



Iterative methods for solving scalar equations ¹

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Abstract

In this paper, we establish new iterative methods for the solution of scalar equations by using the decomposition technique given in [16].

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1 Introduction

In science and engineering, one of the most important problems is to find the solution of nonlinear equations. Numerous iterative methods involving various techniques have been developed to find the approximate roots of nonlinear equations, see [1, 3, 4, 5, 6, 10, 11, 13] and references therein. These methods can be categorized as one-step, two-step and three-step methods. Chun [5] has proposed some one-step and two-step iterative methods with higher-order convergence by using the decomposition technique of Adomian [2]. Several other iterative methods have also been established for finding the simple zero of nonlinear equations. The leading equations in many problems related to science and technology have roots with multiplicity greater than one. The first contribution in dealing with such equations was made by Schroder [18] who developed the following modified Newton's method:

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}.$$

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During the last two decades, much attention has been devoted by various researchers for solving nonlinear equation with multiple roots. Chun et al. [7, 9], Homeier [12], Osada [17] have developed some techniques to find the multiple roots of nonlinear equations. In the recent years, the researchers have made significant and interesting contribution in this field [8, 14, 15].

In the present paper, we explore new iterative methods using the technique of decomposition [16] for solving scalar equations with roots of multiplicity $m \geq 2$.

Consider the nonlinear equation

$$(1.1) \quad f(x) = 0.$$

It is well known that if α is a root with multiplicity m , then it is also a root of $f'(x) = 0$ with multiplicity $m - 1$, of $f''(x) = 0$ with multiplicity $m - 2$ and so on. Hence if initial guess x_0 is sufficiently close to α , the expressions

$$(1.2) \quad \begin{aligned} & x_0 - m \frac{f(x_0)}{f'(x_0)}, \\ & x_0 - (m - 1) \frac{f'(x_0)}{f''(x_0)}, \\ & x_0 - (m - 2) \frac{f''(x_0)}{f'''(x_0)}, \dots \end{aligned}$$

will have the same value.

Remark 1 *The generalized Newton's formula*

$$(1.3) \quad x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(x_n)},$$

gives a quadratic convergence when the equation $f(x) = 0$ has a pair of double roots in the neighbourhood of x_0 . It may be noted that for the double root α near to x_0 , $f(\alpha) = 0 = f'(\alpha)$.

2 Iterative methods

We can rewrite the nonlinear equation (1.1) as a coupled system:

$$(2.1) \quad f(\gamma) + (x - \gamma) \frac{f'(\gamma) + f'(x)}{2} + g(x) = 0,$$

$$(2.2) \quad g(x) = f(x) - f(\gamma) - (x - \gamma) \frac{f'(\gamma) + f'(x)}{2},$$

where γ is the initial approximation for a zero of (1.1).

By taking $g(x) = 0$, we can rewrite (2.1) in the following form:

$$(2.3) \quad \begin{aligned} x &= \gamma - 2 \frac{f(\gamma)}{f'(\gamma)} - (x - \gamma) \frac{f'(x)}{f'(\gamma)} \\ &= c + N(x), \end{aligned}$$

where

$$(2.4) \quad c = \gamma - 2 \frac{f(\gamma)}{f'(\gamma)},$$

and

$$(2.5) \quad N(x) = -(x - \gamma) \frac{f'(x)}{f'(\gamma)}.$$

Here $N(x)$ is a nonlinear operator.

As in [7], the solution of (2.3) has the series form,

$$(2.6) \quad x = \sum_{i=0}^{\infty} x_i.$$

The nonlinear operator $N(x)$ can be decomposed as it has been shown in [7]. Also the series $\sum_{i=0}^{\infty} x_i$ converges absolutely and uniformly to a unique solution of equation (2.3) if the nonlinear operator

$$(2.7) \quad N(x) = N\left(\sum_{i=0}^{\infty} x_i\right) = N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^i x_j\right) \right\}$$

is a contraction.

Combining (2.3), (2.6) and (2.7), we have

$$(2.8) \quad \sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty} \left\{ N\left(\sum_{j=0}^i x_j\right) \right\}.$$

Thus we have the following iterative scheme:

$$\begin{aligned} x_0 &= c, \\ x_1 &= N(x_0), \\ x_2 &= N(x_0 + x_1), \\ &\vdots \end{aligned}$$

$$(2.9) \quad x_{n+1} = N(x_0 + x_1 + \dots + x_n), \quad n = 1, 2, \dots$$

Then

$$x_1 + x_2 + \dots + x_{n+1} = N(x_0) + N(x_0 + x_1) + \dots + N(x_0 + x_1 + \dots + x_n), \quad n = 1, 2, \dots,$$

and

$$(2.10) \quad x = c + \sum_{i=1}^{\infty} x_i.$$

From (2.4), (2.5), and (2.9), we have

$$(2.11) \quad x_0 = c = \gamma - 2 \frac{f(\gamma)}{f'(\gamma)},$$

and

$$(2.12) \quad \begin{aligned} x_1 &= N(x_0) \\ &= -(x_0 - \gamma) \frac{f'(x_0)}{f'(\gamma)} \\ &= 2 \frac{f(\gamma)f'(x_0)}{f'^2(\gamma)}. \end{aligned}$$

It follows from (2.4), (2.9), and (2.10), that

$$(2.13) \quad \begin{aligned} x &\approx x_0 \\ &= c \\ &= \gamma - 2 \frac{f(\gamma)}{f'(\gamma)}. \end{aligned}$$

This enables us to suggest the following method for solving the nonlinear equation (1.1).

Algorithm 1 For the given x_0 compute the approximate solution x_{n+1} by the iterative schemes:

$$x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \quad n = 0, 1, 2, \dots,$$

which is known as the generalized Newton's formula and is quadratically convergent.

Again by using (2.4), (2.5), (2.9) and (2.10), we conclude that

$$\begin{aligned}
 (2.14) \quad x &\approx x_0 + x_1 \\
 &= c + x_1 \\
 &= x_0 + N(x_0) \\
 &= \gamma - 2 \frac{f(\gamma)}{f'(\gamma)} + 2 \frac{f(\gamma)f'(x_0)}{f'^2(\gamma)}.
 \end{aligned}$$

Using (2.14), we can suggest the following two-step iterative method for solving nonlinear equation (1.1) as:

Algorithm 2 For the given x_0 compute the approximate solution x_{n+1} by the iterative schemes:

$$\begin{aligned}
 y_n &= x_n - 2 \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, n = 0, 1, 2, \dots, \\
 x_{n+1} &= y_n + 2 \frac{f(x_n)f'(y_n)}{f'^2(x_n)}.
 \end{aligned}$$

Again, using (2.5), (2.11) and (2.12), we can calculate

$$\begin{aligned}
 (2.15) \quad N(x_0 + x_1) &= -(x_0 + x_1 - \gamma) \frac{f'(x_0 + x_1)}{f'(\gamma)} \\
 &= 2 \frac{f(\gamma)}{f'(\gamma)} \left(1 - \frac{f'(x_0)}{f'(\gamma)} \right) \frac{f'(x_0 + x_1)}{f'(\gamma)}.
 \end{aligned}$$

From (2.10), (2.11), (2.12), and (2.15), we get

$$\begin{aligned}
 (2.16) \quad x &\approx x_0 + x_1 + x_2 \\
 &= c + N(x_0) + N(x_0 + x_1) \\
 &= c + N(x_0) + N(x_0 + x_1) \\
 &= \gamma - 2 \frac{f(\gamma)}{f'(\gamma)} + 2 \frac{f(\gamma)f'(x_0)}{f'^2(\gamma)} - \left(\gamma - 2 \frac{f(\gamma)}{f'(\gamma)} + 2 \frac{f(\gamma)f'(x_0)}{f'^2(\gamma)} - \gamma \right) \frac{f'(x_0 + x_1)}{f'(\gamma)}.
 \end{aligned}$$

Using (2.16), we can suggest and analyze the following three-step iterative method for solving nonlinear equation (1.1).

Algorithm 3 For a given x_0 , compute the approximate solution by the iterative schemes:

$$\begin{aligned} y_n &= x_n - 2 \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, n = 0, 1, 2, \dots, \\ z_n &= 2 \frac{f(x_n)f'(y_n)}{f'^2(x_n)}, \\ x_{n+1} &= x_n \frac{f'(y_n + z_n)}{f'(x_n)} + (y_n + z_n) \left(1 - \frac{f'(y_n + z_n)}{f'(x_n)} \right). \end{aligned}$$

3 Convergence Analysis

In this section, the convergence analysis of Algorithm 2 and 3 is given.

Theorem 1 Assume that the function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval D has a multiple root $\alpha \in D$ of multiplicity 2. Let $f(x)$ be sufficiently smooth in the neighborhood of the root α , then the order of convergence of the methods defined by Algorithm 2 and 3 is 2.

Proof. Let α be a root of $f(x)$ of multiplicity 2, then by expanding $f(x_n)$, $(f'(x_n))^2$, in Taylor's series about α , we obtain

$$(3.1) \quad f(x_n) = c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5),$$

$$(3.2) \quad f'(x_n) = 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5),$$

$$(3.3) \quad (f'(x_n))^2 = 4c_2^2 e_n^2 + 12c_2 c_3 e_n^3 + (16c_2 c_4 + 9c_3^2) e_n^4 + O(e_n^5),$$

where $e_n = x_n - \alpha$ and $C_k = \frac{f^{(k)}(\alpha)}{k!}$, $k = 2, 3, \dots$

Using (3.1) and (3.2), we have

$$(3.4) \quad \frac{f(x_n)}{f'(x_n)} = \frac{1}{2} e_n - \frac{1}{4} \frac{c_3}{c_2} e_n^2 + \left(-\frac{1}{2} \frac{c_4}{c_2} + \frac{3}{8} \frac{c_3^2}{c_2^2} \right) e_n^3 + \left(-\frac{3}{4} \frac{c_5}{c_2} + \frac{5}{4} \frac{c_3 c_4}{c_2^2} - \frac{9}{16} \frac{c_3^3}{c_2^3} \right) e_n^4 + O(e_n^5).$$

Thus

$$(3.5) \quad y_n = x_n - 2 \frac{f(x_n)}{f'(x_n)} = \alpha + \frac{1}{2} \frac{c_3}{c_2} e_n^2 + \left(\frac{c_4}{c_2} - \frac{3}{4} \frac{c_3^2}{c_2^2} \right) e_n^3 + \left(\frac{3}{2} \frac{c_5}{c_2} - \frac{5}{2} \frac{c_3 c_4}{c_2^2} + \frac{9}{8} \frac{c_3^3}{c_2^3} \right) e_n^4 + O(e_n^5).$$

Expanding $f'(y_n)$ by Taylor's series about α , we get

$$(3.6) \quad f'(y_n) = \frac{c_3^2}{c_2} e_n^3 + \left(\frac{5c_3 c_4}{c_2} - \frac{15}{4} \frac{c_3^3}{c_2^2} \right) e_n^4 + \left(\frac{9c_3 c_5}{c_2} - \frac{24c_4 c_3^2}{c_2^2} + \frac{87}{8} \frac{c_3^4}{c_2^3} + \frac{6c_4^2}{c_2} \right) e_n^5 + O(e_n^6).$$

Using (3.3), (3.5) and (3.6), the error tem becomes

$$(3.7) \quad e_{n+1} = \frac{c_3}{c_2} e_n^2 + \left(\frac{2c_4}{c_2} - \frac{5}{2} \frac{c_3^2}{c_2^2} \right) e_n^3 + \left(\frac{3c_5}{c_2} - \frac{17}{2} \frac{c_3 c_4}{c_2^2} + \frac{6c_3^3}{c_2^3} \right) e_n^4 + O(e_n^5).$$

This completes the proof.

Now for Algorithm 3, we use (3.1), (3.2) and (3.6) to compute z_n as follows

$$(3.8) \quad z_n = 2 \frac{f(x_n) f'(y_n)}{f'^2(x_n)} = \frac{1}{2} \frac{c_3^2}{c_2^2} e_n^3 + \left(\frac{5}{2} \frac{c_3 c_4}{c_2^2} - \frac{23}{8} \frac{c_3^3}{c_2^3} \right) e_n^4 + O(e_n^5).$$

Using (3.5) and (3.8), we have

$$(3.9) \quad y_n + z_n = \alpha + \frac{c_3}{c_2} e_n^2 + \left(\frac{2c_4}{c_2} - \frac{5}{2} \frac{c_3^2}{c_2^2} \right) e_n^3 + \left(\frac{3c_5}{c_2} - \frac{17}{2} \frac{c_3 c_4}{c_2^2} + \frac{6c_3^3}{c_2^3} \right) e_n^4 + O(e_n^5).$$

Expanding $f'(y_n + z_n)$ by Taylor's series about α , we obtain

$$(3.10) \quad f'(y_n + z_n) = \frac{c_3^2}{c_2} e_n^3 + \left(\frac{5c_3 c_4}{c_2} - \frac{5}{4} \frac{c_3^3}{c_2^2} \right) e_n^4 + \left(\frac{9c_3 c_5}{c_2} - \frac{75c_3^4}{8c_2^3} + \frac{6c_4^2}{c_2} - \frac{3c_4 c_3^2}{c_2 c_2^2} \right) e_n^5 + O(e_n^6).$$

Thus

$$(3.11) \quad \frac{f'(y_n + z_n)}{f'(x_n)} = \frac{1}{2} \frac{c_3^2}{c_2^2} e_n^2 + \left(\frac{5}{2} \frac{c_3 c_4}{c_2^2} - \frac{11}{8} \frac{c_3^3}{c_2^3} \right) e_n^3 + \left(\frac{9}{2} \frac{c_3 c_5}{c_2^2} - \frac{21c_3^4}{8c_2^4} + \frac{3c_4^2}{c_2^2} - \frac{25c_4 c_3^2}{4c_2^3} \right) e_n^4 + O(e_n^5).$$

Now using (3.4), (3.5), (3.10) and (3.11), we have the error tem as

$$(3.12) \quad e_{n+1} = \frac{2c_3}{c_2} e_n^2 + \left(-\frac{15}{2} \frac{c_3^2}{c_2^2} + \frac{4c_4}{c_2} \right) e_n^3 + O(e_n^4).$$

The last equation shows that the convergence order of algorithm 3 is 2.

4 Numerical Examples

In this section, we present some numerical examples to illustrate the validity and efficiency of the methods proposed by us. We compare our methods with classical Newton's method (CW), generalized Newton's method (GNM), Chun et al. (BNM1 and BNM2) (see [7] page 1577, Equations (35) and (36)). All the computations for above mentioned methods, are performed using software Maple 13 and $\varepsilon = 10^{-10}$ as tolerance and also the following criteria is used for estimating the zero:

- (i) $\delta = |x_{n+1} - x_n| < \varepsilon$,
- (ii) $|f(x_n)| < \varepsilon$,
- (iii) Maximum numbers of iterations = 500.

$f(x)$	x_0	Method	No.of iterations	$x[k]$	$f(x_n)$	δ
$(x-2)^2$	1.97	NM	10	1.9999707031250000	$8.58e-10$	$2.93e-05$
		GNM	1	2.0000000000000000	$0.00e+00$	$3.00e-02$
		BNM1	1	2.0000000000000000	$0.00e+00$	$3.00e-02$
		BNM2	1	2.0000000000000000	$0.00e+00$	$3.00e-02$
		NS1	1	2.0000000000000000	$0.00e+00$	$3.00e-02$
		NS2	1	2.0000000000000000	$0.00e+00$	$3.00e-02$
$(\arctan(x))^2$	0.02	NM	10	0.0000195243074874	$3.81e-10$	$1.95e-05$
		GNM	1	-0.0000053329067398	$2.84e-11$	$2.00e-02$
		BNM1	1	0.0000026713622895	$7.14e-12$	$2.00e-02$
		BNM2	1	0.0000026617546045	$7.08e-12$	$2.00e-02$
		NS1	1	-0.0000106700806580	$1.14e-10$	$2.00e-02$
		NS2	1	-0.0000213515480087	$4.56e-10$	$2.00e-02$
$(e^x - 4x^2)^2$	-0.39	NM	12	-0.4077725133649210	$2.72e-10$	$4.20e-06$
		GNM	2	-0.4077767961647673	$1.16e-13$	$3.05e-04$
		BNM1	2	-0.4077767094044696	$1.78e-27$	$1.69e-05$
		BNM2	1	-0.407777243328640	$1.59e-11$	$1.78e-02$
		NS1	2	-0.4077774521231989	$8.51e-12$	$6.31e-04$
		NS2	2	-0.4077831682193126	$6.43e-10$	$1.31e-03$
$x^3 - x^2 - x + 1$	1.03	NM	11	1.0000148669755619	$4.42e-10$	$1.49e-05$
		GNM	2	1.0000000121033821	$2.93e-16$	$2.20e-04$
		BNM1	1	1.0000055621052384	$6.19e-11$	$3.00e-02$
		BNM2	1	1.0000009278935850	$1.72e-12$	$3.00e-02$
		NS1	2	1.0000000940022467	$1.77e-14$	$4.34e-04$
		NS2	2	1.0000007245500737	$1.05e-12$	$8.51e-04$
$(\sin(x) - \cos(x))^2$	0.7	NM	12	0.7853773818407633	$8.64e-10$	$2.08e-05$
		GNM	2	0.7853981633944398	$1.81e-23$	$2.08e-04$
		BNM1	2	0.7853981633972462	$8.17e-26$	$1.07e-04$
		BNM2	2	0.7853981633972653	$6.70e-26$	$1.03e-04$
		NS1	2	0.7853981633487803	$4.74e-21$	$4.18e-04$
		NS2	2	0.7853981626072611	$1.25e-18$	$8.40e-04$
$(x^2 + \ln x^2 + \sin(x)^2)^2$	0.62	NM	13	0.6636667994551911	$8.32e-10$	$5.43e-06$
		GNM	2	0.6636721958511422	$3.27e-14$	$4.19e-04$
		BNM1	2	0.6636722298834851	$3.77e-29$	$1.81e-05$
		BNM2	2	0.6636722298834768	$2.54e-27$	$3.14e-05$
		NS1	2	0.6636719672474031	$1.95e-12$	$8.22e-04$
		NS2	2	0.6636702205106710	$1.14e-10$	$1.61e-03$
$x^3 - 3x + 2$	1.06	NM	12	1.0000149394272540	$6.70e-10$	$1.49e-05$
		GNM	2	1.0000000565392868	$9.59e-15$	$5.82e-04$
		BNM1	2	1.0000000000000007	$1.51e-30$	$1.94e-05$
		BNM2	1	1.0000033833477819	$3.43e-11$	$6.00e-02$
		NS1	2	1.0000004348814882	$5.67e-13$	$1.14e-03$
		NS2	2	1.0000033111293263	$3.29e-11$	$2.23e-03$
$f(x)$	x_0	Method	No.of iterations	$x[k]$	$f(x_n)$	δ
$(x - e^{\cos x})^2$	1.23	NM	13	1.3029548560201398	$4.26e-10$	$9.15e-06$
		GNM	2	1.3029638361281054	$1.39e-13$	$9.27e-04$
		BNM1	2	1.3029640012159444	$2.37e-26$	$7.09e-05$
		BNM2	2	1.3029640012160073	$1.40e-28$	$4.04e-05$
		NS1	2	1.3029627487769376	$7.99e-12$	$1.81e-03$
		NS2	2	1.3029546324863432	$4.47e-10$	$3.50e-03$
$(x - \tan x)^2$	0.05	NM	1	0.0416722242070108	$5.83e-10$	$8.33e-03$
		GNM	1	0.0333444484140216	$1.53e-10$	$1.67e-02$
		BNM1	1	0.0288991908845572	$6.47e-11$	$2.11e-02$
		BNM2	1	0.0255627464167388	$3.10e-11$	$2.44e-02$
		NS1	1	0.0355381781052461	$2.24e-10$	$1.44e-02$
		NS2	1	0.0381580210068805	$3.43e-10$	$1.18e-02$
$(x - \sin x)^2$	0.06	NM	1	0.0499987998456958	$4.34e-10$	$1.00e-02$
		GNM	1	0.0399975996913915	$1.14e-10$	$2.00e-02$
		BNM1	1	0.0346644425641644	$4.82e-11$	$2.53e-02$
		BNM2	1	0.0306651148409437	$2.31e-11$	$2.93e-02$
		NS1	1	0.0426315727995842	$1.67e-10$	$1.74e-02$
		NS2	1	0.0457776128026630	$2.56e-10$	$1.42e-02$
$(x^2 + \sin(\frac{x}{5}) - \frac{1}{4})^2$	0.37	NM	11	0.4099740214194201	$3.36e-10$	$1.80e-05$
		GNM	2	0.4099948386255551	$8.27e-12$	$1.70e-03$
		BNM1	2	0.4099920179505115	$1.55e-21$	$2.37e-04$
		BNM2	2	0.4099920179891372	$2.43e-33$	$3.20e-05$
		NS1	2	0.4100186071366364	$7.35e-10$	$3.68e-03$
		NS2	3	0.4099922798234558	$7.12e-14$	$2.58e-04$
$(x^2 + 7x - 30)^2$	3.03	NM	14	3.0000018395052141	$5.72e-10$	$1.84e-06$
		GNM	2	3.0000000003653009	$2.26e-17$	$6.89e-05$
		BNM1	1	3.0000004727268600	$3.78e-11$	$3.00e-02$
		BNM2	1	3.0000000032477361	$1.78e-15$	$3.00e-02$
		NS1	2	3.0000000028956322	$1.42e-15$	$1.37e-04$
		NS2	2	3.0000000228996557	$8.86e-14$	$2.73e-04$

5 Conclusions

In the present work, we have proposed two new iterative methods (NS1 and NS2) with convergence order 2 for finding the multiple roots of nonlinear equations. The

numerical results presented in the Table given in the previous section reveal that our iterative methods not only perform better than Newton's method but are even comparable with the methods developed by Chun et al. (BNM1 and BNM2) with convergence order 3. The idea and technique employed in this paper can be developed to higher-order multi-step iterative methods for solving nonlinear equations having roots with multiplicity greater than one.

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