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A transforming property for a subclass of analytic functions obtained by using a Libera type integral operator [1]

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Abstract

In this note we give a transforming property, by using a Libera type integral operator, for a subclass of analytic functions related with the n-uniformly starlike functions of order γ and type α.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f$ is univalent in $U\}$ and $S = \{f \in A : f$ is univalent in $U\}$.

Let D^n be the Sălăgean differential operator (see [8]) defined as:

$$(1)\qquad \begin{aligned} &D^n : A \to A\,, \quad n \in \mathbb{N} \ \text{ and } \ D^0 f(z) = f(z) \\ &D^1 f(z) = Df(z) = zf'(z)\,, \quad D^n f(z) = D(D^{n-1} f(z)). \end{aligned}$$

Remark 1 *If $f \in S$, $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$ then $D^n f(z) = z + \sum\limits_{j=2}^{\infty} j^n a_j z^j$.*

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Let denote by S^* the class of functions analytic in U, with $f(0) = 0,\ f'(0) = 1$ and which are starlike (with respect to the origin) in U, namely

$$S^* = \{f \in H(U); f(0) = f'(0) - 1 = 0,$$

$$Re\frac{zf'(z)}{f(z)} > 0, z \in U\}.$$

(see [8], for example).

Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exist a function w, which is analytic in U, $w(0) = 0$, $|w(z)| < 1$, $z \in U$, such that

$$f(z) = g(w(z)),\ (\forall) z \in U.$$

We denote by $\prec$ the subordination relation.

Remark 2 *Let f be an analytic function in U and g be an analytic and univalent function in U. Then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Let β and γ be two complex numbers, h a univalent function in U and $p(z) = h(0) + p_1 z + ...$ an analytic function in U which satisfy the subordination:

(2) $$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z).$$

This first order differential subordination is called a Briot-Bouquet differential subordination.

The differential subordinations method (also called admissible functions method) was introduced by the P. T. Mocanu and S. S. Miller in [4] and [5]. From the variously and very important results obtained by the authors (in [6] and [7]) by using the differential subordinations method, we recall here only one result regarding the Briot-Bouquet differential subordinations:

Theorem 1 *Let h be a convex function in U such that $Re[\beta h(z) + \gamma] > 0$, $z \in U$. If p is an analytic function in U, with $p(0) = h(0)$, and p satisfy the Briot-Bouquet differential subordination (2), then $p \prec h$.*

2 Prelininary results

I. Magdaş introduce in [3] the n-uniformly starlike of order γ and type α functions:

Definition 1 *We say that a function $f \in A$ is n-uniformly starlike of order γ and type α, where $\alpha \geq 0,\ \gamma \in [-1,1),\ \alpha+\gamma \geq 0$ and $n \in \mathbb{N}$ if*

$$Re\frac{D^{n+1}f(z)}{D^n f(z)} \geq \alpha\left|\frac{D^{n+1}f(z)}{D^n f(z)}-1\right|+\gamma, \tag{3}$$

for all $z \in U$.

We denote by $US_n(\alpha,\gamma)$ the class of all this functions.

Geometric interpretation of the relation (3): $f \in US_n(\alpha,\gamma)$ if and only if $D^{n+1}f(z)/D^n f(z)$ take all values in the domain $D_{\alpha,\gamma}$, where $D_{\alpha,\gamma}$ take all values in the domain $D_{\alpha,\gamma}$, where $D_{\alpha,\gamma}$ is:

i) a elliptic region:

$$\frac{\left(u-\frac{\alpha^2-\gamma}{\alpha^2-1}\right)^2}{\left[\frac{\alpha(1-\gamma)}{\alpha^2-1}\right]^2}+\frac{v^2}{\left(\frac{1-\gamma}{\sqrt{\alpha^2-1}}\right)^2}<1,\ \text{for } \alpha>1;$$

ii) a parabolic region:

$$v^2<2(1-\gamma)u-(1-\gamma^2),\ \text{for } \alpha=1;$$

iii) a hyperbolic region:

$$\frac{\left(u-\frac{\gamma-\alpha^2}{1-\alpha^2}\right)^2}{\left[\frac{\alpha(1-\gamma)}{1-\alpha^2}\right]^2}+\frac{v^2}{\left(\frac{1-\gamma}{\sqrt{1-\alpha^2}}\right)^2}>1 \text{ and } u>0,\ \text{for } 0<\alpha<1;$$

iv) the half-plane $u>\gamma$, for $\alpha=0$

Remark 3 *We remember that for the functions $f \in US_n(\alpha,\gamma)$ we have*

$$Re\left\{D^{n+1}f(z)/D^n f(z)\right\}>(\alpha+\gamma)/(\alpha+1),$$

and thus

$$US_n(\alpha,\gamma) \subset S^*.$$

This mean that the functions from $US_n(\alpha,\gamma)$ are univalent.

Remark 4 *For different choices of the parameters n,α and γ subclasses, of functions of uniformly type, previously studied by researchers in the field are obtained. For example, the subclass of uniformly convex of type α and order γ and n-uniformly starlike of type α (for details see [3]).*

Remark 5 *If we consider $h \in \mathcal{H}(\mathcal{U})$ a convex function, with $h(0) = 1$, which maps the unit disc U into the convex domain $D_{\alpha,\gamma}$, we can also define the class $US_n(\alpha, \gamma)$ such as the class of analytic functions $f \in A$ which satisfied the subordination relation:*

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec h(z), \tag{4}$$

for all $z \in U$.

3 Main results

In [1] the auhtors define the following Libera type integral operator:

$$L : A \to A$$

$$f(z) = L(F)(z) = \frac{1+a}{z^a} \int_0^z F(t)\left(t^{a-1} + t^{a+1}\right) dt\ , \tag{5}$$

where $F(z) \in A$ and $a \in \mathbb{R}^*$

Remark 6 *The above definition was inspired by the well-known integral operator defined by R. J. Libera in [2] like this:*

$$\mathbb{L} : A \to A\,,\, \mathbb{L}(f)(z) = \frac{2}{z} \int_0^z f(t)dt\,.$$

Now, we will consider the subclass of analytic functions $f \in A$ which satisfied the subordination relation:

$$\frac{D^{n+1}[f(z)(1+z^2)]}{D^n[f(z)(1+z^2)]} \prec h(z), \tag{6}$$

for all $z \in U$, where $h(z)$ in defined in Remark 5. Let denote this subclass with B.

The aim of this note is to prove that by using the integral operator defined by (5) the subclass B will be tranform in the class $US_n(\alpha, \gamma)$:

Theorem 2 *If $F(z) \in B$ then $f(z) = L(F)(z) \in US_n(\alpha, \gamma)$,where $\alpha \geq 0$ and $\gamma > 0$.*

Proof. We know that $F(z) \in B$ if and only if $\dfrac{D^{n+1}[F(z)(1+z^2)]}{D^n[F(z)(1+z^2)]}$ take all values in the convex domain included in right half plane $D_{\alpha,\gamma}$.

By differentiating in (5) we obtain

$$(1 + a) \cdot F(z)(1 + z^2) = a \cdot f(z) + zf'(z)\,.$$

By means of the application of the linear operator D^{n+1} we obtain:

$$(1+a)D^{n+1}[F(z)(1+z^2)] = a \cdot D^{n+1}f(z) + D^{n+1}(zf'(z))$$

or

$$(1+a)D^{n+1}[F(z)(1+z^2)] = a \cdot D^{n+1}f(z) + D^{n+2}f(z)$$

Similarly, by means of the application of the linear operator D^n we obtain:

$$(1+a)D^{n}[F(z)(1+z^2)] = a \cdot D^{n}f(z) + D^{n+1}f(z)$$

Thus we have:

$$\frac{D^{n+1}[F(z)(1+z^2)]}{D^n[F(z)(1+z^2)]} = \frac{D^{n+2}f(z) + aD^{n+1}f(z)}{D^{n+1}f(z) + aD^n f(z)} =$$

(7)
$$= \frac{\dfrac{D^{n+2}f(z)}{D^{n+1}f(z)} \cdot \dfrac{D^{n+1}f(z)}{D^n f(z)} + a \cdot \dfrac{D^{n+1}f(z)}{D^n f(z)}}{\dfrac{D^{n+1}f(z)}{D^n f(z)} + a}$$

With notation $\dfrac{D^{n+1}f(z)}{D^n f(z)} = p(z)$ cu $p(z) = 1 + p_1 z + ...$ we have:

$$zp'(z) = z \cdot \left(\frac{D^{n+1}f(z)}{D^n f(z)} \right)' =$$

$$= \frac{z\left(D^{n+1}f(z)\right)' \cdot D^n f(z) - D^{n+1}f(z) \cdot z\left(D^n f(z)\right)'}{\left(D^n f(z)\right)^2} =$$

$$= \frac{D^{n+2}f(z) \cdot D^n f(z) - \left(D^{n+1}f(z)\right)^2}{\left(D^n f(z)\right)^2}$$

and

$$\frac{1}{p(z)} \cdot zp'(z) = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - p(z)$$

From the above relations we obtain:

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = p(z) + \frac{1}{p(z)} \cdot zp'(z)$$

From (7) we have:

$$\frac{D^{n+1}[F(z)(1+z^2)]}{D^n[F(z)(1+z^2)]} = \frac{p(z) \cdot \left(zp'(z) \cdot \frac{1}{p(z)} + p(z) \right) + a \cdot p(z)}{p(z) + a} =$$

$$= p(z) + \frac{1}{p(z)+a} \cdot zp'(z) \tag{8}$$

If we consider the function $h(z)$ univalent in U, with $h(0) = 1$, which maps the unit disc into the convex domain included in right half plane $D_{\alpha,\gamma}$, then from $\dfrac{D^{n+1}[F(z)(1+z^2)]}{D^n[F(z)(1+z^2)]}$ take all values in $D_{\alpha,\gamma}$, using (8) we obtain:

$$p(z) + \frac{1}{p(z)+a} \cdot zp'(z) \prec h(z).$$

From her construction, we have $Re\, h(z) > 0$ and $a \in \mathbb{R}^*$ and thus $Re\,(h(z)+a) > 0$. In this conditions from Theorem 1 we obtain $p(z) \prec h(z)$.

From here we obtain that $\dfrac{D^{n+1}f(z)}{D^n f(z)}$ take all values in the convex domain included in right half plane $D_{\alpha,\gamma}$, or

$$f(z) = L(F)(z) \in US_n(\alpha,\gamma)\,,$$

where $\alpha \geq 0$ and $\gamma > 0$.

This completes the proof of the theorem.

Remark 7 *Using Remark 4 it is easy to see that similar results as in the above theorem can be obtained for the subclasses mentioned there.*

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