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Ruled surfaces obtained by curves in the lightlike cone [1]

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Abstract

In this paper, we consider the ruled surfaces obtained the lightlike curve in the 2-dimensional lightlike cone according to the frame given in [7]. We obtain necessary and sufficient conditions for the ruled surfaces to be minimal and to have a harmonic Gauss map. Then we see that the ruled surface is minimal if and only if it has a harmonic Gauss map.

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1 Introduction

Ruled surface is one of the well-knows surfaces in Euclidean 3-space. A ruled surface S is a surface which contains at least one 1-parameter family of straight lines. Thus the parametrization of ruled surface can be given by $X : U \to S$ of the form

$$X(u,v) = \alpha(u) + v\gamma(u),$$

where α and γ are curves in $\mathbb{R}^3$. Here X is called a ruled patch. The curve α is called the directrix or base curve of the ruled surface, and γ is called the director curve [6].

Similarly, ruled surfaces can be defined in Minkowski 3-space $\mathbb{R}^3_1$. B-scroll is an example of ruled surfaces in Minkowski 3-space $\mathbb{R}^3_1$. This is a surface which can be parametrized as a ruled surface in $\mathbb{R}^3_1$ with null directrix curve and null ruling, i.e. $X(u,v) = x(u) + vB(u)$, $x(u)$ being a null curve and $B(u)$ a null vector field along $x(u)$ satisfying $\langle x', B\rangle = -1$. Many properties of B-scrolls in $\mathbb{R}^3_1$ have been studied by several researchers [1],[4],[5]. Also in [9], the authors studied ruled surfaces in $\mathbb{R}^5_2$.

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On the other hand, in [7], the author studies curves in the lightlike cone. The author obtains the conformally invariant arc length in the $(n+1)$-dimensional lightlike cone and then characterize some curves in the 2-dimensional lightlike cone and 3-dimensional lightlike cone.

In this paper, we consider the ruled surfaces obtained the lightlike curve in the 2-dimensional lightlike cone according to the frame given in [7]. We obtain necessary and sufficient conditions for the ruled surfaces to be minimal and to have a harmonic Gauss map. Then we see that the ruled surface is minimal if and only if it has a harmonic Gauss map.

2 Preliminaries

The Minkowski space-time $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$.

Let $v \in \mathbb{E}_1^3 \backslash \{0\}$ be a vector. Then v is called
(**i**) spacelike if $g(v, v) > 0$,
(**ii**) timelike if $g(v, v) < 0$,
(**iii**) null (lightlike) if $g(v, v) = 0$. In particular, the vector $v = 0$ is said to be a spacelike.

The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$.

Let $v, w \in \mathbb{E}_1^3$ be vectors. If $g(v, w) = 0$, then v and w are said to be orthogonal.

Let $\alpha(s)$ be a curve in $\mathbb{E}_1^3$. Then the curve $\alpha(s)$ is called spacelike, timelike or null (lightlike), if its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null [8].

Let $\alpha(s)$ be a spacelike or timelike curve in $\mathbb{E}_1^3$. If $g(\alpha'(s), \alpha'(s)) = \pm 1$, then the curve α is said to be parameterized by the arc-length parameter s.

Let $\alpha(s)$ be a null curve in $\mathbb{E}_1^3$. If $g(\alpha''(s), \alpha''(s)) = 1$, then the null curve α is said to be parameterized by pseudo-arc s [2].

The pseudo-Riemann lightlike cone is defined by

$$\mathbb{Q}^2\,(c) = \left\{ x \in \mathbb{E}_1^3 : g\,(x - c, x - c) = 0 \right\}.$$

When $c = 0$, we denote $\mathbb{Q}^2\,(0)$ by $\mathbb{Q}^2$.

([7])For the regular curve $x\,(s) \subset \mathbb{Q}^2 \subset \mathbb{E}_1^3$ with asymptotic orthonormal frame $\{x\,(s)\,,\ \alpha\,(s)\,,\ y\,(s)\}$ and the cone curvature function $\kappa\,(s)$, the Frenet formulas of the curve $x\,(s)$ in $\mathbb{Q}^2$ can be written as

$$(1) \qquad \begin{cases} x'\,(s) &= \alpha\,(s)\,, \\ \alpha'\,(s) &= \kappa\,(s)\, x\,(s) - y\,(s)\,, \\ y'\,(s) &= -\kappa\,(s)\,\alpha\,(s)\,, \end{cases}$$

where for all s,

$$g\,(x\,(s)\,, x\,(s)) = g\,(y\,(s)\,, y\,(s)) = g\,(x\,(s)\,, \alpha\,(s)) = g\,(y\,(s)\,, \alpha\,(s)) = 0,$$

$$g(\alpha(s), \alpha(s)) = g(x(s), y(s)) = 1,$$
$$g(\alpha(s), \beta(s)) = g(\alpha(s), y(s)) = g(\beta(s), y(s)) = 0.$$

We recall some well-known formulas for the surfaces in $\mathbb{E}_1^3$. Let M be a surface of $\mathbb{E}_1^3$, the standart connection D on $\mathbb{E}_1^3$ induces the Levi-Civita connection $\triangledown$ on M. We have the following Gauss formula

$$D_X Y = \nabla_X Y + h(X, Y),$$

and the Weingarten formula

$$D_X \xi = -A_\xi X + {}^{\perp}\nabla_X\, \xi,$$

where $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma\left(TM^{\perp}\right)$. Then ∇ is the Levi-Civita connection of M, h is the second fundamental form, A_ξ is the shape operator, and ${}^{\perp}\nabla$ is the normal connection. We note that

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The mean curvature vector field $\overrightarrow{H}$, the mean curvature H and the Gauss curvature of M are given respectively by

$$\overrightarrow{H} = \frac{1}{2}(h(e_1, e_1) + h(e_2, e_2)) \quad \text{and} \quad K = \det A$$

where $\{e_1, e_2\}$ is an orthonormal basis on M.

A surface is called minimal if $H = 0$ identically. A surface is called flat if $K = 0$ identically ([3]).

The Laplacian $\triangle$ on M is given by

$$\triangle = -\frac{1}{\sqrt{G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{G} g^{ij} \frac{\partial}{\partial x_j} \right) \tag{2}$$

where $G = \det(g_{ij})$ and $\left(g^{ij}\right)$ is the inverse matrix of (g_{ij}), which is the local components of the metric on M.

3 Ruled surfaces in the lightlike cone

Let $x(s) \subset \mathbb{Q}^2 \subset \mathbb{E}_1^3$ be a curve with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ and the cone curvature function $\kappa(s)$. We will consider the following ruled surface

$$M: \quad F(s, u) = x(s) + u\alpha(s), \quad u > 0. \tag{3}$$

Then

$$F_s = \alpha + u\kappa x - uy, \quad F_u = \alpha$$

and

$$g(F_s, F_s) = 1 - 2u^2\kappa, \quad g(F_u, F_s) = g(F_u, F_u) = 1.$$

Remark 1. *If $\kappa > 0$, then M is a timelike surface and if $\kappa < 0$, then M is a spacelike surface.*

Case 1. Assume that $\kappa > 0$ which implies that M is a Lorentzian surface. Then we find

$$\begin{aligned} e_1 &= F_u = \alpha, \\ e_2 &= \frac{1}{u\sqrt{2\kappa}}\left(F_s - F_u\right) = \frac{1}{\sqrt{2\kappa}}\left(\kappa x - y\right). \end{aligned}$$

Then $\{e_1, e_2\}$ is an orthonormal frame field on M of signature $(+,-)$. Set

$$e_3 = \frac{1}{\sqrt{2\kappa}}\left(\kappa x + y\right).$$

Then e_3 is a normal vector field to M of signature $(+)$. Then we obtain

$$\begin{aligned} D_{e_1}e_1 &= D_{e_1}e_2 = 0, \\ D_{e_2}e_2 &= \frac{1}{u}\left(\frac{\kappa'}{4\kappa}x + \frac{\kappa'}{4\kappa^2}y + \alpha\right). \end{aligned}$$

The components of the second fundamental form h are given as follows

$$h_{11} = h_{12} = 0, \quad h_{22} = \frac{\kappa'}{u\left(2\kappa\right)^{3/2}}.$$

Theorem 1. *The mean curvature vector $\overrightarrow{H}$ of M is given by*

$$\overrightarrow{H} = -\frac{\kappa'(s)}{8u\kappa(s)}x(s) - \frac{\kappa'(s)}{8u\kappa^2(s)}y(s).$$

Proof. The mean curvature vector $\overrightarrow{H}$ satisfies

$$\begin{aligned} \overrightarrow{H} &= \frac{1}{2}\left(h_{11} - h_{22}\right)e_3 \\ &= -\frac{\kappa'(s)}{8u\kappa(s)}x(s) - \frac{\kappa'(s)}{8u\kappa^2(s)}y(s). \end{aligned}$$

Corollary 1. *The surface M is minimal if and only if κ is a non-zero constant.*

Proof. The mean curvature H of M is given by

$$H = \frac{\kappa'(s)}{u\left(2\kappa(s)\right)^{3/2}}.$$

M is minimal if and only if $H = 0$, which leads that κ is a non-zero constant.

Now we define the Gauss map $G(s,u)$ of M by

$$G(s,u) = \frac{1}{\sqrt{2\kappa}}\left(\kappa x + y\right). \tag{4}$$

By using (4) and (2), we have

$$\Delta G = \frac{u(\kappa')^2 - u\kappa\kappa'' - \kappa\kappa'}{4u^3\kappa^2\sqrt{2\kappa}}x + \frac{u\kappa\kappa'' - 2u(\kappa')^2 + \kappa\kappa'}{4u^3\kappa^3\sqrt{2\kappa}}y - \frac{2\kappa^2\kappa'}{4u^2\kappa^3\sqrt{2\kappa}}\alpha. \tag{5}$$

Then we give the following theorem.

Theorem 2. *M has a harmonic Gauss map if and only if κ is a non-zero constant.*

Proof. M has a harmonic Gauss map if and only if $\Delta G = 0$, which leads that κ is a non-zero constant.

From above theorem, we give the following corollary.

Corollary 2. *The surface M is minimal if and only if M has a harmonic Gauss map.*

Case 2. Assume that $\kappa < 0$. Then we find

$$\begin{aligned} e_1 &= F_u = \alpha, \\ e_2 &= \frac{1}{u\sqrt{-2\kappa}}(F_s - F_u) = \frac{1}{\sqrt{-2\kappa}}(\kappa x - y). \end{aligned}$$

Then $\{e_1, e_2\}$ is an orthonormal frame field on M of signature $(+,+)$. Set

$$e_3 = \frac{1}{\sqrt{-2\kappa}}(\kappa x + y).$$

Then e_3 is a normal vector field to M of signature $(-)$. Then we obtain

$$\begin{aligned} D_{e_1}e_1 &= D_{e_1}e_2 = 0, \\ D_{e_2}e_2 &= -\frac{1}{u}\left(\frac{\kappa'}{4\kappa}x + \frac{\kappa'}{4\kappa^2}y + \alpha\right). \end{aligned}$$

The components of the second fundamental form h are given as follows

$$h_{11} = h_{12} = 0, \quad h_{22} = \frac{\kappa'}{u(-2\kappa)^{3/2}}.$$

Theorem 3. *The mean curvature vector $\overrightarrow{H}$ of M is given by*

$$\overrightarrow{H} = \frac{\kappa'(s)}{8u\kappa(s)}x(s) + \frac{\kappa'(s)}{8u\kappa^2(s)}y(s).$$

Proof. The mean curvature vector $\overrightarrow{H}$ satisfies

$$\begin{aligned} \overrightarrow{H} &= \frac{1}{2}(h_{11} + h_{22})e_3 \\ &= \frac{\kappa'(s)}{8u\kappa(s)}x(s) + \frac{\kappa'(s)}{8u\kappa^2(s)}y(s). \end{aligned}$$

Corollary 3. *The surface M is minimal if and only if κ is a non-zero constant.*

Proof. The mean curvature H of M is given by

$$H = \frac{\kappa'(s)}{u(-2\kappa(s))^{3/2}}.$$

M is minimal if and only if $H = 0$, which leads that κ is a non-zero constant.

Now we define the Gauss map $G(s,u)$ of M by

$$G(s,u) = \frac{1}{\sqrt{-2\kappa}}(\kappa x + y). \tag{6}$$

By using (6) and (2), we have

$$\Delta G = \frac{u(\kappa')^2 - u\kappa\kappa'' - \kappa\kappa'}{4u^3\kappa^2\sqrt{-2\kappa}}x + \frac{u\kappa\kappa'' - 2u(\kappa')^2 + \kappa\kappa'}{4u^3\kappa^3\sqrt{-2\kappa}}y - \frac{2\kappa^2\kappa'}{4u^2\kappa^3\sqrt{-2\kappa}}\alpha. \tag{7}$$

Then we give the following theorem.

Theorem 4. *M has a harmonic Gauss map if and only if κ is a non-zero constant.*

Proof. M has a harmonic Gauss map if and only if $\Delta G = 0$, which leads that κ is a non-zero constant.

From above theorem, we give the following corollary.

Corollary 4. *The surface M is minimal if and only if M has a harmonic Gauss map.*

Remark 2. *Let $x(s) \subset \mathbb{Q}^2 \subset \mathbb{E}_1^3$ be a curve with asymptotic orthonormal frame $\{x(s), \alpha(s), y(s)\}$ and the cone curvature function $\kappa(s)$. Then we can consider ruled surfaces of other types: $x(s) + uy(s)$ and $x(s) + ux(s)$. But their induced metrics are degenerate, that is, they are lightlike surfaces.*

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