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More accurate estimates for the Wallis' ratio [1]

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Abstract

The aim of this work is to give an improvement of Mahmoud's inequality from [M. Mahmoud, A. Talat, H. Moustafa, R.P. Agarwal, Completely monotonic functions involving Bateman's G-function, *J. Comput. Anal. Appl.*, vol. 29, no. 5, 2021, 970-986] about some lower and upper bounds for Wallis' ratio.

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1 Introduction and motivation

One of the most important ratios, Wallis' ratio

$$P_n = \frac{1 \cdot 3 \cdot ... \cdot (2n-1)}{2 \cdot 4 \cdot ... \cdot 2n} \tag{1}$$

and

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} = \frac{1}{(2n+1) \cdot P_n^2}, \tag{2}$$

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have many applications in pure mathematics, applied statistics, statistical physics and quantum mechanics. These ratios are closely connected to the gamma function, that is given by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{3}$$

for $x > 0$, since

$$P_n = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n+1)}. \tag{4}$$

While John Wallis [41] got, in 1655, the formula for all integers $n \geq 1$

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi n}}, \tag{5}$$

as Kazarinoff [18] demonstrated firstly, in 1956, that

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{2}\right)}} < P_n \leq \frac{1}{\sqrt{\pi\left(n + \frac{1}{4}\right)}} \tag{6}$$

for all integers $n \geq 0$. Then Watson [42] and Gautschi [15] demonstrated independently, in 1959, the similar result as Kazarinoff via n continuous case. Also, Gautschi [15] obtained the inequality for all integers $n \geq 1$

$$\sqrt{n} < \frac{\Gamma(n+1)}{\Gamma\left(n + \frac{1}{2}\right)} < \sqrt{n+1}.$$

Later, in 1962, Chu [10] improved Kazarinoff's inequalities (6) as

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{16n-4}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n + \frac{1}{4} - \frac{1}{(4n-2)^2}\right)}},$$

while Boyd [3], in 1967, and Slavić [36], in 1975, obtained

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32n - \frac{64n-148}{8n+11}}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n + \frac{1}{4} + \frac{1}{32n+32}\right)}}.$$

Later, in 1983, Kershaw [19] showed the following double inequality for all integers $n \geq 1$

$$\frac{1}{\sqrt{\left(n + \frac{\sqrt{3}}{2} - \frac{1}{2}\right)}} < \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma(n+1)} < \frac{1}{\sqrt{\pi\left(n + \frac{1}{4}\right)}}.$$

Later on, in 1983, Panaitopol [33] refined (5) for all integers $n \geq 1$

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}} < P_n \leq \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}.$$

Then, in 2002, Chen and Qi [4] refined the lower bound in Wallis' inequality as

$$\frac{1}{\sqrt{\pi\left(n+\frac{2}{7}\right)}} < P_n \leq \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}} \tag{7}$$

using that the sequence

$$y_n = \frac{1}{P_n\sqrt{\pi\left(n+\frac{2}{7}\right)}}$$

is strictly increasing for all integers $n \geq 2$,

$$\lim_{n\to\infty} y_n = 1$$

and by direct computing left inequality (7) is valid for $n = 1$. Then, in 2004, Zhao De Jun [39] obtained an accurate estimate giving the following inequality:

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < P_n \leq \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{3}}\right)}}. \tag{8}$$

Then, in 2004, Lampret [20] got the following Wallis' inequality for all integers $n \geq 1$

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{11}{40n}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{5n}\right)}}. \tag{9}$$

Then, in 2004, Chen and Qi [4] obtained the bounds of Wallis' inequality as

$$\frac{1}{\sqrt{\pi\left(n+A\right)}} < P_n \leq \frac{1}{\sqrt{\pi\left(n+B\right)}}$$

for $n \in \mathbb{N}$, using that the sequence

$$Q_n = \left[\frac{\Gamma\left(n+1\right)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 - n$$

is strictly decreasing and

$$\lim_{n\to\infty} Q_n = \frac{1}{4}$$

where $A = \frac{4}{\pi} - 1$ and $B = \frac{1}{4}$, while, in 2007, Sun and Qu [37] determined the same result with another proof. Hirschhorn [17] showed, in 2005, the Wallis' inequality for all integers $n \geq 1$

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{8}{3}}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{7}{3}}\right)}}$$

as Păltănea [34] got, in 2007, for all integers $n \geq 1$

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\sqrt{\frac{2n}{2n+1}}}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\sqrt{\frac{2n+1}{2n+2}}}}.$$

In 2006, Zhao and Wu [40] improved the upper bound of (8), demonstrating that for $0 < \varepsilon < \frac{1}{2}$, it holds

$$P_n \leq \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}+\varepsilon}\right)}} \tag{10}$$

whenever $n \geq n^*(\varepsilon)$, where $n^*(\varepsilon)$ is the maximal solution of the equation

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

Later on, in 2007, Zhang, Xu and Situ [38] obtained the double inequality

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} < P_n \leq \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}}$$

for all integers $n \geq 1$. Then, in 2010, Mortici [25] obtained the double sharp inequalities for all $x \geq 1$

$$\sqrt{x+\frac{1}{4}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \omega\sqrt{x+\frac{1}{4}}$$

and

$$\mu\sqrt{x+\frac{1}{2}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x+\frac{1}{2}},$$

using that the function $f_a : (0,\infty) \to \mathbb{R}$,

$$f_a(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln(x+a), \quad a \geq 0$$

is completely monotonic for every $a \in \left[0, \frac{1}{4}\right]$ and $-f_a$ is completely monotonic for every $a \in \left[\frac{1}{2}, \infty\right)$, where $\omega = \frac{4}{\sqrt{5\pi}} = 1,009253...$ and $\mu = \frac{2\sqrt{2}}{\sqrt{3\pi}} = 0,921317...$ are the best possible constants. Then, in 2010, Mortici [28] proved the following approximation formula for Wallis's ratio

$$P_n \approx \frac{1}{\sqrt{\pi}\sqrt[8]{n^4+n^3+\frac{1}{2}n^2+\frac{1}{8}n}}.$$

Mortici [29] set, in 2012, the double inequality

$$\frac{\alpha}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < P_n \leq \frac{\beta}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} \tag{11}$$

using that the function

$$g_a(x)=\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma\left(x+1\right)}\sqrt{x\left(1+\frac{1}{4x-a}\right)}$$

is logarithmically completely monotonic on $[1,\infty)$, for every $1/2\leq a\leq 2$ and where $\alpha=1$ and $\beta=\frac{3\sqrt{7\pi}}{14}=1,00049...$ are the best possible constants. Then, in 2012, Mortici [30] obtained the double inequality

$$\frac{1}{\sqrt{\pi n\exp\left(\frac{n}{4n^2-1}+\frac{11n}{30(4n^2-1)^2}-\frac{23}{60n(4n^2-1)}-\frac{493}{107520n^3(n^2-1)^2}\right)}} < P_n$$
$$< \frac{1}{\sqrt{\pi n\exp\left(\frac{n}{4n^2-1}+\frac{11n}{30(4n^2-1)^2}-\frac{23}{60n(4n^2-1)}\right)}}.$$

Then, in 2012, Chen and Lin [6] set the following approximation for Wallis's ratio

$$P_n\approx\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}-\frac{1}{128n^2}-\frac{5}{2048n^3}+\frac{23}{8192n^4}+...\right)}},$$

while the same formula were obtained by Chen [9], Chen and Paris [8], Chen and Lin [7], Lin, Deng and Chen [22], Elezović, Lin and Vukšić [14]. Then, in 2013, S. Guo, J.-G. Xu and F. Qi [16] established that the double inequality

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n} < P_n \leq \frac{4}{3}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n}$$

is sharp for all integers $n\geq 2$ and the constants $\sqrt{\frac{e}{\pi}}$ and $\frac{4}{3}$ are the best possible, using the fact that the function

$$f(x)=\frac{x^{x+\frac{1}{2}}}{e^x\Gamma(x+1)}$$

is strictly logarithmically concave and strictly increasing from $(0,\infty)$ onto $\left(0,\frac{1}{\sqrt{2\pi}}\right)$. Then, in 2013, Qi and Mortici [35] showed that

$$\sqrt{\frac{e}{\pi}}\left[1-\frac{1}{2\left(n+1/3\right)}\right]^{n+1/3}\frac{1}{\sqrt{n}} < P_n$$

$$< \sqrt{\frac{e}{\pi}}\left[1-\frac{1}{2\left(n+1/3\right)}\right]^{n+1/3}\frac{1}{\sqrt{n}}\exp\left(\frac{1}{144n^3}\right)$$

for every integer $n \geq 1$, using the property of a function that is strictly convex and another is strictly concave and two sequences converge to 0 and are strictly increasing or strictly decreasing, respectively. Later, in 2015, Dumitrescu [13] got the following approximation for Wallis' ratio

$$P_n \approx \frac{1}{\sqrt{\pi n}}\sqrt[10]{1-\frac{5}{4n}+\frac{25}{32n^2}-\frac{35}{128n^3}+\frac{75}{2048n^4}-\frac{3}{8192n^5}}.$$

Then, in 2015, Cristea established in [11] the following inequality:

$$P_n \leq \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}+\frac{1}{48n^2}-\frac{1}{2880n^3}},$$

for all integers $n \geq 1$, while, in 2015, Cristea proved in [12] the following inequality:

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+3+\frac{7}{10n}-\frac{7}{40n^2}-\frac{809}{2800n^3}}} \leq P_n$$

$$\leq \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+3+\frac{7}{10n}-\frac{7}{40n^2}}},$$

for all integers $n \geq 2$. Later, in 2016, Mortici and Cristea proved in [31] the following inequality:

$$\frac{\exp\left(\frac{3}{512n^3}-\frac{51}{32768n^5}\right)}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} \leq P_n \leq \frac{\exp\left(\frac{3}{512n^3}\right)}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}},$$

for all integers $n \geq 2$.

In 2021, Mahmoud, Talat, Moustafa, and Agarwal in [23] have proved the following inequality

$$P_n < \frac{1}{\sqrt{\pi n}}\exp\left[\left(\frac{1}{2\sqrt{2}}\right)\left(\arctan\left(2\sqrt{2}n\right)-\frac{\pi}{2}\right)\right],$$

for all integers $n \geq 1$.

Motivated by Mahmoud, Talat, Moustafa, and Agarwal [23] and Mortici and Cristea [31], I propose the following approximation, for all integers $n \geq 1$:

$$P_n \approx \frac{1}{\sqrt{\pi n}}\exp\left[\left(\frac{1}{2\sqrt{2}}\right)\left(\arctan\left(2\sqrt{2}n\right)-\frac{\pi}{2}\right)\right]$$

$$\cdot\exp\left(-\frac{3}{2560n^5}+\frac{33}{28672n^7}\right). \tag{12}$$

In this work, we want to establish some lower and upper bounds for P_n using the above approximation. Some computations made in this paper were performed using Maple software.

2 The Results

In this section we show that

$$P_n \approx \frac{1}{\sqrt{\pi n}} \exp\left[\left(\frac{1}{2\sqrt{2}}\right)\left(\arctan\left(2\sqrt{2}n\right) - \frac{\pi}{2}\right)\right]$$
$$\cdot \exp\left(-\frac{3}{2560n^5} + \frac{33}{28672n^7}\right),$$

as $n \to \infty$, is the most accurate among all approximations of the following form

$$P_n \approx \frac{1}{\sqrt{\pi n}} \exp\left[\left(\frac{1}{2\sqrt{2}}\right)\left(\arctan\left(2\sqrt{2}n\right) - \frac{\pi}{2}\right)\right] \beta(n), \tag{13}$$

where $\beta(n) \to 1$ as $n \to \infty$. We want to find the sequences $\beta(n)$ of the form:

$$\beta(n) = \exp\left(\frac{a}{n^5} + \frac{b}{n^7}\right), \tag{14}$$

where a, b are certain real numbers. To find the best approximation (13), we set the relative error sequence v_n by the following formulas, for every $n \geq 1$:

$$P_n = \frac{1}{\sqrt{\pi n}} \exp\left[\left(\frac{1}{2\sqrt{2}}\right)\left(\arctan\left(2\sqrt{2}n\right) - \frac{\pi}{2}\right)\right] \beta(n) \exp(v_n)$$

or

$$P_n = \frac{1}{\sqrt{\pi n}} \exp\left(-\frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}n\right)\right) \beta(n) \exp(v_n)$$

and we take into account an approximation (13) better as the speed of convergence of v_n is higher. We obtain

$$v_n - v_{n+1} = \ln\frac{2n+2}{2n+1} + \frac{1}{2}\ln\frac{n}{n+1} + \frac{1}{2\sqrt{2}}\operatorname{arccot}\left(2\sqrt{2}n\right)$$
$$-\frac{1}{2\sqrt{2}}\operatorname{arccot}\left(2\sqrt{2}n + 2\sqrt{2}\right) + \left(\frac{a}{(n+1)^5} + \frac{b}{(n+1)^7}\right) - \left(\frac{a}{n^5} + \frac{b}{n^7}\right),$$

and then

$$v_n - v_{n+1} = \left(-5a - \frac{3}{512}\right)\frac{1}{n^6} + \left(15a + \frac{9}{512}\right)\frac{1}{n^7}$$
$$+\left(-35a - 7b - \frac{135}{4096}\right)\frac{1}{n^8} + \left(70a + 28b + \frac{51}{1024}\right)\frac{1}{n^9}$$
$$+\left(-126a - 84b - \frac{10\,827}{163\,840}\right)\frac{1}{n^{10}} + O\left(\frac{1}{n^{11}}\right). \tag{15}$$

The sequence v_n is the fastest possible when $v_n - v_{n+1}$ is the fastest possible; that is when the first coefficients in (15) are zero [24]. We obtain $a = -\frac{3}{2560}$, $b = \frac{33}{28672}$. Then, (15) is written in this form

$$v_n - v_{n+1} = -\frac{495}{32\,768n^{10}} + O\left(\frac{1}{n^{11}}\right).$$

As we claimed, (13) is the best approximation among all approximations (13). Now, we introduce some bounds for P_n related to this approximation (13):

Theorem 1 *We have the following inequality for every integer $n \geq 1$:*

$$\frac{1}{\sqrt{\pi n}} \exp\left(-\frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}n\right)\right) \cdot \exp\left(-\frac{3}{2560n^5}\right) < P_n \tag{16}$$
$$< \frac{1}{\sqrt{\pi n}} \exp\left(-\frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}n\right)\right) \cdot \exp\left(-\frac{3}{2560n^5} + \frac{33}{28672n^7}\right).$$

Proof. Double inequality (16) is equivalent to $a_n > 0$ for left side inequality and $b_n < 0$ for the right side inequality, where

$$a_n = \ln P_n + \frac{1}{2} \ln \pi n + \frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}n\right) + \frac{3}{2560n^5}$$

and

$$b_n = \ln P_n + \frac{1}{2} \ln \pi n + \frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}n\right) + \frac{3}{2560n^5} - \frac{33}{28672n^7}.$$

As a_n and b_n converge to 0, it suffices to prove that a_n is strictly decreasing and b_n is strictly increasing. Then, we get $a_{n+1} - a_n = f(n)$ and $b_{n+1} - b_n = g(n)$, where

$$f(x) = \ln \frac{2x+1}{2x+2} + \frac{1}{2} \ln \frac{x+1}{x} + \frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}\,(x+1)\right)$$
$$-\frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}x\right) + \frac{3}{2560\,(x+1)^5} - \frac{3}{2560x^5}$$

and

$$g(x) = \ln \frac{2x+1}{2x+2} + \frac{1}{2} \ln \frac{x+1}{x} + \frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}\,(x+1)\right)$$
$$-\frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}x\right) + \left(\frac{3}{2560\,(x+1)^5} - \frac{33}{28672\,(x+1)^7}\right)$$
$$-\left(\frac{3}{2560x^5} - \frac{33}{28672x^7}\right).$$

Then, we obtain

$$f'(x) = \frac{3P(x)}{512\,(x+1)^6\,(16x+8x^2+9)^1\,(8x^2+1)^1\,(2x+1)^1\,x^6} > 0$$

and

$$g'(x) = \left(-\frac{3}{4096}\right) \cdot \frac{Q(x)}{(x+1)^{-8}\,(16x+8x^2+9)^{-1}\,(8x^2+1)^{-1}\,(2x+1)^{-1}\,x^{-8}} < 0,$$

where

$$P(x) = 1408x^8 + 5632x^7 + 9372x^6 + 8404x^5 + 4543x^4 + 1650x^3 + 451x^2 + 88x + 9$$

and

$$Q(x) = 26\,400x^{10} + 132\,000x^9 + 294\,360x^8 + 385\,440x^7 + 330\,836x^6 + 197\,868x^5$$
$$+86\,146x^4 + 28\,192x^3 + 6924x^2 + 1166x + 99$$

are two polynomials with positive coefficients and $x \geq 1$. Thus, f is strictly increasing on $[1, \infty)$ and g is strictly decreasing on $[1, \infty)$, with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, so $f < 0$ and $g > 0$ on $[1, \infty)$. The proof is completed. □

3 Further improvements

Using (4), the double inequality (16) becomes

$$\frac{1}{\sqrt{n}} \exp\left(-\frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}n\right)\right) \exp\left(-\frac{3}{2560n^5}\right) < \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$$
$$< \frac{1}{\sqrt{n}} \exp\left(-\frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}n\right)\right) \exp\left(-\frac{3}{2560n^5} + \frac{33}{28672n^7}\right).$$

Now, we are in a position to give the following theorem:

Theorem 2 *The functions* $F : [1, \infty) \to \mathbb{R}$ *and* $G : [1, \infty) \to \mathbb{R}$, *given by*

$$F(x) = \ln\Gamma\left(x + \frac{1}{2}\right) - \ln\Gamma(x+1) + \frac{1}{2}\ln x + \frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}x\right) + \frac{3}{2560x^5}$$

and

$$G(x) = \ln\Gamma\left(x + \frac{1}{2}\right) - \ln\Gamma(x+1) + \frac{1}{2}\ln x + \frac{1}{2\sqrt{2}} \operatorname{arccot}\left(2\sqrt{2}x\right)$$
$$+\frac{3}{2560x^5} - \frac{33}{28672x^7}$$

satisfy the following assertions:

i) F is strictly convex, strictly decreasing, with $\lim_{x\to\infty} F(x) = 0$.

ii) G is strictly concave, strictly increasing, with $\lim_{x\to\infty} G(x) = 0$.

Proof. Using the result of Alzer [2], who showed that the following functions are completely monotonic on $(0, \infty)$ for all integers $n \geq 1$:

$$F_n(x) = \ln x - \left(x - \frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - \sum_{i=1}^{2n} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}}$$

and

$$G_n(x) = -\ln x + \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{i=1}^{2n-1} \frac{B_{2i}}{2i\,(2i-1)\,x^{2i-1}},$$

where B_i's are the Bernoulli numbers given by

$$\frac{x}{e^x - 1} = \sum_{i=1}^{\infty} B_i \frac{x^i}{i!}$$

and

$$\psi(x) = \ln x - \frac{1}{2x} - \sum_{i=1}^{\infty} \frac{B_{2i}}{2ix^{2i}}.$$

Then, we obtain bounds for ψ' function for every real number $x > 0$:

$$a(x) < \psi'(x) < b(x), \tag{17}$$

where

$$a(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}$$

and

$$b(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9} + \frac{5}{66x^{11}}.$$

Then, we get

$$F''(x) = \psi'(x + \frac{1}{2}) - \psi'(x+1) + \frac{9}{256x^7} - \frac{1}{2x^2} + \frac{16x}{16x^2 + 64x^4 + 1}$$

and

$$G''(x) = \psi'(x + \frac{1}{2}) - \psi'(x+1) + \frac{9}{256x^7} - \frac{1}{2x^2} - \frac{33}{512x^9} + \frac{16x}{16x^2 + 64x^4 + 1}.$$

Using (17), we have $F''(x) > u(x)$ and $G''(x) < v(x)$, where

$$u(x) = \frac{P_1(x)}{295\,680\,(2x+1)^9\,(x+1)^7\,(8x^2+1)^2\,x^7} > 0$$

and

$$v(x) = \left(-\frac{1}{591\,360}\right) \cdot \frac{Q_1(x)}{(x+1)^{11}\,(2x+1)^9\,(8x^2+1)^2\,x^9} < 0,$$

with

$$\begin{aligned} P_1(x) = {} & 624\,476\,160x^{22} + 9679\,380\,480x^{21} + 68\,836\,544\,000x^{20} + 298\,370\,320\,640x^{19} \\ & + 879\,027\,045\,120x^{18} + 1876\,032\,429\,312x^{17} + 3029\,499\,056\,320x^{16} \\ & + 3820\,404\,596\,512x^{15} + 3853\,840\,911\,648x^{14} + 3170\,160\,000\,000x^{13} \end{aligned}$$

$$+2160\,922\,472\,358x^{12}+1237\,093\,794\,077x^{11}+601\,174\,398\,131x^{10}$$
$$+249\,821\,399\,947x^{9}+89\,088\,681\,379x^{8}+27\,239\,707\,458x^{7}+7098\,013\,230x^{6}$$
$$+1556\,628\,150x^{5}+281\,122\,380x^{4}+40\,280\,625x^{3}+4293\,135x^{2}+301\,455x+10\,395$$

and

$$Q_1(x)=5836\,902\,400x^{22}+149\,227\,366\,400x^{21}+1090\,581\,273\,600x^{20}$$
$$+4300\,386\,077\,184x^{19}+11\,124\,501\,203\,456x^{18}+20\,732\,654\,447\,104x^{17}$$
$$+29\,411\,567\,984\,256x^{16}+32\,950\,459\,780\,416x^{15}+29\,947\,971\,350\,352x^{14}$$
$$+22\,539\,413\,889\,320x^{13}+14\,268\,197\,095\,948x^{12}+7684\,829\,737\,666x^{11}$$
$$+3549\,295\,702\,160x^{10}+1411\,790\,986\,029x^{9}+483\,921\,124\,995x^{8}$$
$$+142\,452\,211\,440x^{7}+35\,726\,187\,420x^{6}+7525\,945\,350x^{5}$$
$$+1300\,698\,630x^{4}+177\,428\,790x^{3}+17\,893\,260x^{2}+1181\,565x+38\,115$$

are two polynomials with positive coefficients and $x \geq 1$. Then it results, that F is strictly convex and G is strictly concave. Furthermore, F' is strictly increasing and G' is strictly decreasing on $[1,\infty)$ and $\lim_{x\to\infty} F'(x)=\lim_{x\to\infty} G'(x)=0$, so $F'(x)<0$ and $G'(x)>0$ on $[1,\infty)$. It follows that F is strictly decreasing and G is strictly increasing on $[1,\infty)$ and $\lim_{x\to\infty} F(x)=\lim_{x\to\infty} G(x)=0$. The proof is completed.$\square$

Corollary 1 *The following double inequality holds for $x \geq 1$:*

$$\frac{1}{\sqrt{\pi x}}\exp\left(-\frac{1}{2\sqrt{2}}\operatorname{arccot}\left(2\sqrt{2}x\right)\right)\exp\left(-\frac{3}{2560x^5}\right)<\frac{1}{\sqrt{\pi x}}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}$$
$$<\frac{1}{\sqrt{\pi x}}\exp\left(-\frac{1}{2\sqrt{2}}\operatorname{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}+\frac{33}{28672x^7}\right).$$

Proof. Using the **Theorem 2 (i)**, we get that F is strictly convex, strictly decreasing, with $\lim_{x\to\infty} F(x)=0$, then $F(x)>0$. It follows that

$$\frac{1}{\sqrt{\pi x}}\exp\left(-\frac{1}{2\sqrt{2}}\operatorname{arccot}\left(2\sqrt{2}x\right)\right)\exp\left(-\frac{3}{2560x^5}\right)<\frac{1}{\sqrt{\pi x}}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}.$$

Using the **Theorem 2 (ii)**, we obtain that G is strictly concave, strictly increasing, with $\lim_{x\to\infty} G(x)=0$, then $G(x)<0$. It follows that

$$\frac{1}{\sqrt{\pi x}}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}<\frac{1}{\sqrt{\pi x}}\exp\left(-\frac{1}{2\sqrt{2}}\operatorname{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}+\frac{33}{28672x^7}\right).$$

Corollary 2 *i) For all real numbers $x \geq 1$, we have the following double inequality:*

$$\frac{\alpha}{\sqrt{\pi x}}\exp\left(-\frac{1}{2\sqrt{2}}\text{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}+\frac{33}{28672x^7}\right)<\frac{1}{\sqrt{\pi x}}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}$$
$$<\frac{1}{\sqrt{\pi x}}\exp\left(-\frac{1}{2\sqrt{2}}\text{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}+\frac{33}{28672x^7}\right),$$

where the constant $\alpha=\exp(G(1))=\frac{1}{2}\sqrt{\pi}\cdot\exp\left(\frac{1}{4}\sqrt{2}\text{arccot}2\sqrt{2}+\frac{3}{143\,360}\right)=0.999\,389$ *is sharp.*

ii) For all real numbers $x\geq 1$*, we have the following double inequality:*

$$\frac{1}{\sqrt{\pi x}}\exp\left(-\frac{1}{2\sqrt{2}}\text{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}\right)<\frac{1}{\sqrt{\pi x}}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)}$$
$$<\frac{\beta}{\sqrt{\pi x}}\exp\left(-\frac{1}{2\sqrt{2}}\text{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}\right),$$

where the constant $\beta=\exp(F(1))=\frac{1}{2}\sqrt{\pi}\cdot\exp\left(\frac{1}{4}\sqrt{2}\text{arccot}2\sqrt{2}+\frac{3}{2560}\right)=1.000\,54$ *is sharp.*

Proof. Using the **Theorem 2 (ii)**, we obtain G is strictly concave, strictly increasing, with $\lim_{x\to\infty}G(x)=0$, then

$$G(1)\leq G(x)<0.$$

Then, we obtain

$$\frac{\alpha}{\sqrt{\pi x}}\cdot\exp\left(-\frac{1}{2\sqrt{2}}\text{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}+\frac{33}{28672x^7}\right)\leq\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma\left(x+1\right)}$$
$$<\frac{1}{\sqrt{\pi x}}\cdot\exp\left(-\frac{1}{2\sqrt{2}}\text{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}+\frac{33}{28672x^7}\right),$$

where

$$\alpha=\exp(G(1))=\frac{1}{2}\sqrt{\pi}\cdot\exp\left(\frac{1}{4}\sqrt{2}\text{arccot}2\sqrt{2}+\frac{3}{143\,360}\right)=0.999\,389.$$

Using the **Theorem 2 (i)**, we obtain F is strictly convex, strictly decreasing, with $\lim_{x\to\infty}F(x)=0$, then

$$F(1)\geq F(x)>0.$$

Then, we obtain

$$\frac{1}{\sqrt[\pi]{x}}\cdot\exp\left(-\frac{1}{2\sqrt{2}}\text{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}\right)\leq\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma\left(x+1\right)}$$
$$<\frac{\beta}{\sqrt[\pi]{x}}\cdot\exp\left(-\frac{1}{2\sqrt{2}}\text{arccot}\left(2\sqrt{2}x\right)\right)\cdot\exp\left(-\frac{3}{2560x^5}\right),$$

where

$$\beta = \exp(F(1)) = \frac{1}{2}\sqrt{\pi} \cdot \exp\left(\frac{1}{4}\sqrt{2}\mathrm{arccot}2\sqrt{2} + \frac{3}{2560}\right) = 1.000\,54.$$

This ends the proof.□

Let us consider the following sequences: by Wallis

$$w_n = \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}},$$

by Kazarinoff

$$k_{1n} = \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}},$$

by Gautschi and Watson

$$g_{1n} = \frac{1}{\sqrt{\pi\left(n+1\right)}},$$

by Chu

$$j_n = \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}-\frac{1}{(4n-2)^2}\right)}},$$

by Boyd and Slavić

$$b_n = \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n+32}\right)}},$$

by Kershaw

$$k_{2n} = \frac{1}{\sqrt{\pi\left(n+\frac{\sqrt{3}}{2}-\frac{1}{2}\right)}},$$

by Panaitopol

$$p_{1n} = \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}\right)}},$$

by Zhao De Jun and Mortici

$$z_{1n} = \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}},$$

by Lampret

$$l_n = \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{5n}\right)}},$$

by Chen and Qi

$$y_{1n} = \frac{1}{\sqrt{\pi\left(n+\frac{2}{7}\right)}},$$

$$y_{2n} = \frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}},$$

by Hirchhorn

$$h_n = \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\left(1-\frac{1}{4n+\frac{7}{3}}\right)}},$$

by Păltănea

$$p_{2n} = \frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)\sqrt{\frac{2n}{2n+1}}}},$$

by Zhao and Wu

$$z_{2n} = \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}+\frac{1}{n+1}}\right)}},$$

where in (10), we put $\varepsilon = \frac{1}{n+1}$ and for all integers $n \geq 1$, by Zhang, Xu and Situ

$$z_{3n} = \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}},$$

by Guo, Xu and Qi

$$x_n = \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n \frac{\sqrt{n-1}}{n},$$

by Mortici

$$r_{1n} = \frac{1}{\sqrt{\pi n \exp\left(\frac{n}{4n^2-1}+\frac{11n}{30(4n^2-1)^2}-\frac{23}{60n(4n^2-1)}\right)}},$$

$$r_{2n} = \frac{1}{\sqrt{\pi}\sqrt[8]{n^4+n^3+\frac{1}{2}n^2+\frac{1}{8}n}},$$

by Qi and Mortici

$$q_n = \sqrt{\frac{e}{\pi}}\left[1-\frac{1}{2\left(n+1/3\right)}\right]^{n+1/3}\frac{1}{\sqrt{n}},$$

by Chen, Lin, Paris, Deng, Elezović and Vukšić

$$y_{3n} = \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}+\frac{1}{32n}-\frac{1}{128n^2}-\frac{5}{2048n^3}+\frac{23}{8192n^4}\right)}},$$

by Dumitrescu

$$d_n = \frac{1}{\sqrt{\pi n}} \sqrt[10]{1 - \frac{5}{4n} + \frac{25}{32n^2} - \frac{35}{128n^3} + \frac{75}{2048n^4} - \frac{3}{8192n^5}},$$

by Mortici and Cristea

$$r_{3n} = \frac{\exp\left(\frac{3}{512n^3} - \frac{51}{32768n^5}\right)}{\sqrt{\pi n\left(1 + \frac{1}{4n - \frac{1}{2}}\right)}},$$

by Mahmoud, Talat, Moustafa, and Agarwal

$$t_n = \frac{1}{\sqrt{\pi n}} \exp\left[\left(\frac{1}{2\sqrt{2}}\right)\left(\arctan\left(2\sqrt{2}n\right) - \frac{\pi}{2}\right)\right],$$

by Cristea

$$c_{1n} = \frac{1}{\sqrt{e\pi n}}\left(1 + \frac{1}{2n}\right)^{n - \frac{1}{12n} + \frac{1}{48n^2} - \frac{1}{2880n^3}},$$

$$c_{2n} = \frac{1}{\sqrt{e\pi n}}\left(1 + \frac{1}{2n}\right)^{n - \frac{1}{12n + 3 + \frac{7}{10n} - \frac{7}{40n^2} - \frac{809}{2800n^3}}},$$

$$c_{3n} = \frac{1}{\sqrt{\pi n}} \exp\left[\left(\frac{1}{2\sqrt{2}}\right)\left(\arctan\left(2\sqrt{2}n\right) - \frac{\pi}{2}\right)\right] \cdot \exp\left(-\frac{3}{2560n^5} + \frac{33}{28672n^7}\right),$$

$$c_{4n} = \frac{1}{\sqrt{\pi n}} \exp\left(-\frac{1}{2\sqrt{2}}\operatorname{arccot}\left(2\sqrt{2}n\right)\right) \cdot \exp\left(-\frac{3}{2560n^5}\right).$$

We give the following tables with the above sequences:

n	$\lvert\ln\frac{P_n}{w_n}\rvert$	$\lvert\ln\frac{P_n}{k_{1n}}\rvert$	$\lvert\ln\frac{P_n}{g_{1n}}\rvert$
2	$4.968\,11 \times 10^{-2}$	$2.999\,2 \times 10^{-3}$	$0.140\,842$
100	$1.243\,78 \times 10^{-3}$	$1.554\,69 \times 10^{-6}$	$3.725\,17 \times 10^{-3}$
1000	$1.249\,38 \times 10^{-4}$	$1.561\,72 \times 10^{-8}$	$3.747\,5 \times 10^{-4}$
10000	$1.249\,94 \times 10^{-5}$	$1.562\,42 \times 10^{-10}$	$3.749\,75 \times 10^{-5}$

n	$\lvert\ln\frac{P_n}{j_n}\rvert$	$\lvert\ln\frac{P_n}{b_n}\rvert$	$\lvert\ln\frac{P_n}{k_{2n}}\rvert$
2	$9.210\,46 \times 10^{-3}$	$6.897\,29 \times 10^{-4}$	$2.214\,14 \times 10^{-2}$
100	$1.586\,18 \times 10^{-6}$	$1.152\,31 \times 10^{-8}$	$5.767\,91 \times 10^{-4}$
1000	$1.564\,85 \times 10^{-8}$	$31.169\,90 \times 10^{-11}$	$5.797\,92 \times 10^{-5}$
10000	$1.562\,73 \times 10^{-10}$	$1.171\,68 \times 10^{-14}$	$5.800\,94 \times 10^{-6}$

n	$\lvert\ln\frac{P_n}{p_{1n}}\rvert$	$\lvert\ln\frac{P_n}{z_{1n}}\rvert$	$\lvert\ln\frac{P_n}{l_n}\rvert$
2	$4.610\,19 \times 10^{-4}$	$6.908\,52 \times 10^{-4}$	$2.999\,2 \times 10^{-3}$
100	$3.908\,53 \times 10^{-9}$	$5.859\,22 \times 10^{-9}$	$2.427\,75 \times 10^{-4}$
1000	$3.906\,49 \times 10^{-12}$	$5.859\,37 \times 10^{-12}$	$2.492\,75 \times 10^{-5}$
10000	$3.906\,27 \times 10^{-15}$	$5.859\,37 \times 10^{-15}$	$2.499\,28 \times 10^{-6}$

n	$\lvert \ln \frac{P_n}{y_{1n}} \rvert$	$\lvert \ln \frac{P_n}{y_{2n}} \rvert$	$\lvert \ln \frac{P_n}{h_n} \rvert$
2	4.87498×10^{-3}	2.13865×10^{-3}	1.21029×10^{-3}
100	1.76540×10^{-4}	1.14340×10^{-4}	5.21651×10^{-7}
1000	1.78367×10^{-5}	1.16011×10^{-5}	5.20920×10^{-9}
10000	1.78551×10^{-6}	1.16179×10^{-6}	5.20842×10^{-11}

n	$\lvert \ln \frac{P_n}{p_{2n}} \rvert$	$\lvert \ln \frac{P_n}{z_{2n}} \rvert$	$\lvert \ln \frac{P_n}{z_{3n}} \rvert$
2	6.10483×10^{-3}	1.81856×10^{-3}	1.11983×10^{-3}
100	3.10941×10^{-6}	2.50807×10^{-8}	1.03889×10^{-8}
1000	3.12344×10^{-8}	2.53594×10^{-11}	1.04139×10^{-11}
10000	3.12484×10^{-10}	2.53875×10^{-14}	1.04164×10^{-14}

n	$\lvert \ln \frac{P_n}{x_n} \rvert$	$\lvert \ln \frac{P_n}{r_{1n}} \rvert$	$\lvert \ln \frac{P_n}{r_{2n}} \rvert$
2	0.360047	1.66876×10^{-5}	6.06146×10^{-6}
100	5.02936×10^{-3}	2.29247×10^{-17}	9.61916×10^{-16}
1000	5.00292×10^{-4}	2.29252×10^{-24}	9.75098×10^{-22}
10000	5.00029×10^{-5}	3.78653×10^{-29}	1.04761×10^{-27}

n	$\lvert \ln \frac{P_n}{q_n} \rvert$	$\lvert \ln \frac{P_n}{y_{3n}} \rvert$	$\lvert \ln \frac{P_n}{d_n} \rvert$
2	8.20746×10^{-4}	1.15016×10^{-6}	2.27382×10^{-5}
100	6.94314×10^{-9}	3.92037×10^{-16}	9.36997×10^{-16}
1000	6.94433×10^{-12}	4.03126×10^{-22}	9.24540×10^{-22}
10000	6.94443×10^{-15}	4.16519×10^{-28}	6.4371×10^{-28}

n	$\lvert \ln \frac{P_n}{r_{3n}} \rvert$	$\lvert \ln \frac{P_n}{t_n} \rvert$	$\lvert \ln \frac{P_n}{c_{1n}} \rvert$
2	7.06719×10^{-6}	2.98205×10^{-5}	3.26954×10^{-5}
100	1.18573×10^{-17}	1.17176×10^{-13}	1.17430×10^{-13}
1000	1.18577×10^{-24}	1.17187×10^{-18}	1.17213×10^{-18}
10000	6.31089×10^{-29}	1.17188×10^{-23}	1.17192×10^{-23}

n	$\lvert \ln \frac{P_n}{c_{2n}} \rvert$	$\lvert \ln \frac{P_n}{c_{3n}} \rvert$	$\lvert \ln \frac{P_n}{c_{4n}} \rvert$
2	3.26954×10^{-5}	2.19116×10^{-6}	6.80063×10^{-6}
100	1.17430×10^{-13}	1.67808×10^{-21}	1.15078×10^{-17}
1000	1.17213×10^{-18}	1.13596×10^{-28}	1.15085×10^{-24}
10000	1.17192×10^{-23}	5.04871×10^{-29}	5.04871×10^{-29}

Using the values from the above tables, we note that the sequences given by Cristea $(c_{3n})_{n\geq 2}$ and $(c_{4n})_{n\geq 2}$ have more accuracy.

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