



Eigenstructure and Voronovskaja type formula for a sequence of integral operators ¹

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Dedicated to professor Vijay Gupta on the occasion of his 60th birthday

Abstract

The composition $\mathcal{F}_n$ of Rathore and Gamma operators was considered in the literature. We introduce a generalization of $\mathcal{F}_n$. For it we determine the eigenstructure and establish the corresponding Voronovskaja type formula.

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1 Introduction

Integral operators acting on some subspaces of $C[0, \infty)$ as well as compositions of such operators were considered in several papers and investigated from various points of view. The composition $\mathcal{F}_n$ of Rathore and Gamma operators was introduced and studied in [5]. Some properties of it can be found in [2]. An extension $\mathcal{F}_n^{[c]}$ of $\mathcal{F}_n$ is considered in [1]. In this paper we describe the eigenstructure of $\mathcal{F}_n^{[c]}$ and establish the corresponding Voronovskaja type formula, generalizing some results from [2].

The operators mentioned above are explicitly presented in what follows.

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The Rathore operator is defined by (see [4])

$$W_n f(x) = \frac{n^{nx}}{\Gamma(nx)} \int_0^\infty e^{-nt} t^{nx-1} f(t) dt.$$

The Gamma operator is described in [3, 9.1.11] as follows

$$G_n f(x) = \frac{(nx)^{n+1}}{n!} \int_0^\infty \frac{1}{s^{n+2}} e^{-\frac{nx}{s}} f(s) ds.$$

As mentioned above, the following operator was introduced in [5]

$$\mathcal{F}_n := W_n \circ G_n.$$

The explicit form of the operator $\mathcal{F}_n$ is (see [5])

$$(1) \quad \mathcal{F}_n f(x) = \frac{(nx)_{n+1}}{n!} \int_0^\infty \frac{t^{nx-1}}{(1+t)^{n+nx+1}} f(t) dt.$$

Denote $e_j(t) = t^j$, $t \geq 0$, $j = 0, 1, \dots$. We have (see [5])

$$(2) \quad \mathcal{F}_n e_j(x) = \frac{(nx)^{1, \bar{j}}}{(n)^{1, \underline{j}}},$$

where $a^{c, \bar{k}} := \prod_{l=0}^{k-1} (a + cl)$, $a^{c, \underline{k}} := \prod_{l=0}^{k-1} (a - cl)$, $a^{c, \bar{0}} := 1$, $a^{c, \underline{0}} := 1$.

Let $c > 0$. In [1] an extension of this operator was considered as follows

$$\mathcal{F}_n^{[c]}(f(t); x) := \mathcal{F}_{n/c} \left(f \left(\frac{t}{c} \right); cx \right), \quad c > 0.$$

2 Eigenstructure of the operators $\mathcal{F}_n^{[c]}$

From (2) we get

$$(3) \quad \mathcal{F}_n^{[c]} e_\nu(x) = \frac{(nx)^{1, \bar{\nu}}}{n^{c, \underline{\nu}}}.$$

Therefore, the eigenvalues of $\mathcal{F}_n^{[c]}$ are the numbers defined as

$$\mu_{n,k}^{[c]} = \frac{n^k}{n^{c, \underline{k}}}, \quad k = 0, \dots, n.$$

Let

$$(4) \quad u_{n,k}^{[c]}(x) := \sum_{\nu=0}^k \alpha_{n,k,\nu}^{[c]} x^\nu,$$

be the monic eigenpolynomial associated with $\mu_{n,k}^{[c]}$, $k = 0, \dots, n$. In particular,

$$(5) \quad \alpha_{n,k,k}^{[c]} = 1, \quad k = 0, \dots, n.$$

Since $\mathcal{F}_n^{[c]} e_i = e_i$, $i = 0, 1$, it follows that $u_{n,0}^{[c]} = e_0$, $u_{n,1}^{[c]} = e_1$, $u_{n,2}^{[c]} = x^2 + \frac{1}{c}x$, and consequently,

$$(6) \quad \alpha_{n,0,0}^{[c]} = 1, \quad \alpha_{n,1,0}^{[c]} = 0, \quad \alpha_{n,1,1}^{[c]} = 1.$$

From (3) and (5) one has

$$(7) \quad \sum_{\nu=0}^k \alpha_{n,k,\nu}^{[c]} \frac{(nx)^{1,\bar{\nu}}}{n^{c,\underline{\nu}}} = \mu_{n,k}^{[c]} \sum_{\nu=0}^k \alpha_{n,k,\nu}^{[c]} x^\nu.$$

Using the Stirling numbers of the first kind $s(j, i)$, we can write

$$(8) \quad (nx)^{1,\bar{\nu}} = \sum_{i=0}^{\nu} s(\nu, i) (-1)^{\nu-i} n^i x^i,$$

Relations (7) and (8) lead to

$$\sum_{\nu=0}^k \sum_{i=0}^{\nu} \alpha_{n,k,\nu}^{[c]} \frac{s(\nu, i) (-1)^{\nu-i} n^i x^i}{n^{c,\underline{\nu}}} = \frac{n^k}{n^{c,\underline{k}}} \sum_{i=0}^k \alpha_{n,k,i}^{[c]} x^i.$$

Consequently,

$$\sum_{i=0}^k \sum_{\nu=i}^k \alpha_{n,k,\nu}^{[c]} \frac{s(\nu, i) (-1)^{\nu-i} n^i}{n^{c,\underline{\nu}}} x^i = \sum_{i=0}^k \frac{n^k}{n^{c,\underline{k}}} \alpha_{n,k,i}^{[c]} x^i,$$

and so,

$$\sum_{\nu=i}^k \alpha_{n,k,\nu}^{[c]} \frac{s(\nu, i) (-1)^{\nu-i} n^i}{n^{c,\underline{\nu}}} = \frac{n^k}{n^{c,\underline{k}}} \alpha_{n,k,i}^{[c]}, \quad i = 0, \dots, k.$$

This can be written under the form

$$(9) \quad \alpha_{n,k,i}^{[c]} \frac{n^i}{n^{c,\underline{i}}} + \sum_{\nu=i+1}^k \alpha_{n,k,\nu}^{[c]} \frac{s(\nu, i) (-1)^{\nu-i} n^i}{n^{c,\underline{\nu}}} = \frac{n^k}{n^{c,\underline{k}}} \alpha_{n,k,i}^{[c]}, \quad i = 0, \dots, k.$$

From (9) and (5) we get the following result.

Theorem 1 *The coefficients $\alpha_{n,k,i}^{[c]}$, $i = k-1, k-2, \dots, 1, 0$, can be obtained recurrently from the relations*

$$(10) \quad \alpha_{n,k,i}^{[c]} = \frac{n^{c,\underline{k}} n^{c,\underline{i}}}{n^k n^{c,\underline{i}} - n^i n^{c,\underline{k}}} \sum_{\nu=i+1}^k \alpha_{n,k,\nu}^{[c]} \frac{s(\nu, i) (-1)^{\nu-i} n^i}{n^{c,\underline{\nu}}}, \quad i = 0, \dots, k-1,$$

$$(11) \quad \alpha_{n,k,k}^{[c]} = 1, \quad k = 0, \dots, n.$$

In the sequel we will be concerned with the limit of $\alpha_{n,k,i}^{[c]}$.

Theorem 2 *For each $k \geq 0$ and $i = 0, \dots, k$ there exists the limit*

$$l_{k,i}^{[c]} := \lim_{n \rightarrow \infty} \alpha_{n,k,i}^{[c]}.$$

Moreover,

$$(12) \quad l_{k,i}^{[c]} = \frac{i(i+1)}{c(k-i)(k+i-1)} l_{k,i+1}^{[c]}, \quad i = 0, \dots, k-1,$$

and

$$(13) \quad l_{k,i}^{[c]} = \left(-\frac{1}{c}\right)^{k-i} \prod_{l=1}^{k-i} \frac{(k-l+1)(k-l)}{l(l-2k+1)}, \quad i = 0, \dots, k.$$

Proof. From (10) we obtain

$$(14) \quad \alpha_{n,k,i}^{[c]} = \frac{n^{c,k} n^{c,i}}{n^k n^{c,i} - n^i n^{c,k}} \left\{ \sum_{\nu=i+2}^k \alpha_{n,k,\nu}^{[c]} \frac{s(\nu, i)(-1)^{\nu-i} n^i}{n^{c,\nu}} - \alpha_{n,k,i+1}^{[c]} \frac{s(i+1, i) n^i}{n^{c,i+1}} \right\},$$

$i = 0, \dots, k-1$.

Relation (11) leads to $l_{k,k}^{[c]} = 1$. Combining this fact with (14), we get (12).

Now a proof by induction yields (13).

3 Voronovskaja formula for $\mathcal{F}_n^{[c]}$

In this section we give a Voronovskaja type formula for the operators $\mathcal{F}_n^{[c]}$. Using relation (3), the moments and the central moments up to order four are given in the next lemmas.

Denote $\Psi_x(t) = t - x$.

Lemma 1 *The operators $\mathcal{F}_n^{[c]}$ verify*

- a) $\mathcal{F}_n^{[c]}(e_0; x) = 1,$
- b) $\mathcal{F}_n^{[c]}(e_1; x) = x,$
- c) $\mathcal{F}_n^{[c]}(e_2; x) = \frac{x(nx+1)}{n-c},$
- d) $\mathcal{F}_n^{[c]}(e_3; x) = \frac{x(nx+1)(nx+2)}{(n-c)(n-2c)},$
- d) $\mathcal{F}_n^{[c]}(e_4; x) = \frac{x(nx+1)(nx+2)(nx+3)}{(n-c)(n-2c)(n-3c)}.$

Lemma 2 *The operators $\mathcal{F}_n^{[c]}$ verify*

- a) $\mathcal{F}_n^{[c]}(\Psi_x; x) = 0,$
- b) $\lim_{n \rightarrow \infty} n\mathcal{F}_n^{[c]}(\Psi_x^2; x) = x(cx + 1),$
- c) $\lim_{n \rightarrow \infty} n^2\mathcal{F}_n^{[c]}(\Psi_x^3; x) = 2x(cx + 1)(2cx + 1),$
- d) $\lim_{n \rightarrow \infty} n^2\mathcal{F}_n^{[c]}(\Psi_x^4; x) = 3x^2(cx + 1)^2.$

Theorem 3 *Let $x \in [0, \infty)$ and f continuous and bounded on $[0, \infty)$ such that there exists $f''(x)$ finite. Then*

$$(15) \quad \lim_{n \rightarrow \infty} n \left(\mathcal{F}_n^{[c]} f(x) - f(x) \right) = \frac{x(cx + 1)}{2} f''(x).$$

Proof. Using Lemma 2 and a general approximation theorem for positive linear operators due to Sikkema [6], we obtain the Voronovskaja type result (15).

References

- [1] A. M. Acu, F. Sofonea, I. Rasa, *Composition of some positive linear integral operators*, (submitted).
- [2] A. M. Acu, I. Raşa, Andra Seserman, *Composition and decomposition of positive linear operators*, (VIII), Axioms 2023, 12, 228.
- [3] Z. Ditzian, V. Totik, *Moduli of Smoothness*, Springer, New York, 1987.
- [4] R. K. S. Rathore, *Linear combinations of linear positive operators and generating relations on special functions*, Ph. D. Thesis, Delhi, 1973.
- [5] F. Sofonea, *On a sequence of linear and positive operators*, General Mathematics, vol. 16, no. 4, 2008, 155-165.
- [6] P. C. Sikkema, *On some linear positive operators*, Indag. Math., vol. 32, 1970, 327-337.

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