum_{k=0}^{\infty} P(n,k)\theta(x-k)$, where $\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ represents Heaviside′s function.

As $M(\beta_n) = \dfrac{nq}{p}$ and $D(\beta n) = \dfrac{\sqrt{nq}}{p}$, from (5), we obtain

$$F^*_{n,\beta}(x) = \sum_{k=0}^{\infty} P(n,k)\theta\left(x\frac{\sqrt{nq}}{p} + \frac{nq\sqrt{nq}}{p^2} - k\right) = \sum_{k=0}^{\left[x\frac{\sqrt{nq}}{p} - \frac{nq\sqrt{nq}}{p^2}\right]} \binom{n+k-1}{k} p^n \cdot q^k.$$

From Theorem 2 it follows that

$$\lim_{n\to\infty} \sum_{k=0}^{\left[x\frac{\sqrt{nq}}{p} + \frac{nq\sqrt{nq}}{p^2}\right]} \binom{n+k-1}{k} p^n \cdot q^k = \Phi(x), \ (\forall)\, x \geq 0. \tag{7}$$

Remark 3. *As the function $\Phi(x)$ is bounded for $x \geq 0$, we can consider*

$$\lim_{n\to\infty} \frac{F_{n,\beta^*(x)}}{\Phi(x)} = 1, \ for\, all\ x\ real\, positive. \tag{8}$$

Let $(s_n)_{n\in\mathbb{N}}$ by any sequence of real numbers.

With the help of the result from (7) and (8), we can build the following regular transformation of the sequence $(s_n)_{n\in\mathbb{N}}$:

$$T_n^{(1)}(p;x) = \sum_{k=0}^{\left[x\frac{\sqrt{nq}}{p} + \frac{nq\sqrt{nq}}{p^2}\right]} c_{n,k} \cdot s_k \tag{9}$$

where $c_{n,k}(x) = \dfrac{1}{\Phi(x)} \binom{n+k-1}{k} p^n \cdot q^k, x \geq 0, q = 1-p, p \in (0,1)$.

For $x = 0$, we get

$$T_n^{(1)}(p;0) = 2p^n \sum_{k=0}^{\left[\frac{nq\sqrt{nq}}{p^2}\right]} \binom{n+k-1}{k} q^k s_k, \ q = 1-p, \ p \in (0,1). \tag{10}$$

Using the formula

$$\frac{1}{(1-z)^{\alpha}} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} z^k, \ |z|<1, \tag{11}$$

we can determine the following regular transformation of the sequence $(s_n)_{n\in\mathbb{N}}$

$$T_n^{(2)}(z) = (1-z)^{\alpha} \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} z^k s_k, \ |z|<1. \tag{12}$$

In (10), we put, in place of n, the quantity $n\lambda$, with $\lambda>0$, $q=z$ and in (12) we consider $\alpha = n\lambda$ with $\lambda>0$.

$$T_n^{(1)}(z,\lambda;0) = 2(1-z)^{n\lambda} \sum_{k=0}^{\left[\frac{n\lambda z\sqrt{n\lambda z}}{(1-z)^2}\right]} \binom{n\lambda+k-1}{k} z^k s_k, \ z\in(0,1), \lambda>0. \tag{13}$$

$$T_n^{(2)}(z,\lambda) = (1-z)^{n\lambda} \sum_{k=0}^{\infty} \binom{n\lambda+k-1}{k} z^k s_k, \ |z|<1, \lambda>0. \tag{14}$$

Remark 4. *The transformation* $\frac{1}{2}T_n^{(1)}(z,\lambda;0)$ *is a finite section of the transformation* $T_n^{(2)}(z,\lambda)$ *for* $z\in(0,1), \lambda>0$.

For $z=\frac{1}{2}$, we obtain

$$T_n^{(1)}\left(\frac{1}{2},\lambda;0\right) = \frac{1}{2^{n\lambda-1}} \sum_{k=0}^{\left[n\lambda\sqrt{2n\lambda}\right]} \binom{n\lambda+k-1}{k} \frac{s_k}{2^k},$$

$$T_n^{(2)}\left(\frac{1}{2},\lambda\right) = \frac{1}{2^{n\lambda}} \sum_{k=0}^{\infty} \binom{n\lambda+k-1}{k} \frac{s_k}{2^k}, \ \lambda>0.$$

3 Application

Knowing that the transformation $T_n(z) = (1-z)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k s_k$ with $|z|<1$ of the sequence $(s_n)_{n\in\mathbb{N}}$ is regular, we consider the following transformation of the sequence $(s_n)_{n\in\mathbb{N}}$ defined by

$$T_n^{(1)}(z) = (1-z)^n A \sum_{k=0}^{N_n(z)} \binom{n+k-1}{k} z^k s_k, \ \textit{with}\, z\in(0,1).$$

What conditions do A and $N_n(z)$ have to satisfy so that the transformation $T_n^{(1)}$ is regular?

Answer: From (9), for $q = z$, we get

$$T_n^{(1)}(z;x) = \frac{(1-z)^n}{\Phi(x)} \sum_{k=0}^{\left[\frac{x\sqrt{nz}}{1-x} + \frac{nz\sqrt{nx}}{(1-z)^2}\right]} \binom{n+k-1}{k} z^k s_k,\ z \geq 0,\ z \in (0,1).$$

Consequently, we consider

$$A = \frac{1}{\Phi(x)},\ N_n(z) = \left[\sqrt{n}\frac{x\sqrt{z}}{1-z} + n\sqrt{n}\frac{z\sqrt{z}}{(1-z)^2}\right]$$

with $z \geq 0$ and $z \in (0,1)$.

We have:

$$\lim_{n\to\infty} \frac{N_n(z)}{n^a} = 0,\ for\, a > 3/2$$

$$\lim_{n\to\infty} \frac{N_n(z)}{n^{3/2}} = \frac{z\sqrt{z}}{(1-z)^2},\ for\, z \in (0,1).$$

References

[1] B. V. Gnedenko, *The theory of probability*, Mir Publishers, Moskow, 1976.

[2] G. H. Hardy, *Divergent series*, Oxford, 1949.

[3] G. H. Mihoc, G. Ciucu, V. Craiu, *Teoria probabilităţilor şi statistică matematică*, EDP Bucureşti, 1970.

[4] A. Reny, *Probabilistic methods in analysis*, Mat. Lopok 18, 1967.

Daniel Florin Sofonea
Lucian Blaga University of Sibiu
Faculty of Sciences
Department of Mathematics and Informatics
No. 5-7, Dr. I. Raţiu Street, 550012, Sibiu, Romania
e-mail: florin.sofonea@ulbsibiu.ro

Ioan Ţincu
Lucian Blaga University of Sibiu
Faculty of Sciences
Department of Mathematics and Informatics
No. 5-7, Dr. I. Raţiu Street, 550012, Sibiu, Romania
e-mail: ioan.tincu@ulbsibiu.ro