General Mathematics Vol. 32, No. 2 (2024), 1–15

DOI: 10.2478/gm-2024-0008

sciendo

Estimates of the Initial Taylor-Maclaurin Coefficients of a general subclass of Bi-Bazilevič type functions [1]

Nagamangala Sathyananda Tejas, Halit Orhan, Dasanur Shivanna Raju, Nanjundan Magesh

Abstract

This paper introduces a novel subclass of generalized bi-Bazilevič functions, which unifies and bridges several well-known subclasses, such as bi-starlike and bi-convex functions. We derive bounds for the initial coefficients of the defined class through the generating function of Horadam polynomials and establish estimates for the famous Fekete–Szegö inequality. Additionally, we provide relevant connections to previously established results. By specializing the parameters in our main results, we obtain further interesting findings that extend the scope of this study.

2010 Mathematics Subject Classification: 30C45, 30C50, 30C80, 33C45.

Key words and phrases: Analytic functions, Bi-univalent functions, Bi-Bazilevič functions, Bi-Starlike and Bi-Convex functions, Fekete-Szegö estimate, Horadam polynomials.

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions f defined on the unit disk $\mathcal{U} = \{\zeta : \zeta \in \mathbb{C}, \, |\zeta| < 1\}$. The subclass $\mathcal{A}$ consists of functions $f \in \mathcal{H}$ of the form

$$(1) \qquad f(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n, \qquad \zeta \in \mathcal{U},$$

[1] *Received 3 December, 2024*

Accepted for publication (in revised form) 21 December, 2024

subject to the normalization conditions $f(0) = 0$ and $f'(0) = 1$. Additionally, let $\mathcal{S}$ represent the class of all functions in $\mathcal{A}$ that are univalent in $\mathcal{U}$.

For two functions f_1 and f_2 in $\mathcal{H}$, we say f_1 is subordinate to f_2 in $\mathcal{U}$ if $\exists$ an analytic function ϕ with $\phi(0) = 0$ and $|\phi(\zeta)| < 1$, satisfying $f_1(\zeta) = f_2(\phi(\zeta))$. This subordination is denoted by $f_1 \prec f_2$. If f_2 is univalent in $\mathcal{U}$, then

$$f_1 \prec f_2 \qquad (z \in \mathcal{U}) \;\Leftrightarrow\; f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathcal{U}) \subset f_2(\mathcal{U}).$$

The class of starlike functions, $\mathcal{S}^*$, and the class of convex functions, $\mathcal{C}$, are among the most well-studied subclasses of $\mathcal{S}$. These subclasses are defined as follows:

$$\mathcal{S}^* = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right) > 0, \quad \zeta \in \mathcal{U} \right\}$$

and

$$\mathcal{C} = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} \right) > 0, \quad \zeta \in \mathcal{U} \right\}.$$

Another noteworthy subclass of $\mathcal{S}$, which serves as a bridge between $\mathcal{S}^*$ and $\mathcal{C}$, was introduced by Miller et al. in [21] and Darus and Thomas in [8]. These subclasses are defined as:

$$\mathcal{M}(\kappa) = \left\{ f \in \mathcal{S} : \operatorname{Re}\left((1-\kappa)\frac{\zeta f'(\zeta)}{f(\zeta)} + \kappa\left(1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} \right) \right) > 0, \quad \zeta \in \mathcal{U} \right\}$$

and

$$\mathcal{M}^\nu = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\left(1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} \right)^\nu \left(\frac{\zeta f'(\zeta)}{f(\zeta)} \right)^{1-\nu} \right) > 0, \quad \zeta \in \mathcal{U} \right\}.$$

The class $\mathcal{M}(\kappa)$ was initially studied in [17, 22], where it was shown that $\mathcal{M}(\kappa) \subset \mathcal{S}^*$. Similarly, [8] established that $\mathcal{M}^\nu \subset \mathcal{S}^*$.

A function $f \in \mathcal{A}$ is called a Bazilevič function [6] of type (κ, ν), where $\kappa > 0$ and $\nu \in \mathbb{R}$, if it is represented as

$$f(\zeta) = \left((\kappa + i\nu) \int_0^\zeta [f_1(t)]^\kappa p(t) t^{i\nu - 1} dt \right)^{1/(\kappa + i\nu)}, \quad \zeta \in \mathcal{U}, \tag{2}$$

where $f_1 \in \mathcal{S}^*$, $p \in \mathcal{H}$ with $p(0) = 1$, and $\operatorname{Re}[p(\zeta)] > 0$ for all $\zeta \in \mathcal{U}$. The class of Bazilevič functions of type (κ, ν) is denoted by $\mathcal{B}(\kappa, \nu)$. Bazilevič [6] proved that $\mathcal{B}(\kappa, \nu)$ is a subclass of $\mathcal{S}$. The definition in (2) can also be expressed equivalently as:

$$\mathcal{B}(\kappa, \nu) = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{\zeta f'(\zeta)}{f(\zeta)} \left(\frac{f(\zeta)}{f_1(\zeta)} \right)^\kappa \left(\frac{f(\zeta)}{\zeta} \right)^{i\nu} \right) > 0, \quad \zeta \in \mathcal{U} \right\},$$

where $f_1 \in \mathcal{S}^*$. By choosing suitable choice of parameters, the following subclasses of $\mathcal{S}$ can be derived:

1. The class $\mathcal{B}(\kappa,0)\equiv\mathcal{B}(\kappa)$, was introduced and examined by Singh [29] known as the Bazilevič functions of type κ.
2. The class $\mathcal{B}(1,0)\equiv\mathcal{K}$ corresponds to the class of close-to-convex functions.
3. When $f_1(\zeta)=\zeta$ and $\nu=0$, the resulting class $\mathcal{B}_1(\kappa)$ was introduced and analyzed by Singh [29]. Furthermore, $\mathcal{B}_1(0)\equiv\mathcal{S}^*$ and $\mathcal{B}_1(1)\equiv\mathcal{R}$ represent the class of functions whose derivative has a positive real part in $\mathcal{U}$.

The Bazilevič function class $\mathcal{B}(\kappa,\nu)$ is regarded as one of the largest subclasses of univalent functions, incorporating several prominent subclasses of $\mathcal{S}$. The following inclusion relationships are well-established (see [20, 9, 28]):

$$\mathcal{C}\subset\mathcal{S}^*\subset\mathcal{K}\subset\mathcal{B}(\kappa)\subset\mathcal{B}(\kappa,\nu)\subset\mathcal{S}.$$

Considerable attention has been given to the study of Bazilevič functions. For more detailed information, one can refer to [11, 12, 16, 23, 24, 35, 25, 26] and [36].

Fitri and Thomas [12] introduced the class $\mathcal{B}_1^\nu(\kappa)$ as an analogue of $\mathcal{M}^\nu$. A function $f\in\mathcal{A}$, satisfying $f(\zeta)\neq 0$ and $f'(\zeta)\neq 0$, is said to belong to the class $\mathcal{B}_1^\nu(\kappa)$ if

(3)
$$\mathrm{Re}\left(\left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^\kappa}+\frac{\zeta f''(\zeta)}{f'(\zeta)}+(\kappa-1)\left(\frac{\zeta f'(\zeta)}{f(\zeta)}-1\right)\right]^\nu\left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^\kappa}\right]^{1-\nu}\right)>0,\quad \zeta\in\mathcal{U},$$

where $\kappa\geq 0$ and $\nu\geq 0$.

In [12, Theorem 1, p. 3], it was established that $\mathcal{B}_1^\nu(\kappa)\subset\mathcal{B}(\kappa)$, which implies that all functions in the class $\mathcal{B}_1^\nu(\kappa)$ are univalent in $\mathcal{U}$.

The Koebe one-quarter theorem [9] asserts that every function $f\in\mathcal{S}$ possesses an inverse $f^{-1}=g$, defined by

$$g(f(\zeta))=\zeta,\qquad \zeta\in\mathcal{U}, \tag{4}$$

and

$$f(g(\omega))=\omega,\qquad |\omega|<r_0(f);\quad r_0(f)\geq\frac{1}{4}, \tag{5}$$

where

$$g(\omega)=\omega+\sum_{n=2}^{\infty}a_n\omega^n=\omega-b_2\omega^2+(2b_2^2-b_3)\omega^3-(5b_2^3-5b_2b_3+b_4)\omega^4+\cdots. \tag{6}$$

If a function $f\in\mathcal{A}$ and its inverse $f^{-1}=g$ are univalent in $\mathcal{U}$, then f is said to be bi-univalent in $\mathcal{U}$. The class of all such functions is denoted by Σ. In 1967, Lewin [18] introduced the class Σ of bi-univalent analytic functions and established that every $f\in\Sigma$ satisfies $|a_2|<1.51$. Subsequently, in 1980, Brannan and Clunie

[7] conjectured that $|a_2| \leq \sqrt{2}$ for all $f \in \Sigma$. Later, in 1985, Kedzierawski [15] confirmed this conjecture for a special case in which both f and $f^{-1} = g$ are starlike functions. This result led to the following bounds:

$$|a_2| \leq \begin{cases} 1.5894 & \text{if } f \text{ and } g \in \mathcal{S}, \\ \sqrt{2} & \text{if } f \text{ and } g \in \mathcal{S}^*, \\ 1.224 & \text{if } f \in \mathcal{C} \text{ and } g \in \mathcal{S}, \\ 1.507 & \text{if } f \in \mathcal{S}^* \text{ and } g \in \mathcal{S}. \end{cases}$$

The study by Srivastava et al. [33] serves as a significant reference for revisiting the study of Σ and presenting notable examples of functions in Σ. Some well-known examples of functions in the class Σ include:

$$-\log(1-\zeta), \quad \frac{\zeta}{1-\zeta}, \quad \text{and} \quad \log\sqrt{\frac{1+\zeta}{1-\zeta}}.$$

It is significant that the famous Koebe function does not belong to Σ. Moreover, the following functions,

$$\frac{2\zeta-\zeta^2}{2} \quad \text{and} \quad \frac{\zeta}{1-\zeta^2},$$

are members of $\mathcal{S}$, yet they do not belong to Σ.

In recent mathematical literature, researchers have made significant progress in deriving initial coefficient estimates $|a_2|$ and $|a_3|$ for functions in the classes $\mathcal{S}$ and Σ, extending the foundational work of Srivastava et al. [33]. However, determining general coefficient bounds $|a_n|$ for $n \in \mathbb{N}$ with $n \geq 3$ remains an active area of research, offering valuable opportunities to explore the subfamilies of Σ. These studies also focus on functions associated with certain orthogonal polynomials, such as Horadam polynomials, Hermite polynomials, Lucas polynomials, Laguerre polynomials, Chebyshev polynomials and so on (see [3, 30, 37, 2, 38, 5, 4, 1, 19, 27]). The Horadam polynomials, in particular, have garnered significant attention due to their broad applicability, as they include several well-known polynomials as special cases.

Hörçum and Koçer [14] provided an in-depth study of the Horadam polynomials $h_n(y)$, which are defined by the recurrence relation:

$$h_n(y) = pyh_{n-1}(y) + qh_{n-2}(y), \qquad n \geq 3 \text{ and } n \in \mathbb{N}, \tag{7}$$

with the initial terms:

$$h_1(y) = a \qquad \text{and} \qquad h_2(y) = by, \tag{8}$$

where a, b, p, and q are real constants. The recurrence relation (7) leads to the characteristic equation:

$$t^2 - pyt - q = 0,$$

which has the roots:

$$y_1 = \frac{py + \sqrt{p^2y^2 + 4q}}{2} \qquad \text{and} \qquad y_2 = \frac{py - \sqrt{p^2y^2 + 4q}}{2}.$$

As noted earlier, Horadam polynomials reduces to several well-known polynomials. Further details regarding these reductions are available in [13, 14]. Moreover, the generating function of the Horadam polynomials $h_n(y)$, as derived in [14], is given by:

$$\Pi(y,\zeta) := \sum_{n=1}^{\infty} h_n(y)\zeta^{n-1} = \frac{a + (b - ap)y\zeta}{1 - py\zeta - q\zeta^2}. \tag{9}$$

Inspired by the aforementioned literature, particularly those of Abirami et al. [3] and Srivastava et al. [30], we define a subclass of Σ as follows:

Definition 1 *For $\nu \geq 0$, $\kappa \geq 0$, a function $f \in \Sigma$ of the form (1) is said to belong to the class $\mathcal{GB}_\Sigma(\kappa,\nu;y)$ if:*

$$\left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^\kappa} + \frac{\zeta f''(\zeta)}{f'(\zeta)} + (\kappa - 1)\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right)\right]^\nu \left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^\kappa}\right]^{1-\nu} \prec \Pi(y,\zeta) + 1 - a,$$

and for $g(\omega) = f^{-1}(\omega)$ defined by (6),

$$\left[\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^\kappa} + \frac{\omega g''(\omega)}{g'(\omega)} + (\kappa - 1)\left(\frac{\omega g'(\omega)}{g(\omega)} - 1\right)\right]^\nu \left[\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^\kappa}\right]^{1-\nu} \prec \Pi(y,\omega) + 1 - a,$$

holds for all $\zeta, \omega \in \mathcal{U}$. Here, a and b are real constants as defined in (8).

Remark 1 *It is worth noting that subclasses of analytic bi-univalent functions, such as bi-Bazilevič, bi-Starlike, bi-Convex functions and so on can be derived by assigning specific values to κ and ν in Definition 1:*

1. *The class $\mathcal{GB}_\Sigma(0,0;y) \equiv \mathcal{S}^*_\Sigma(y)$ consists of functions $f \in \Sigma$ satisfying:*

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \prec \Pi(y,\zeta) + 1 - a \quad \textit{and} \quad \frac{\omega g'(\omega)}{g(\omega)} \prec \Pi(y,\omega) + 1 - a,$$

   were discussed in [30, 3].

2. *The class $\mathcal{GB}_\Sigma(0,1;y) \equiv \mathcal{K}_\Sigma(y)$ consists of functions $f \in \Sigma$ satisfying:*

$$1 + \frac{\zeta f''(\zeta)}{f'(\zeta)} \prec \Pi(y,\zeta) + 1 - a \quad \textit{and} \quad 1 + \frac{\omega g''(\omega)}{g'(\omega)} \prec \Pi(y,\omega) + 1 - a,$$

   was studied in [3].

3. *The class $\mathcal{GB}_\Sigma(\kappa,0;y) \equiv \mathcal{R}_\Sigma(y)$ consists of functions $f \in \Sigma$ satisfying:*

$$f'(\zeta) \prec \Pi(y,\zeta) + 1 - a \quad \textit{and} \quad g'(\omega) \prec \Pi(y,\omega) + 1 - a.$$

4. *The class* $\mathcal{GB}_\Sigma(1,1;y) \equiv \mathcal{T}_\Sigma(y)$ *consists of functions* $f \in \Sigma$ *satisfying:*

$$f'(\zeta) + \frac{\zeta f''(\zeta)}{f'(\zeta)} \prec \Pi(y,\zeta) + 1 - a \quad \text{and} \quad g'(\omega) + \frac{\omega g''(\omega)}{g'(\omega)} \prec \Pi(y,\omega) + 1 - a.$$

5. *The class* $\mathcal{GB}_\Sigma(0,\nu;y) \equiv \mathcal{M}_\Sigma(\nu,y)$ *consists of functions* $f \in \Sigma$ *satisfying:*

$$\left[\frac{\zeta f'(\zeta)}{f(\zeta)}\right]^{1-\nu}\left[1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right]^{\nu} \prec \Pi(y,\zeta) + 1 - a \quad \text{and}$$

$$\left[\frac{\omega g'(\omega)}{g(\omega)}\right]^{1-\nu}\left[1 + \frac{\omega g''(\omega)}{g'(\omega)}\right]^{\nu} \prec \Pi(y,\omega) + 1 - a,$$

was considered in [3].

6. *The class* $\mathcal{GB}_\Sigma(\kappa,0;y) \equiv \mathcal{B}_\Sigma(\kappa,y)$ *consists of functions* $f \in \Sigma$ *satisfying:*

$$\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^{\kappa}} \prec \Pi(y,\zeta) + 1 - a \quad \text{and} \quad \frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^{\kappa}} \prec \Pi(y,\omega) + 1 - a.$$

For all the above classes, $\zeta, \omega \in \mathcal{U}$*,* f *is of the form* (1)*,* $g = f^{-1}$ *is defined by* (6) *and* a *and* b *are real constants as defined in* (8)*.*

Remark 2 *The geometric properties of the function in the class* $\mathcal{GB}_\Sigma(\kappa,\nu;y)$ *depend on the specific values assigned to the parameters. For instance, if we set* $a = p = y = 1$*,* $b = 2$*, and* $q = 0$*, we obtain*

$$\left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^{\kappa}} + \frac{\zeta f''(\zeta)}{f'(\zeta)} + (\kappa-1)\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right)\right]^{\nu}\left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^{\kappa}}\right]^{1-\nu} \prec \frac{1+\zeta}{1-\zeta} \qquad (\zeta \in \mathcal{U}).$$

In this case, the function f *maps the open unit disk* $\mathcal{U}$ *onto the half-plane defined by Re* $\left(\frac{+\zeta}{-\zeta}\right) > 0$*, since the expression* $\left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^{\kappa}} + \frac{\zeta f''(\zeta)}{f'(\zeta)} + (\kappa-1)\left(\frac{\zeta f'(\zeta)}{f(\zeta)} - 1\right)\right]^{\nu}\left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^{\kappa}}\right]^{1-\nu}$ *takes values in the half-plane.*

Remark 3 *By setting* $a = 1$*,* $b = p = 2$*,* $q = -1$*, and letting* $y \to t$*, the generating function in* (9) *simplifies to the generating function of the Chebyshev polynomials of the second kind,* $U_n(t)$*. These polynomials are defined as:*

$$U_n(t) = (n+1)\,{}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-t}{2}\right) = \frac{\sin(n+1)\varphi}{\sin\varphi}, \quad \text{where } t = \cos\varphi,$$

using the celebrated Gauss hypergeometric function ${}_2F_1$*. In this view, the class* $\mathcal{S}^*_\Sigma(y)$ *stated in Remark 1 reduces to the class* $\mathcal{S}^*_\Sigma(t)$*, which has been previously studied in [1, 19].*

2 Coefficient Estimates

In this section, we find the coefficient estimates for the defined class $\mathcal{GB}_\Sigma(\kappa,\nu;y)$ as follows:

Theorem 1 *If $\kappa \geq 0$, $\nu \geq 0$, and $f \in \mathcal{GB}_\Sigma(\kappa,\nu;y)$ is of the form (1), then*

$$|a_2| \leq \frac{\sqrt{2}|by|\sqrt{by}}{\sqrt{|(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)b^2y^2-2(bpy^2+aq)(1+\kappa)^2(1+\nu)^2|}},$$

and

$$|a_3| \leq \frac{|by|}{(2+\kappa)(1+2\nu)} + \frac{b^2y^2}{(1+\kappa)^2(1+\nu)^2},$$

where

$$\sigma_1 = \kappa^2+2\kappa+1, \quad \sigma_2 = 1-\kappa^2, \quad \textit{and} \quad \sigma_3 = \kappa^2+3\kappa+2. \tag{10}$$

Proof. Let $f \in \mathcal{GB}_\Sigma(\kappa,\nu;y)$ be given by (1). Then there exist analytic functions $u(\zeta) = \sum_{n=1}^{\infty} c_n\zeta^n$ and $v(\omega) = \sum_{n=1}^{\infty} d_n\omega^n$, such that

$$u(0)=0, \qquad v(0)=0, \qquad |u(\zeta)|<1, \qquad \text{and} \qquad |v(\omega)|<1, \quad (\zeta,\omega \in \mathcal{U}). \tag{11}$$

Furthermore, it is well known that

$$|c_n| \leq 1 \qquad \text{and} \qquad |d_n| \leq 1. \tag{12}$$

Let

$$F(\zeta) := \left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^\kappa} + \frac{\zeta f''(\zeta)}{f'(\zeta)} + (\kappa-1)\left(\frac{\zeta f'(\zeta)}{f(\zeta)}-1\right)\right]^\nu \left[\frac{\zeta f'(\zeta)}{f^{1-\kappa}(\zeta)\zeta^\kappa}\right]^{1-\nu},$$

and

$$G(\omega) := F^{-1}(\omega) = \left[\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^\kappa} + \frac{\omega g''(\omega)}{g'(\omega)} + (\kappa-1)\left(\frac{\omega g'(\omega)}{g(\omega)}-1\right)\right]^\nu \left[\frac{\omega g'(\omega)}{g^{1-\kappa}(\omega)\omega^\kappa}\right]^{1-\nu}.$$

In view of (1), it can be computed that:

$$F(\zeta) = 1+(1+\kappa)(1+\nu)a_2\zeta + \left(\left[\frac{M\nu^2+N\nu+S}{2}\right]a_2^2 + (\kappa+2)(2\nu+1)a_3\right)\zeta^2 + \cdots, \tag{13}$$

where

$$M = \kappa^2+2\kappa+1, \quad N = -\kappa^2-4\kappa-7, \quad \text{and} \quad S = \kappa^2+\kappa-2.$$

Similarly,

(14)
$$G(\omega) = 1-(1+\kappa)(1+\nu)a_2\omega + \left(\left[\frac{\tilde{M}\nu^2+\tilde{N}\nu+\tilde{S}}{2}\right]a_2^2-(\kappa+2)(2\nu+1)a_3\right)\omega^2+\cdots,$$

where

$$\tilde{M}=\kappa^2+2\kappa+1, \quad \tilde{N}=-\kappa^2+4\kappa+9, \quad \text{and} \quad \tilde{S}=\kappa^2+5\kappa+6.$$

By Definition 1, we can write:

$$F(\zeta)=\Pi(y,v(\zeta))+1-a \quad \text{and} \quad G(\omega)=\Pi(y,u(\omega))+1-a.$$

Equivalently,

$$F(\zeta)=1+h_1(y)-a+h_2(y)u(\zeta)+h_3(y)[u(\zeta)]^2+\cdots, \tag{15}$$

and

$$G(\omega)=1+h_1(y)-a+h_2(y)v(\omega)+h_3(y)[v(\omega)]^2+\cdots. \tag{16}$$

From (15) and (16), and in view of (9), we have:

$$F(\zeta)=1+h_2(y)c_1\zeta+\big[h_2(y)c_2+h_3(y)c_1^2\big]\zeta^2+\cdots, \tag{17}$$

and

$$G(\omega)=1+h_2(y)d_1\omega+\big[h_2(y)d_2+h_3(y)d_1^2\big]\omega^2+\cdots. \tag{18}$$

Comparing the corresponding coefficients in (13) and (17), as well as in (14) and (18), we obtain:

$$(1+\kappa)(1+\nu)a_2 = h_2(y)c_1, \tag{19}$$

$$\left[\frac{M\nu^2+N\nu+S}{2}\right]a_2^2+(\kappa+2)(2\nu+1)a_3 = h_2(y)c_2+h_3(y)c_1^2, \tag{20}$$

$$-(1+\kappa)(1+\nu)a_2 = h_2(y)d_1, \tag{21}$$

$$\left[\frac{\tilde{M}\nu^2+\tilde{N}\nu+\tilde{S}}{2}\right]a_2^2-(\kappa+2)(2\nu+1)a_3 = h_2(y)d_2+h_3(y)d_1^2. \tag{22}$$

From (19) and (21), we get:

$$c_1 = -d_1, \tag{23}$$

$$2(1+\kappa)^2(1+\nu)^2a_2^2 = [h_2(y)]^2(c_1^2+d_1^2). \tag{24}$$

Adding (20) and (22), we get:

$$\left[\frac{(M+\tilde{M})\nu^2+(N+\tilde{N})\nu+(S+\tilde{S})}{2}\right]a_2^2 = h_2(y)(c_2+d_2)+h_3(y)(c_1^2+d_1^2). \tag{25}$$

Substituting (24) into (25), we get:

$$a_2^2 = \frac{[h_2(y)]^3(c_2+d_2)}{(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)[h_2(y)]^2 - 2h_3(y)(1+\kappa)^2(1+\nu)^2}. \tag{26}$$

In view of (8), (12), and (26), we have:

$$|a_2| \leq \frac{\sqrt{2}|by|\sqrt{by}}{\sqrt{|(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)b^2y^2 - 2(bpy^2+aq)(1+\kappa)^2(1+\nu)^2|}}.$$

Subtracting (22) from (20), we get:

$$a_3 = \frac{h_2(y)(c_2-d_2)}{2(2+\kappa)(1+2\nu)} + a_2^2. \tag{27}$$

From (27) and (24), we have:

$$a_3 = \frac{h_2(y)(c_2-d_2)}{2(2+\kappa)(1+2\nu)} + \frac{[h_2(y)]^2(c_1^2+d_1^2)}{2(1+\kappa)^2(1+\nu)^2}. \tag{28}$$

From (8), (12), and (28), we have:

$$|a_3| \leq \frac{|by|}{(2+\kappa)(1+2\nu)} + \frac{b^2y^2}{(1+\kappa)^2(1+\nu)^2}. \tag{29}$$

Theorem 1 leads to the following corollaries, as a consequence of Remark 1.

Corollary 1 *[30, Corollary 1, p. 5] If $f \in \mathcal{S}^*_\Sigma(y)$ and is of the form as in* (1), *then*

$$|a_2| \leq \frac{|by|\sqrt{|by|}}{\sqrt{|(b-p)by^2 - aq|}} \quad \text{and} \quad |a_3| \leq \frac{|by|}{2} + b^2y^2.$$

Corollary 2 *[3, Corollary 3, p. 5] If $f \in \mathcal{K}_\Sigma(y)$ and is of the form as in* (1), *then*

$$|a_2| \leq \frac{|by|\sqrt{|by|}}{\sqrt{|(2b-4p)by^2 - 4aq|}} \quad \text{and} \quad |a_3| \leq \frac{|by|}{6} + \frac{b^2y^2}{4}.$$

Corollary 3 *If $f \in \mathcal{R}_\Sigma(y)$ and is of the form as in* (1), *then*

$$|a_2| \leq \frac{\sqrt{2}|by|\sqrt{|by|}}{\sqrt{|6b^2y^2 - 8(bpy^2+aq)|}} \quad \text{and} \quad |a_3| \leq \frac{|by|}{3} + \frac{b^2y^2}{4}.$$

Corollary 4 *If $f \in \mathcal{T}_\Sigma(y)$ and is of the form as in* (1), *then*

$$|a_2| \leq \frac{\sqrt{2}|by|\sqrt{|by|}}{\sqrt{|10b^2y^2 - 32(bpy^2+aq)|}} \quad \text{and} \quad |a_3| \leq \frac{|by|}{9} + \frac{b^2y^2}{16}.$$

Corollary 5 *If $f \in \mathcal{M}_\Sigma(\nu, y)$ and is of the form as in* (1), *then*

$$|a_2| \leq \frac{\sqrt{2}|by|\sqrt{|by|}}{\sqrt{|(\nu^2+\nu+2)b^2y^2 - 2(bpy^2+aq)(1+\nu)^2|}} \quad \text{and} \quad |a_3| \leq \frac{|by|}{1+2\nu} + \frac{b^2y^2}{(1+\nu)^2}.$$

Corollary 6 *If $f \in \mathcal{B}_\Sigma(\kappa, y)$ and is of the form as in* (1), *then*

$$|a_2| \leq \frac{\sqrt{2}|by|\sqrt{|by|}}{\sqrt{|(\kappa^2+3\kappa+2)b^2y^2 - 2(bpy^2+aq)(1+\kappa)^2|}} \quad \text{and} \quad |a_3| \leq \frac{|by|}{2+\kappa} + \frac{b^2y^2}{(1+\kappa)^2}.$$

3 Fekete-Szegö Estimates

In the following theorem, we derive the famous Fekete-Szegö Estimate for the defined class $\mathcal{GB}_\Sigma(\kappa,\nu;y)$

Theorem 2 *Let $\kappa \geq 0$, $\nu \geq 0$, and $\mu \in \mathbb{R}$. If $f \in \mathcal{GB}_\Sigma(\kappa,\nu;y)$, then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{|by|}{(2+\kappa)(1+2\nu)} \\ \qquad \text{if } |\mu-1| \leq \dfrac{|(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)b^2y^2-2(bpy^2+aq)(1+\kappa)^2(1+\nu)^2|}{2b^2y^2(2+\kappa)(1+2\nu)}, \\ \dfrac{2|by|^3|\mu-1|}{|(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)b^2y^2-2(bpy^2+aq)(1+\kappa)^2(1+\nu)^2|} \\ \qquad \text{if } |\mu-1| \geq \dfrac{|(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)b^2y^2-2(bpy^2+aq)(1+\kappa)^2(1+\nu)^2|}{2b^2y^2(2+\kappa)(1+2\nu)}, \end{cases}$$

where σ_1, σ_2, and σ_3 are the constants defined in (10).

Proof. From (27), for $\mu \in \mathbb{R}$, we write:

$$a_3-\mu a_2^2 = \frac{h_2(y)(c_2-d_2)}{2(\kappa+2)(1+2\nu)} + (1-\mu)a_2^2. \tag{30}$$

From (26) and (30), we have:

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{h_2(y)(c_2-d_2)}{2(\kappa+2)(1+2\nu)} + \frac{(1-\mu)[h_2(y)]^3(c_2+d_2)}{(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)[h_2(y)]^2-2h_3(y)(1+\kappa)^2(1+\nu)^2} \\ &= h_2(y)\left[\left(\Omega+\frac{1}{2(2+\kappa)(1+2\nu)}\right)c_2+\left(\Omega-\frac{1}{2(2+\kappa)(1+2\nu)}\right)d_2\right], \qquad (31) \end{aligned}$$

where

$$\Omega = \frac{(1-\mu)[h_2(y)]^2}{(\sigma_1\nu^2+\sigma_2\nu+\sigma_3)[h_2(y)]^2-2h_3(y)(1+\kappa)^2(1+\nu)^2}.$$

Hence, we can conclude that

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{|h_2(y)|}{(2+\kappa)(1+2\nu)}, & \text{if } 0 \leq |\Omega| \leq \dfrac{1}{2(2+\kappa)(1+2\nu)}, \\ 2|h_2(y)||\Omega|, & \text{if } |\Omega| \geq \dfrac{1}{2(2+\kappa)(1+2\nu)}. \end{cases} \tag{32}$$

In view of (8), the proof of Theorem 2 is now complete.
In the view of Remark 1, we state the following corollaries:

Corollary 7 *[30, Corollary 3, p. 5] If $f \in \mathcal{S}^*_\Sigma(y)$ and $\mu \in \mathbb{R}$, then*

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{|by|}{2}, & \text{if } |\mu-1| \leq \dfrac{|(b-p)by^2-aq|}{2b^2y^2}, \\ \dfrac{|by|^3|\mu-1|}{|(b-p)by^2-aq|}, & \text{if } |\mu-1| \geq \dfrac{|(b-p)by^2-aq|}{2b^2y^2}. \end{cases}$$

Corollary 8 *If $f \in \mathcal{K}_\Sigma(y)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|by|}{6}, & \text{if } |\mu - 1| \leq \dfrac{|(b-2p)by^2 - 2aq|}{3b^2y^2}, \\ \dfrac{|by|^3|\mu-1|}{|(b-2p)by^2 - 2aq|}, & \text{if } |\mu - 1| \geq \dfrac{|(b-2p)by^2 - 2aq|}{3b^2y^2}. \end{cases}$$

Corollary 9 *If $f \in \mathcal{R}_\Sigma(y)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|by|}{3}, & \text{if } |\mu - 1| \leq \dfrac{|3b^2y^2 - 4(bpy^2 + aq)|}{3b^2y^2}, \\ \dfrac{|by|^3|\mu-1|}{|3b^2y^2 - 4(bpy^2 + aq)|}, & \text{if } |\mu - 1| \geq \dfrac{|3b^2y^2 - 4(bpy^2 + aq)|}{3b^2y^2}. \end{cases}$$

Corollary 10 *If $f \in \mathcal{T}_\Sigma(y)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|by|}{9}, & \text{if } |\mu - 1| \leq \dfrac{|(5b-16p)by^2 - 16aq|}{9b^2y^2}, \\ \dfrac{|by|^3|\mu-1|}{|(5b-16p)by^2 - 16aq|}, & \text{if } |\mu - 1| \geq \dfrac{|(5b-16p)by^2 - 16aq|}{9b^2y^2}. \end{cases}$$

Corollary 11 *If $f \in \mathcal{M}_\Sigma(\nu, y)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|by|}{2(1+2\nu)}, \\ \qquad \text{if } |\mu - 1| \leq \dfrac{|(\nu^2+\nu+2)b^2y^2 - 2(bpy^2+aq)(1+\nu)^2|}{4b^2y^2(1+2\nu)}, \\ \dfrac{2|by|^3|\mu-1|}{|(\nu^2+\nu+2)b^2y^2 - 2(bpy^2+aq)(1+\nu)^2|}, \\ \qquad \text{if } |\mu - 1| \geq \dfrac{|(\nu^2+\nu+2)b^2y^2 - 2(bpy^2+aq)(1+\nu)^2|}{4b^2y^2(1+2\nu)}. \end{cases}$$

Corollary 12 *If $f \in \mathcal{B}_\Sigma(\kappa, y)$ and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|by|}{2+\kappa}, \\ \qquad \text{if } |\mu - 1| \leq \dfrac{|(\kappa^2+3\kappa+2)b^2y^2 - 2(bpy^2+aq)(1+\kappa)^2|}{2b^2y^2(2+\kappa)}, \\ \dfrac{2|by|^3|\mu-1|}{|(\kappa^2+3\kappa+2)b^2y^2 - 2(bpy^2+aq)(1+\kappa)^2|}, \\ \qquad \text{if } |\mu - 1| \geq \dfrac{|(\kappa^2+3\kappa+2)b^2y^2 - 2(bpy^2+aq)(1+\kappa)^2|}{2b^2y^2(2+\kappa)}. \end{cases}$$

References

[1] S. Altınkaya, S. Yalçın, *On the Chebyshev polynomial coefficient problem of some subclasses of bi-univalent functions*, Gulf J. Math., vol. 5, no. 3, 2017, 34-40, https://doi.org/10.56947/gjom.v5i3.105.

[2] S. Altınkaya, S. Yalçın, *On the (p,q)-Lucas polynomial coefficient bounds of the bi-univalent function class* Σ, Bol. Soc. Mat. Mex., vol. 25, no. 3, 2018, 567-575, https://doi.org/10.1007/s40590-018-0212-z.

[3] C. Abirami, N. Magesh, J. Yamini, *Initial bounds for certain classes of bi-univalent functions defined by Horadam polynomials*, Abstr. Appl. Anal., Art. ID 7391058, 2020, 1-8, https://doi.org/10.1155/2020/7391058.

[4] I. Al-Shbeil, A. Cătaş, H. M. Srivastava, N. Aloraini, *Coefficient estimates of new families of analytic functions associated with q-Hermite polynomials*, Axioms, vol. 12, no. 1, 2023, 52, 1-14, https://doi.org/10.3390/axioms12010052.

[5] W. Al-Rawashdeh, *A family of analytic functions subordinate to Horadam polynomials*, Eur. J. Pure Appl. Math., vol. 17, no. 1, 2024, 158-170, https://doi.org/10.29020/nybg.ejpam.v17i1.5022.

[6] I. E. Bazilevič, *On a case of integrability in quadratures of the Loewner-Kufarev equation*, Mat. Sb. (N.S.), vol. 37, no. 79, 1955, 471-476.

[7] D. Brannan, J. Clunie, *Aspects of contemporary complex analysis*, Academic Press, New York, 1980.

[8] M. Darus, D. K. Thomas, *κ-logarithmically convex functions*, Indian J. Pure Appl. Math., 29, 1998, 1049-1059.

[9] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, Springer, New York, vol. 259, 1983.

[10] M. Fekete, G. Szegö, *Eine Bemerkung über ungerade schlichte Funktionen*, J. London Math. Soc., vol. 1, no. 2, 1933, 85-89, https://doi.org/10.1112/jlms/s1-8.2.85.

[11] S. A. Fitri, M. Muslikh, *Beta-Bazilevič function*, Sahand Commun. Math. Anal., vol. 21, no. 2, 2024, 179-193, https://doi.org/10.22130/scma.2023.2004270.1351.

[12] S. A. Fitri, D. K. Thomas, *Gamma-Bazilevič functions*, Mathematics, vol. 8, no. 2, 2020, 175, 1-8, https://doi.org/10.3390/math8020175.

[13] A. F. Horadam, J. M. Mahon, *Pell and Pell–Lucas polynomials*, Fibonacci Quart., 23, 1985, 7-20.

[14] T. Horzum, E. G. Koçer, *On some properties of Horadam polynomials*, Int. Math. Forum, vol. 4, no. 25, 2009, 1243-1252.

[15] A. W. Kedzierawski, *Some remarks on bi-univalent functions*, Ann. Univ. Mariae Curie-Sk lodowska Sect. A, 39, 1985, 77-81.

[16] F. R. Keough, S. S. Miller, *On the coefficients of Bazilevič functions*, Proc. Amer. Math. Soc., 30, 1971, 492-496, https://doi.org/10.2307/2037722.

[17] P. K. Kulshrestha, *Coefficients for alpha-convex univalent functions*, Bull. Amer. Math. Soc., 80, 1974, 341-342, https://doi.org/10.1090/S0002-9904-1974-13494-4.

[18] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., 18, 1967, 63-68.

[19] N. Magesh, S. Bulut, *Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions*, Afr. Mat., 29, 2018, 203-209, https://doi.org/10.1007/s13370-017-0535-3.

[20] S. S. Miller, *On the Hardy class of a Bazilevič function and its derivative*, Proc. Amer. Math. Soc., 30, 1971, 125-138, https://doi.org/10.2307/2038236.

[21] S. S. Miller, P. T. Mocanu, M. O. Reade, *All α-convex functions are starlike*, Rev. Roumaine Math. Pure Appl., 17, 1972, 1395-1397.

[22] S. S. Miller, P. T. Mocanu, M. O. Reade, *All α-convex functions are univalent and starlike*, Proc. Amer. Math. Soc., 37, 1973, 553-554, https://doi.org/10.1090/S0002-9939-1973-0313490-3.

[23] K. I. Noor, *Bazilevič functions of type β*, Internat. J. Math. Math. Sci., vol. 5, no. 2, 1982, 411-415, https://doi.org/10.1155/S0161171282000416.

[24] K. I. Noor, S. A. Al-Bany, *On Bazilevič functions*, Internat. J. Math. Math. Sci., vol. 10, no. 1, 1987, 79-88, https://doi.org/10.1155/S0161171287000103.

[25] K. I. Noor, S. A. Shah, A. Saliu, *On generalized gamma-Bazilevič functions*, Asian Euro J Math., vol. 14, no. 6, 2021, 1-8, https://doi.org/10.1142/S1793557121500935.

[26] S. Owa, M. Obradović, *Certain subclasses of Bazilevič functions of type α*, Internat. J. Math. Math. Sci., vol. 9, no. 2, 1986, 347-359, https://doi.org/10.1155/S0161171286000431.

[27] T. Panigrahi, J. Sokól, *Generalized Laguerre polynomial bounds for subclass of bi-univalent functions*, Jordan J. Math & Stat., vol. 14, no. 1, 2021, 127-140, https://doi.org/10.47013/14.1.7.

[28] N. L. Sharma, T. Bulboacă, *Logarithmic coefficient bounds for the class of Bazilevič functions*, Anal. Math. Phys., vol. 14, no. 52, 2024, 1-18, https://doi.org/10.1007/s13324-024-00909-y.

[29] R. Singh, *On Bazilevič functions*, Proc. Am. Math. Soc., 38, 1973, 261-271, https://doi.org/10.2307/2039275.

[30] H. M. Srivastava, Ş. Altınkaya, S. Yalçın, *Certain subclasses of bi-univalent functions associated with the Horadam polynomials*, Iran. J. Sci. Technol. Trans. A Sci., vol. 43, no. 4, 2019, 1873-1879, https://doi.org/10.1007/s40995-018-0647-0.

[31] H. M. Srivastava, S. Gaboury, F. Ghanim, *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Afr. Mat., vol. 28, no. 5-6, 2017, 693-706, https://doi.org/10.1007/s13370-016-0478-0.

[32] H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, S. Hussain, *The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator*, Stud. Univ. Babeş-Bolyai Math., vol. 63, no. 4, 2018, 419-436, https://doi.org/10.24193/subbmath.2018.4.01.

[33] H. M. Srivastava, A. K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Ltt., vol. 23, no. 10, 2010, 1188-1192, https://doi.org/10.1016/j.aml.2010.05.009.

[34] H. M. Srivastava, M. F. Sakar, G. H. Özlem, *Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination*, Filomat, vol. 32, no. 4, 2018, 1313-1322, https://doi.org/10.2298/fil1804313s.

[35] H. M. Srivastava, A. K. Wanas, H. Ö. Güney, *New families of bi-univalent functions associated with the Bazilevič functions and the λ-pseudo-starlike functions*, Iran. J. Sci. Technol. Trans. A Sci., vol. 45, no. 5, 2021, 1799-1804, https://doi.org/10.1007/s40995-021-01176-3.

[36] D. K. Thomas, *On Bazilevič functions*, Trans. Amer. Math. Soc., 132, 1968, 353-361, https://doi.org/10.2307/1994845.

[37] A. K. Wanas, A. Alb Lupas, *Applications of Horadam polynomials on Bazilevič bi-univalent function satisfying subordinate conditions*, J. Phys. Conf. Series, vol. 1294, no. 3, 2019, 1-6, https://doi.org/10.1088/1742-6596/1294/3/032003.

[38] A. K. Wanas, A. Alb Lupas, *Applications of Laguerre polynomials on a new family of bi-prestarlike functions*, Symmetry, vol. 14, no. 4, 2022, 1-10, https://doi.org/10.3390/sym14040645.

Nagamangala Sathyananda Tejas
The National Institute of Engineering
Department of Mathematics
Mysore - 570008, India
e-mail: nstejas@gmail.com

Halit Orhan
Atatürk University
Faculty of Science
Department of Mathematics
25240, Erzurum, Türkiye
e-mail: horhan@atauni.edu.tr

Dasanur Shivanna Raju
Visvesvaraya Technological University
Department of Mathematics
Belagavi - 590018, India
e-mail: rajudsvm@gmail.com

Nanjundan Mages
Government Arts College for Men
Post-Graduate and Research Department of Mathematics
Krishnagiri - 635 001, Tamilnadu, India
e-mail: nmagi_2000@yahoo.co.in