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Hermite-Hadamard Type Inequalities for Fractional Integrals by Means of Linear Functionals [1]

Daniel Ianoşi

Abstract

In this paper we present necessary and sufficient conditions for Hermite-Hadamard type inequalities to hold for linear and positive functionals. In particular, the results from [M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57 (9–10), 2013, 2403-2407] are obtained.

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1 Introduction

Convexity of functions is a property that has been generalized in many ways and for which more and more applications have been found and many applications are still being found.

Definition 1 *The function $f : [a, b] \to \mathbb{R}$ is called a convex function on $[a, b]$ if the following inequality is verified:*

$$f\left(\alpha x + (1-\alpha)y\right) \leq \alpha f(x) + (1-\alpha)f(y),$$

for any $x, y \in [a, b]$ and $\alpha \in [0, 1]$.

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If f is concave, then the inequality holds in the reverse direction.
The inequalities discovered by Hermite and Hadamard for convex functions have been intensively researched and appear frequently in the literature (see [3]). These inequalities give estimates of the mean value of continuous convex functions by stating that if $f : I \to \mathbb{R}$ is a convex function on the interval I of the real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is concave, then both inequalities are reversed. We note that the Hermite-Hadamard inequalities follow easily from Jensen's inequality and can be considered a refinement of the concept of convexity. The Hermite-Hadamard inequalities have received a lot of attention recently, and a wide variety of generalizations and refinements have been presented over time (see [3]). For the treatment of Hermite-Hadamard inequalities from the perspective of linear functionals (see [3], [5]) and their references.

We recall notions from the theory of fractional integral calculus.

Definition 2 *Let $f \in L_1[a,b]$ be a function with $a \geq 0$. The Riemann–Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

and respectively

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, x < b, \tag{1.2}$$

where $\Gamma(x)$ is Gamma function.

We also note that: $J^{0}_{a^+}f(x) = J^{0}_{b^-}f(x) = f(x)$.
For results related to fractional integral inequalities, see [1], [4] and [7].
In [7], results related to Hermite-Hadamard type inequalities are presented using Riemann-Liouville fractional integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$.

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{1.3}$$

with $\alpha > 0$.

Also, in [7] the following lemma, which provides, using fractional integrals, equality (2.1) from Lemma 2.1. from [2] is proved.

Lemma 1 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]=\frac{b-a}{2}\int_{0}^{1}\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(ta+(1-t)b\right)dt. \tag{1.4}$$

Using equality (1.4), the following theorem is proved in [7], which evaluates the right side of the Hermite-Hadamard inequality using fractional integrals.

Theorem 2 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\leq\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{1.5}$$

We note that for $\alpha = 1$ in (1.5) we obtain the inequality (2.3) from Theorem 2.2. from [2].

In [6] U.S. Kirmaci presents the following lemma, by which is proved an equality concerning the left side of the Hermite-Hadamard inequalities.

Lemma 2 *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{1}{b-a}\int_{a}^{b}f(x)dx-f\left(\frac{a+b}{2}\right) \tag{1.6}$$

$$=(b-a)\left[\int_{0}^{\frac{1}{2}}tf'\left(ta+(1-t)b\right)dt+\int_{\frac{1}{2}}^{1}(t-1)f'\left(ta+(1-t)b\right)dt\right].$$

Using this lemma, U.S. Kirmaci proves in [6] the following theorem.

Theorem 3 *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If $f' \in L[a, b]$ and $\mid f' \mid$ is convex on $[a, b]$, then the following equality holds:*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{1.7}$$

The aim of this paper is to establish Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals by means of linear and positive functionals. Estimates for both sides of the Hermite-Hadamard inequality via linear functionals are also given.

2 Hermite-Hadamard inequalities using linear and positive functionals

In this section we give a necessary and sufficient condition for obtaining Hermite-Hadamard type inequalities.

Let $A : L_1[0,1] \to \mathbb{R}$ be a linear and positive functional with $A(e_0) = 1$. We will further use that $A(e_0) = 1$, but we note that the following results also hold if $A(e_0) \neq 1$, putting instead of A the functional $F = \frac{A}{A(e_0)}$.

Theorem 4 *The necessary and sufficient condition for the double inequality to hold:*

$$f\left(\frac{1}{2}\right) \leq A(f) \leq \frac{f(0)+f(1)}{2}, \tag{2.1}$$

for any $f : [0,1] \to \mathbb{R}$ *convex function, is* $A(e_1) = \frac{1}{2}$.

Proof. If the function f is convex it is known that:

$$f(\lambda) \leq A(f)(x) \leq (1-\lambda)f(0) + \lambda f(1), \tag{2.2}$$

for any $\lambda \in [0,1]$.

For $\lambda = \frac{1}{2}$ is obtained $f\left(\frac{1}{2}\right) \leq A(f) \leq \frac{f(0)+f(1)}{2}$, that is (2.1).

If (2.1) holds for any convex function on $[0,1]$, then it also holds for the function $f_1(x) = x$ and it is obtained $\frac{1}{2} \leq A(f_1(x)) \leq \frac{1}{2}$.

Therefore it is obtained $A(e_1) = \frac{1}{2}$.

Remark 1 *If for any* $f \in L_1[a,b]$ *convex function the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq A(f) \leq \frac{f(a)+f(b)}{2}, \tag{2.3}$$

where A *is a linear and positive functional, then* $A(e_1) = \frac{a+b}{2}$.

Remark 2 *If we take the Riemann-Liouville fractional integrals* (1.2)*, then the result* (1.3) *of Theorem 1 is obtained, i.e*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2}.$$

Theorem 5 *Let* $A : L_1[0,1] \to \mathbb{R}$ *be a linear and positive functional with* $A(e_1) = \frac{1}{2}$. *If* f *is differentiable on* $(0,1)$ *and* $|f'|$ *is convex on* $[0,1]$*, then*

$$\left|\frac{f(0)+f(1)}{2} - A(f)\right| \leq \int_0^1 \left[(1-t)\left|f'(0)\right| + t\left|f'(1)\right|\right] \cdot \left|\frac{1}{2} - A\left(u(x-t)\right)\right| dt, \tag{2.4}$$

where u *is Heaviside's function.*

Proof. Let $B(f) = \frac{f(0)+f(1)}{2} - A(f)$ be a linear functional, which has the degree of accuracy greater than or equal to 1, i.e $B(e_0) = B(e_1) = 0$. We use Peano's theorem and we have:

$$B(f) = \int_0^1 B\left(u(x-t)\right) f'(t)dt, \tag{2.5}$$

where B acts on x.

For $B\left(u(x-t)\right) = \frac{u(-t)+u(1-t)}{2} - A\left(u(x-t)\right) = \frac{1}{2} - A\left(u(x-t)\right)$, substituting in (2.5) we get:

$$B(f) = \int_0^1 \left[\frac{1}{2} - A\left(u(x-t)\right)\right] f'(t)dt,$$

from where

$$|B(f)| \leq \int_0^1 \left|\frac{1}{2} - A\left(u(x-t)\right)\right| \left|f'(t)\right| dt. \tag{2.6}$$

Since $|f'(t)| = |f'\left((1-t)\cdot 0 + t\cdot 1\right)| \leq (1-t)\left|f'(0)\right| + t\left|f'(1)\right|$, then from (2.6) it is obtained

$$\left|\frac{f(0)+f(1)}{2} - A(f)\right| \leq \int_0^1 \left[(1-t)\left|f'(0)\right| + t\left|f'(1)\right|\right] \cdot \left|\frac{1}{2} - A\left(u(x-t)\right)\right| dt,$$

that is relation (2.4).

Remark 3 *If the linear functional $A(g) = A\left(g(1-\cdot)\right)$ is considered, then*

$$\left|\frac{g(0)+g(1)}{2} - A(g)\right| \leq \frac{|g'(0)| + |g'(1)|}{2} \int_0^1 \left|\frac{1}{2} - A\left(g(x-t)\right)\right| dt. \tag{2.7}$$

Remark 4 *If we take the Riemann-Liouville fractional integrals, given in* (1.2) *in the relation* (2.7)*, then the relation* (1.5) *is obtained (see Theorem 3 in [7]).*

For the left side of the Hermite-Hadamard inequalities, using the Riemann-Liouville fractional integrals, we have the following results.

Lemma 3 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{2^{\alpha-1}\cdot\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a)\right] - f\left(\frac{a+b}{2}\right)$$

$$= 2^{\alpha-1}(b-a)\left[\int_0^{\frac{1}{2}} t^\alpha f'\left(ta+(1-t)b\right)dt - \int_{\frac{1}{2}}^1 (1-t)^\alpha f'\left(ta+(1-t)b\right)dt\right]. \tag{2.8}$$

Proof. We denote $I_1 = \int_0^{\frac{1}{2}} t^\alpha f'(ta + (1-t)b)\, dt$ and $I_2 = \int_{\frac{1}{2}}^1 (1-t)^\alpha f'(ta + (1-t)b)\, dt$. Integrating by parts we have:

$I_1 = \int_0^{\frac{1}{2}} t^\alpha f'(ta + (1-t)b)\, dt = \frac{\left(\frac{1}{2}\right)^\alpha f\left(\frac{a+b}{2}\right)}{a-b} - \frac{\alpha}{a-b} \int_0^{\frac{1}{2}} t^{\alpha-1} f(ta + (1-t)b)\, dt$. After we change the variable $ta + (1-t)b = x$, we get

$$I_1 = \frac{\left(\frac{1}{2}\right)^\alpha f\left(\frac{a+b}{2}\right)}{a-b} - \frac{\alpha}{a-b} \int_b^{\frac{a+b}{2}} \left(\frac{x-b}{a-b}\right)^{\alpha-1} f(x) \cdot \frac{1}{a-b} dx$$

$$= \frac{\left(\frac{1}{2}\right)^\alpha f\left(\frac{a+b}{2}\right)}{a-b} - \frac{\alpha}{(b-a)^{\alpha+1}} \int_{\frac{a+b}{2}}^b (b-x)^{\alpha-1} f(x)\, dx$$

$$= -\frac{f\left(\frac{a+b}{2}\right)}{2^\alpha(b-a)} + \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b). \tag{2.9}$$

Similarly, we get $I_2 = \int_{\frac{1}{2}}^1 (1-t)^\alpha f'(ta + (1-t)b)\, dt$
$= \frac{f\left(\frac{a+b}{2}\right)}{2^\alpha(b-a)} + \frac{\alpha}{a-b} \int_{\frac{1}{2}}^1 (1-t)^{\alpha-1} f(ta + (1-t)b)\, dt$.
After we change the variable $ta + (1-t)b = x$, we get

$$I_2 = -\frac{\left(\frac{1}{2}\right)^\alpha f\left(\frac{a+b}{2}\right)}{a-b} + \frac{\alpha}{a-b} \int_{\frac{a+b}{2}}^a \left(\frac{a-x}{a-b}\right)^{\alpha-1} f(x) \cdot \frac{1}{a-b} dx$$

$$= \frac{\left(\frac{1}{2}\right)^\alpha f\left(\frac{a+b}{2}\right)}{b-a} - \frac{\alpha}{(b-a)^{\alpha+1}} \int_{\frac{a+b}{2}}^a (x-a)^{\alpha-1} f(x)\, dx$$

$$= \frac{f\left(\frac{a+b}{2}\right)}{2^\alpha(b-a)} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a). \tag{2.10}$$

We have: $I_1 - I_2 = -\frac{f\left(\frac{a+b}{2}\right)}{2^\alpha(b-a)} + \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) - \frac{f\left(\frac{a+b}{2}\right)}{2^\alpha(b-a)} + \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a)$
$= \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] - \frac{f\left(\frac{a+b}{2}\right)}{2^{\alpha-1}(b-a)}$.
Thus, by multiplying both sides by $2^{\alpha-1}(b-a)$ we get

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) + J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) \right] - f\left(\frac{a+b}{2}\right) = 2^{\alpha-1}(b-a)\left(I_1 + (-I_2)\right)$$

$$= 2^{\alpha-1}(b-a) \left[\int_0^{\frac{1}{2}} t^\alpha f'(ta + (1-t)b)\, dt - \int_{\frac{1}{2}}^1 (1-t)^\alpha f'(ta + (1-t)b)\, dt \right].$$

Remark 5 *If in Lemma 3, we take $\alpha = 1$, then equality* (2.8) *becomes equality* (1.6) *of Lemma 2.*

Using Lemma 3 the following result is obtained.

Theorem 6 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{2^{\alpha-1}\cdot\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4(\alpha+1)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{2.11}$$

Proof. Using Lemma 3, we find:

$$\left|\tfrac{2^{\alpha-1}\cdot\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\tfrac{a+b}{2}\right)\right|$$
$$=\left|2^{\alpha-1}(b-a)\right|\cdot\left|\int_0^{\frac{1}{2}}t^{\alpha}f'\left(ta+(1-t)b\right)dt-\int_{\frac{1}{2}}^{1}(1-t)^{\alpha}f'\left(ta+(1-t)b\right)dt\right|$$
$$\leq 2^{\alpha-1}(b-a)\cdot\left[\int_0^{\frac{1}{2}}\left|t^{\alpha}\right|\cdot\left|f'\left(ta+(1-t)b\right)\right|dt+\int_{\frac{1}{2}}^{1}\left|(1-t)^{\alpha}\right|\left|f'\left(ta+(1-t)b\right)\right|dt\right].$$

Further, using the fact that $|f'|$ is convex we have:

$$\left|\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\cdot(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq 2^{\alpha-1}(b-a)\cdot\left[\int_0^{\frac{1}{2}}t^{\alpha}\left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dt+\int_{\frac{1}{2}}^{1}(1-t)^{\alpha}\left(t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right)dt\right]$$

$$=2^{\alpha-1}(b-a)$$

$$\cdot\left[\int_0^{\frac{1}{2}}t^{\alpha+1}\left|f'(a)\right|dt+\int_0^{\frac{1}{2}}t^{\alpha}(1-t)\left|f'(b)\right|dt+\int_{\frac{1}{2}}^{1}(1-t)^{\alpha}t\left|f'(a)\right|dt+\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+1}\left|f'(b)\right|dt\right]$$

$$=2^{\alpha-1}(b-a)\cdot\left(J_1+J_2+J_3+J_4\right).$$

Calculating J_1, J_2, J_3, J_4 is obtained (2.11).

We note that for $\alpha = 1$ in (2.11) we obtain an inequality similar to inequality (2.3) from theorem 2.2. from [2].

For $A : L_1[0,1] \to \mathbb{R}$ a linear and positive functional the result from Theorem 6 is generalized by the following theorem.

Theorem 7 *Let $A : L_1[0,1] \to \mathbb{R}$ be a linear and positive functional with $A(e_1) = \frac{1}{2}$. If f is differentiable on $(0,1)$ and $|f'|$ is convex on $[0,1]$, then*

$$\left|A(f)-f\left(\frac{1}{2}\right)\right|\leq\int_0^{\frac{1}{2}}\left[(1-t)\left|f'(0)\right|+t\left|f'(1)\right|\right]\cdot\left|1-A\left(u(x-t)\right)\right|dt$$

$$+\int_{\frac{1}{2}}^{1}\left[(1-t)\left|f'(0)\right|+t\left|f'(1)\right|\right]\cdot A\left(u(x-t)\right)dt, \tag{2.12}$$

where u is Heaviside's function.

Proof. We consider the functional $B(f) = A(f) - f\left(\frac{1}{2}\right)$, which is a linear functional with the degree of accuracy greater or equal to 1, i.e $B(e_0) = B(e_1) = 0$. From Peano's theorem we have

$$B(f) = \int_0^1 B\left(u(x-t)\right) f'(t)dt, \tag{2.13}$$

where B acts on x.

For $B\left(u(x-t)\right) = A\left(u(x-t)\right) - u\left(\frac{1}{2} - t\right)$, substituting in (2.13) we get:

$$B(f) = \int_0^1 \left[A\left(u(x-t)\right) - u\left(\frac{1}{2} - t\right)\right] f'(t)dt,$$

and we have

$$|B(f)| \le \int_0^1 \left|A\left(u(x-t)\right) - u\left(\frac{1}{2} - t\right)\right| \cdot \left|f'(t)\right| dt. \tag{2.14}$$

Since $|f'(t)| = |f'\left((1-t)\cdot 0 + t \cdot 1\right)| \le (1-t)\left|f'(0)\right| + t\left|f'(1)\right|$, then from (2.14) it is obtained

$$\begin{aligned}|B(f)| = \left|A(f) - f\left(\frac{1}{2}\right)\right| &\le \int_0^1 \left[(1-t)\left|f'(0)\right| + t\left|f'(1)\right|\right] \cdot \left|A\left(u(x-t)\right) - u\left(\frac{1}{2} - t\right)\right| dt \\ &\le \int_0^{\frac{1}{2}} \left[(1-t)\left|f'(0)\right| + t\left|f'(1)\right|\right] \cdot \left|1 - A\left(u(x-t)\right)\right| dt \\ &\quad + \int_{\frac{1}{2}}^1 \left[(1-t)\left|f'(0)\right| + t\left|f'(1)\right|\right] \cdot A\left(u(x-t)\right) dt,\end{aligned}$$

which is relation (2.12).

Remark 6 *If we take the Riemann-Liouville fractional integrals, given in* (1.2) *in the relation* (2.12)*, then the relation* (2.11) *is obtained (see Theorem 6).*

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Daniel Ianoși
Technical University of Cluj-Napoca
Faculty of Automation and Computer Science
Department of Mathematics
Str. George Baritiu, Nr. 26-28, Cluj-Napoca, România
e-mail: daniel.ianosi@gmail.com