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Formalization of Separable Version of Banach–Alaoglu Theorem[1]

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Summary. In this article, we first formalize the weak* sequential compactness in dual normed spaces; then we prove the separable version of Banach–Alaoglu theorem.

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Introduction

This article is the next one in the series developing the theory of normed spaces in Mizar [3], [9] (for similar developments in another theorem provers, see [4] in Isabelle/HOL, [1], [2] in Coq). We present a formalization of the separable case of the Banach–Alaoglu Theorem (also known as Alaoglu's theorem), a central result in functional analysis concerning compactness in dual spaces of a normed vector space [10].

In classical mathematics, the Banach–Alaoglu Theorem asserts that the closed unit ball in the dual of a normed vector space is compact under the weak* topology, with the separable case providing sequential compactness properties that are especially useful in analysis [5], [12]. Stefan Banach proved a version of this theorem for separable normed spaces in 1932, while Leonidas Alaoglu later extended the result to the general case (published in 1940) [7].

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We develop key preliminary results about sequences in dual normed spaces – including convergence and boundedness – and then define and prove the weak* sequential compactness of bounded sets in the dual of a separable normed space. This lays the groundwork for the main formalized statement, which confirms that, when the underlying space is separable, the unit ball in its dual space is sequentially compact in the weak* topology [11].

1. Weak* Sequential Compactness in Dual Normed Spaces

Now we state the propositions:

(1) Let us consider a real normed space X, a sequence v_1 of DualSp X, and points x, y of X. Suppose $v_1\#x$ is convergent and $v_1\#y$ is convergent. Then

 (i) $v_1\#(x+y)$ is convergent, and

 (ii) $\lim(v_1\#(x+y)) = \lim(v_1\#x) + \lim(v_1\#y)$.

(2) Let us consider a real normed space X, a sequence v_1 of DualSp X, a point x of X, and a real number a. Suppose $v_1\#x$ is convergent. Then

 (i) $v_1\#a\cdot x$ is convergent, and

 (ii) $\lim(v_1\#a\cdot x) = a\cdot(\lim(v_1\#x))$.

(3) Let us consider a real normed space X, a subset X_0 of X, and a sequence v_1 of DualSp X. Suppose for every point x of X such that $x \in X_0$ holds $v_1\#x$ is convergent. Let us consider a point x of X. If $x \in \mathrm{Lin}(X_0)$, then $v_1\#x$ is convergent.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every point x of X for every subset X_1 of X for every linear combination L of X_1 such that $X_1 \subseteq X_0$ and $\overline{\overline{\text{the support of } L}} \leqslant \$_1$ and $x \in \mathrm{Lin}(X_1)$ and $x = \sum L$ holds $v_1\#x$ is convergent. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number n, $\mathcal{P}[n]$. Consider l being a linear combination of X_0 such that $x = \sum l$. □

(4) Let us consider a real Banach space X, subsets X_0, X_1 of X, and a sequence v_1 of DualSp X. Suppose $X_1 =$ the carrier of $\mathrm{Lin}(X_0)$ and X_1 is dense and $\|v_1\|$ is bounded and for every point x of X such that $x \in X_0$ holds $v_1\#x$ is convergent. Then v_1 is weakly* convergent. The theorem is a consequence of (3).

(5) Let us consider a real linear space V, a vector v of V, and a real number a. Suppose $a \neq 0$. Then there exists a linear combination l of V such that

 (i) $l(v) = a$, and

(ii) the support of $l = \{v\}$.

PROOF: Reconsider $z_1 = 0$ as an element of $\mathbb{R}$. Define $\mathcal{F}(\text{vector of } V) = z_1$. Consider f being a function from the carrier of V into $\mathbb{R}$ such that $f(v) = a$ and for every vector u of V such that $u \neq v$ holds $f(u) = \mathcal{F}(u)$. $\{v\} \subseteq$ the support of f. The support of $f \subseteq \{v\}$. □

(6) Let us consider a real linear space X. Then $\Omega_{\mathrm{Lin}(\Omega_X)} = \Omega_X$.

PROOF: For every object x, $x \in \Omega_{\mathrm{Lin}(\Omega_X)}$ iff $x \in \Omega_X$. □

(7) Let us consider a real Banach space X, and a sequence f of DualSp X. Then f is weakly* convergent if and only if $\|f\|$ is bounded and there exist subsets X_0, X_1 of X such that $X_1 =$ the carrier of $\mathrm{Lin}(X_0)$ and X_1 is dense and for every point x of X such that $x \in X_0$ holds $f\#x$ is convergent. The theorem is a consequence of (6) and (4).

Let X be a real normed space and X_0 be a non empty subset of DualSp X. We say that X_0 is weakly* sequentially compact if and only if

(Def. 1) for every sequence s_1 of X_0, there exists a sequence s_2 of DualSp X such that s_2 is subsequence of s_1 and weakly* convergent and w*-lim$(s_2) \in X_0$.

2. Separable Version of Banach–Alaoglu Theorem

Now we state the proposition:

(8) BANACH ALAOGLU THEOREM (SEPARABLE CASE):

Let us consider a real Banach space X, a real number M, and a non empty subset X_0 of DualSp X. Suppose X is separable and $0 \leqslant M$ and $\overline{\mathrm{Ball}}(0_{\mathrm{DualSp}\,X}, M) = X_0$. Then X_0 is weakly* sequentially compact.

PROOF: For every sequence s_1 of X_0, there exists a sequence s_2 of DualSp X such that s_2 is subsequence of s_1 and weakly* convergent and w*-lim$(s_2) \in X_0$ by [6, (18)], [8, (26),(15)]. □

Let X be a real normed space, f be a partial function from X to $\mathbb{R}$, and x_0 be a point of X. We say that f is weakly continuous in x_0 if and only if

(Def. 2) $x_0 \in \mathrm{dom}\, f$ and for every real number e such that $0 < e$ there exists a real number d and there exists a subset Y of DualSp X such that $0 < d$ and Y is finite and $Y \neq \emptyset$ and for every point x of X such that $x \in \mathrm{dom}\, f$ and for every point y of DualSp X such that $y \in Y$ holds $|y(x - x_0)| < d$ holds $|f(x) - f(x_0)| < e$.

Let X_0 be a subset of X. We say that f is weakly continuous on X_0 if and only if

(Def. 3) $X_0 \subseteq \mathrm{dom}\, f$ and for every point x_0 of X such that $x_0 \in X_0$ for every real number e such that $0 < e$ there exists a real number d and there exists a subset Y of DualSp X such that $0 < d$ and Y is finite and $Y \neq \emptyset$ and for every point x of X such that $x \in X_0$ and for every point y of DualSp X such that $y \in Y$ holds $|y(x - x_0)| < d$ holds $|f(x) - f(x_0)| < e$.

Now we state the proposition:

(9) Let us consider a real normed space X, a partial function f from X to $\mathbb{R}$, and a subset X_0 of X. Then f is weakly continuous on X_0 if and only if $X_0 \subseteq \operatorname{dom} f$ and for every point x_0 of X such that $x_0 \in X_0$ holds $f \restriction X_0$ is weakly continuous in x_0.

Let X be a real normed space, f be a partial function from DualSp X to $\mathbb{R}$, and x_0 be a point of DualSp X. We say that f is weakly* continuous in x_0 if and only if

(Def. 4) $x_0 \in \operatorname{dom} f$ and for every real number e such that $0 < e$ there exists a real number d and there exists a subset Y of X such that $0 < d$ and Y is finite and $Y \neq \emptyset$ and for every point x of DualSp X such that $x \in \operatorname{dom} f$ and for every point y of X such that $y \in Y$ holds $|(x - x_0)(y)| < d$ holds $|f(x) - f(x_0)| < e$.

Let X_0 be a subset of DualSp X. We say that f is weakly* continuous on X_0 if and only if

(Def. 5) $X_0 \subseteq \operatorname{dom} f$ and for every point x_0 of DualSp X such that $x_0 \in X_0$ for every real number e such that $0 < e$ there exists a real number d and there exists a subset Y of X such that $0 < d$ and Y is finite and $Y \neq \emptyset$ and for every point x of DualSp X such that $x \in X_0$ and for every point y of X such that $y \in Y$ holds $|(x - x_0)(y)| < d$ holds $|f(x) - f(x_0)| < e$.

Now we state the propositions:

(10) Let us consider a real normed space X, a partial function f from DualSp X to $\mathbb{R}$, and a subset X_0 of DualSp X. Then f is weakly* continuous on X_0 if and only if $X_0 \subseteq \operatorname{dom} f$ and for every point x_0 of DualSp X such that $x_0 \in X_0$ holds $f \restriction X_0$ is weakly* continuous in x_0.

(11) Let us consider a real normed space X, a partial function f from X to $\mathbb{R}$, a point x_0 of X, and a sequence x of X. Suppose f is weakly continuous in x_0 and $\operatorname{rng} x \subseteq \operatorname{dom} f$ and x is weakly convergent and w-lim$(x) = x_0$. Then

   (i) $f_* x$ is convergent, and

   (ii) $\lim(f_* x) = f(x_0)$.

Proof: For every real number e such that $0 < e$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|(f_* x)(m) - f(x_0)| < e$. $\square$

(12) Let us consider a real normed space X, a partial function f from DualSp X to $\mathbb{R}$, a point x_0 of DualSp X, and a sequence x of DualSp X. Suppose f is weakly* continuous in x_0 and $\operatorname{rng} x \subseteq \operatorname{dom} f$ and x is weakly* convergent and w*-lim$(x) = x_0$. Then

(i) $f_* x$ is convergent, and

(ii) $\lim(f_* x) = f(x_0)$.

PROOF: For every real number e such that $0 < e$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|(f_* x)(m) - f(x_0)| < e$. □

(13) Let us consider a real normed space X, a partial function f from DualSp X to $\mathbb{R}$, and a non empty subset S of DualSp X. Suppose S is weakly* sequentially compact and f is weakly* continuous on S. Then

(i) there exists a real number r such that for every point x of DualSp X such that $x \in S$ holds $|f(x)| \leqslant r$, and

(ii) there exists a point x_0 of DualSp X such that $x_0 \in S$ and for every point x of DualSp X such that $x \in S$ holds $f(x) \leqslant f(x_0)$, and

(iii) there exists a point v_0 of DualSp X such that $v_0 \in S$ and for every point x of DualSp X such that $x \in S$ holds $f(v_0) \leqslant f(x)$.

PROOF: There exists a real number r such that for every point x of DualSp X such that $x \in S$ holds $|f(x)| \leqslant r$. Consider r being a real number such that for every point x of DualSp X such that $x \in S$ holds $|f(x)| \leqslant r$. Reconsider $Y = \operatorname{rng}(f \restriction S)$ as a non empty, real-membered set. Consider s being an object such that $s \in S$. For every real number z such that $z \in Y$ holds $|z| < r + 1$. There exists a point x_0 of DualSp X such that $x_0 \in S$ and for every point x of DualSp X such that $x \in S$ holds $f(x) \leqslant f(x_0)$. Set $N = \inf Y$.

Define $\mathcal{Q}$[natural number, point of DualSp X] $\equiv$ $\$_2 \in S$ and $|(f \restriction S)(\$_2) - N| < \frac{1}{\$_1 + 1}$. For every element x of $\mathbb{N}$, there exists an element y of DualSp X such that $\mathcal{Q}[x, y]$. Consider s_1 being a function from $\mathbb{N}$ into DualSp X such that for every element n of $\mathbb{N}$, $\mathcal{Q}[n, s_1(n)]$. Consider s_2 being a sequence of DualSp X such that s_2 is subsequence of s_1 and weakly* convergent and w*-lim$(s_2) \in S$. $f \restriction S$ is weakly* continuous in w*-lim(s_2). $f \restriction S_* s_2$ is convergent and $\lim(f \restriction S_* s_2) = (f \restriction S)(\text{w*-lim}(s_2))$. Consider K being an increasing sequence of $\mathbb{N}$ such that $s_2 = s_1 \cdot K$. □

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