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Application of Complex Classes to Number Theory

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Summary. The paper illustrates how the formal framework proposed earlier for complex numbers can be applied to integer and natural numbers, facilitating proofs by means of clusters' registrations for Mizar adjectives. The contents of the article are closely related to the series of Mizar articles "Elementary Number Theory Problems" (with MML identifiers `NUMBER*`), but it is not solving any of "250 Problems in Elementary Number Theory" by Wacław Sierpiński book.

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Introduction

The article explores the use of complex number classes in the Mizar proof assistant to simplify proofs in elementary number theory. It builds on prior work defining such classes in Mizar [14] and applies those constructs to integers, natural, and real numbers, showing how properties like divisibility, modular arithmetic, and relative primality can be formalized and verified using these constructions. Although the paper does not solve any of specific problems from Sierpiński's "250 Problems in Elementary Number Theory" [12], its formalized results connect closely to the series of Elementary Number Theory Problems [8] within the Mizar Mathematical Library [1] as initiated in [9]. Furthermore, it demonstrates various verified propositions about integers and real numbers

using automated proof techniques [10], [11]. Even if all the proofs are strictly rigorous, they can be treated as an area of recreational mathematics, like [13].

Specific advantage of registrations [2], [7] is in formulating some notions in terms of attributes, like `heavy`, `light`, and `weightless` in Sect. 1, `zero` in Sect. 2, `square` or `square-free` in Sect. 3. Relatively new construction of a reduction [4] also makes automation more efficient; an ellipsis [5] allows Mizar statements to be more compact leaving out their obvious parts, still remaining rigorous, with expected future potential for more flexible Mizar checker via unfolding definitions [3].

1. Preliminaries

Let a be a real number. One can check that $\operatorname{frac} a$ is light and non negative and there exists an even integer which is weightless and there exists an odd integer which is weightless and every integer which is non weightless is also heavy and there exists an integer which is heavy, positive, and odd and there exists an integer which is heavy, even, and negative. Now we state the proposition:

(1) Let us consider non weightless integers a, b. If a and b are relatively prime, then $a \nmid b$ and $b \nmid a$.

Let a be a non integer real number. One can verify that $\operatorname{frac} a$ is light and positive. Let b be a light, positive real number. Let us observe that $\lceil b \rceil$ is weightless and positive and $\lfloor b \rfloor$ is zero. Let us observe that $\operatorname{frac} b$ reduces to b. Let a be an integer. Note that $\lfloor a + b \rfloor$ reduces to a and $\lceil a - b \rceil$ reduces to a and $\operatorname{frac}(a + b)$ reduces to b.

Let a be a heavy, positive real number. Observe that $\lfloor a \rfloor$ is positive. Let a be a heavy, negative real number. Let us observe that $\lceil a \rceil$ is negative. Let a be an integer and b be a light, negative real number. One can check that $\lfloor a - b \rfloor$ reduces to a and $\lceil a + b \rceil$ reduces to a.

2. Inequalities in Rounding Products and Fractions

Now we state the proposition:

(2) Let us consider positive real numbers a, b. Then $\lfloor a \rfloor \cdot \lfloor b \rfloor \leqslant \lfloor a \rfloor \cdot b \leqslant a \cdot b$.

Let us consider positive real numbers a, b. Now we state the propositions:

(3) $\lfloor a \rfloor \cdot \lfloor b \rfloor \leqslant \lfloor a \cdot b \rfloor$. The theorem is a consequence of (2).

(4) $\frac{\lfloor a \rfloor}{b} \leqslant \frac{a}{b}$.

(5) Let us consider a positive real number a, and a heavy, positive real number b. Then $\frac{a}{\lfloor b \rfloor} \geqslant \frac{a}{b}$.

(6) Let us consider an integer a, and a non zero integer b. Then $b \mid a$ if and only if $a \bmod b = 0$.

(7) Let us consider a positive real number a, and a natural number n. Then $\lfloor n \cdot a \rfloor \geqslant n \cdot \lfloor a \rfloor$.

Let a be an integer. Let us observe that $a \bmod -1$ is zero.

Let b be a weightless integer. Observe that $a \bmod b$ is zero.

Let b be a non zero natural number. Let us note that $a^b \bmod a$ is zero.

Let a be an odd integer and b be a non zero, even integer. Observe that $a \bmod b$ is odd.

Let a be an even integer. Observe that $a \bmod b$ is even and $a^4 \bmod 8$ is zero.

Let a, b be odd integers. One can check that $\frac{a^2-b^2}{2}$ is even and $\frac{a^2+b^2}{2}$ is odd.

3. On Squares and Roots

Let a be square integer. One can check that $\sqrt[2]{a}$ is natural.

Let us observe that $\sqrt[2]{a} \cdot \sqrt[2]{a}$ reduces to a.

Let a be an even, a square integer. Observe that $\sqrt[2]{a}$ is even and $\frac{a}{2}$ is even.

Let b be an odd, a square integer. Let us note that $a - b$ is non square.

Let a be an odd, a square integer. Observe that $\sqrt[2]{a}$ is odd and $\frac{a+b}{2}$ is odd and $\frac{a-b}{2}$ is even and there exists square integer which is even and there exists square integer which is odd and there exists square natural number which is even. Now we state the proposition:

(8) Let us consider a natural number a. Then a is a square and square-free if and only if $a = 1$.

PROOF: If a is a square and square-free, then $a = 1$. □

Note that there exists a natural number which is square-free and square.

Let a be an even, a square integer. Let us note that $\frac{a}{4}$ is a square and integer.

Let a be a non zero, a square integer. One can verify that $2 \cdot a$ is non square and $a + a$ is non square.

Let a be an integer and b, c be non zero natural numbers. Let us note that $(a \bmod b) \bmod b \cdot c$ reduces to $a \bmod b$.

Let b be a non trivial natural number. Observe that $1 \bmod b$ reduces to 1. Let a be a natural number. Let us note that $a \cdot b + 1 \bmod b$ reduces to 1. Let n be a natural number. Note that $(a \cdot b + 1)^n \bmod b$ reduces to 1.

Let n be an even natural number. Observe that $(b-1)^n \bmod b$ reduces to 1.

Let n be an odd natural number. One can check that $(b-1)^n \bmod b$ reduces to $b - 1$. Now we state the proposition:

(9) Let us consider a natural number a, and a prime number p. Then

(i) $a^{p-1} \bmod p = 0$, or

(ii) $a^{p-1} \bmod p = 1$.

Let a be a natural number. Observe that $a \bmod 2$ is a square and $a^2 \bmod 3$ is a square and $a^2 \bmod 4$ is a square and $a^2 \bmod 5$ is a square and $a^2 \bmod 8$ is a square.

Let p be a prime natural number. One can verify that $a^{p-1} \bmod p$ is a square.

4. Fractional Part

Now we state the proposition:

(10) Let us consider a non integer real number a. Then $\operatorname{frac} a + \operatorname{frac}(-a) = 1$.

Let a be a non integer real number. Observe that $\operatorname{frac} a + \operatorname{frac}(-a)$ is non zero and trivial. Now we state the propositions:

(11) Let us consider a non integer real number a. Then $-1 < \operatorname{frac} a - \operatorname{frac}(-a) < 1$. The theorem is a consequence of (10).

(12) Let us consider a non integer real number a. Then $\operatorname{frac} a = \operatorname{frac}(-a)$ if and only if $2 \cdot a$ is an odd integer. The theorem is a consequence of (10).

Let us consider real numbers a, b. Now we state the propositions:

(13) $\operatorname{frac}(a \cdot b) = \operatorname{frac}(a \cdot \operatorname{frac} b + b \cdot \operatorname{frac} a - \operatorname{frac} a \cdot \operatorname{frac} b)$.

(14) $\operatorname{frac}(a \cdot b) = \operatorname{frac}(\lfloor a \rfloor \cdot \operatorname{frac} b + \lfloor b \rfloor \cdot \operatorname{frac} a + \operatorname{frac} a \cdot \operatorname{frac} b)$.

(15) Let us consider a real number a, and an integer b. Then $\operatorname{frac}(a \cdot b) = \operatorname{frac}(b \cdot \operatorname{frac} a)$. The theorem is a consequence of (14).

Let us consider a real number a. Now we state the propositions:

(16) $\operatorname{frac}(a \cdot a) = \operatorname{frac}(2 \cdot a \cdot \operatorname{frac} a - \operatorname{frac} a \cdot \operatorname{frac} a)$. The theorem is a consequence of (13).

(17) $\operatorname{frac}(a \cdot a) = \operatorname{frac}(2 \cdot \lfloor a \rfloor \cdot \operatorname{frac} a + \operatorname{frac} a \cdot \operatorname{frac} a)$. The theorem is a consequence of (14).

Let us consider a positive real number a. Now we state the propositions:

(18) If $\operatorname{frac} a = \frac{1}{2}$, then $\operatorname{frac}(2 \cdot a) = 0$. The theorem is a consequence of (15).

(19) If $\frac{1}{2} > \operatorname{frac} a$, then $\operatorname{frac}(2 \cdot a) = 2 \cdot \operatorname{frac} a$. The theorem is a consequence of (15).

(20) If $\frac{1}{2} < \operatorname{frac} a$, then $\operatorname{frac}(2 \cdot a) < \operatorname{frac} a$. The theorem is a consequence of (15).

5. DIVISIBILITY REVISITED

Let us consider an integer a and a non zero integer b. Now we state the propositions:

(21) If $b \nmid a$, then $(a \operatorname{div} b) + (-a \operatorname{div} b) = -1$.

(22) $b \nmid a$ if and only if $(a \bmod b) + (-a \bmod b) = b$.
PROOF: If $b \nmid a$, then $(a \bmod b) + (-a \bmod b) = b$. If $b \mid a$, then $(a \bmod b) + (-a \bmod b) = 0$. □

Now we state the propositions:

(23) Let us consider integers a, b. Then
(i) $(a \bmod b) + (-a \bmod b) = 0$, or
(ii) $(a \bmod b) + (-a \bmod b) = b$.
The theorem is a consequence of (6) and (22).

(24) Let us consider an even integer a, and an odd integer b. Then $a \operatorname{div} b$ is odd if and only if $a \bmod b$ is odd.

(25) Let us consider odd integers a, b. Then $a \bmod b$ is odd if and only if $a \operatorname{div} b$ is even.

(26) Let us consider an integer a, and an odd integer b. Suppose $b \nmid a$. Then $a \bmod b$ is odd if and only if $-a \bmod b$ is even. The theorem is a consequence of (22).

(27) Let us consider a non zero integer a, and an integer b. Then $a \mid b$ if and only if $a \mid b \bmod a$.
PROOF: If $a \mid b$, then $a \mid b \bmod a$. □

(28) Let us consider a non weightless integer a, and a non weightless, odd integer b. Suppose a and b are relatively prime. Then $b + a \bmod b \neq b - a \bmod b$. The theorem is a consequence of (1), (6), (27), and (26).

(29) Let us consider an integer a, and an even integer b. Suppose $b \nmid a$. If $a \bmod b$ is odd, then $-a \bmod b$ is odd. The theorem is a consequence of (22).

(30) Let us consider non zero integers a, b. Suppose $b \bmod a = -b \bmod a$. Then
(i) a is even, or
(ii) $a \mid b$.
The theorem is a consequence of (26).

(31) Let us consider natural numbers a, b. Suppose a and b are relatively prime. Let us consider a non trivial natural number n. Then $\max(a \bmod n, b \bmod n) > 0$.

(32) Let us consider square integer s. Then $s \bmod 3$ is a trivial natural number.

(33) Let us consider square natural numbers a, b. If $\frac{a+b}{2}$ is a square, then $a \bmod 3 = b \bmod 3$. The theorem is a consequence of (32).

(34) Let us consider odd natural numbers a, b. Suppose a and b are relatively prime. If $\frac{a^2+b^2}{2}$ is a square, then $3 \nmid a \cdot b$. The theorem is a consequence of (33), (31), and (32).

(35) Let us consider integers a, b. Then $a \operatorname{div} b = -a \operatorname{div} -b$.

(36) Let us consider square natural numbers a, b. Suppose a and b are relatively prime. If $\frac{a-b}{2}$ is a square, then $b \bmod 3 = 1$. The theorem is a consequence of (32) and (31).

(37) Let us consider an odd natural number a. Then $3 \mid 2^a + 1$.

(38) Let us consider integers a, b, c, d. If $\gcd(a \cdot b, c \cdot d) = 1$, then $\gcd(a, c) = 1$.

(39) Let us consider integers a, b, and non zero natural numbers m, n. Then a and b are relatively prime if and only if a^m and b^n are relatively prime. PROOF: If a and b are relatively prime, then a^m and b^n are relatively prime. □

(40) Let us consider square natural numbers a, b. If a and b are relatively prime, then $3 \nmid a + b$.

(41) Let us consider odd, a square natural numbers a, b. If a and b are relatively prime, then $3 \nmid \frac{a+b}{2}$. The theorem is a consequence of (40).

6. On (co)Prime Factorization

Now we state the propositions:

(42) Let us consider natural numbers a, b, c, d. Suppose a and c are relatively prime and b and d are relatively prime. If $a \cdot b = c \cdot d$, then $a = d$ and $b = c$.

(43) Let us consider natural numbers a, b, c. Suppose a and b are relatively prime. If $c \mid a \cdot b$, then $(\gcd(a, c)) \cdot (\gcd(b, c)) = c$.

(44) Let us consider natural numbers a, b, c, d. Suppose a and b are relatively prime. If $a \cdot b = c \cdot d$, then $a \cdot b = (\gcd(a, c)) \cdot (\gcd(b, c)) \cdot (\gcd(a, d)) \cdot (\gcd(b, d))$. The theorem is a consequence of (43).

(45) Let us consider positive real numbers a, b, c, d. If $a \cdot b = c \cdot d$ and $a \cdot c = b \cdot d$, then $a = d$.

(46) Let us consider integers a, b. If a and b are relatively prime, then $\gcd((a - b) \cdot (a + b), a \cdot b) = 1$.

(47) Let us consider an odd integer a, and an even integer b. Suppose a and b are relatively prime. Then $\gcd((a-b)\cdot(a+b), 2\cdot a\cdot b) = 1$. The theorem is a consequence of (46).

(48) Let us consider an even integer a, and an integer b. If a and b are relatively prime, then $a+b$ and $a-b$ are relatively prime.

(49) Let us consider an even integer a, an integer b, and a non zero natural number n. Suppose a and b are relatively prime. Then $b^n + a^n$ and $b^n - a^n$ are relatively prime. The theorem is a consequence of (48).

(50) Let us consider natural numbers a, b, and non zero natural numbers c, d. Suppose $a \bmod c\cdot d = b \bmod c\cdot d$. Then $a \bmod c = b \bmod c$.

Let us consider non zero natural numbers a, b. Now we state the propositions:

(51) $a - b \bmod 2\cdot b = a + b \bmod 2\cdot b$.

(52) If $a^2 - b^2 \in \mathbb{N}$, then $a^2 - b^2 \bmod 2\cdot b = a^2 + b^2 \bmod 2\cdot b$. The theorem is a consequence of (51) and (50).

(53) $(a-b)^2 \bmod 4\cdot a\cdot b = (a+b)^2 \bmod 4\cdot a\cdot b$.

(54) Let us consider odd natural numbers a, b. Then $(\frac{a-b}{2})^2 \bmod a\cdot b = (\frac{a+b}{2})^2 \bmod a\cdot b$.

(55) Let us consider natural numbers a, b, c. Suppose $a^2 + b^2 = c^2$. Then

(i) $b^2 \bmod a = c^2 \bmod a$, and

(ii) $a^2 \bmod b = c^2 \bmod b$.

(56) Let us consider a natural number a, a non trivial natural number b, and a non zero natural number c. If $a \bmod b\cdot c = 1$, then $a \bmod b = 1$.

(57) Let us consider a natural number a, a non zero natural number b, and non zero natural numbers m, n. Suppose $m \geqslant n$. If $a \bmod b^m = 1$, then $a \bmod b^n = 1$. The theorem is a consequence of (56).

Let us consider an integer i. Now we state the propositions:

(58) (i) $i^2 \bmod 4 = 0$, or

(ii) $i^2 \bmod 4 = 1$.

(59) (i) $i^2 \bmod 8 = 0$, or

(ii) $i^2 \bmod 8 = 1$, or

(iii) $i^2 \bmod 8 = 4$.

(60) (i) $i^4 \bmod 8 = 0$, or

(ii) $i^4 \bmod 8 = 1$.

The theorem is a consequence of (59).

Let us consider odd integers a, b. Now we state the propositions:

(61) (i) $a + b \bmod 4 = 2$, or

(ii) $a - b \bmod 4 = 2$.

(62) (i) $a + b \bmod 4 = 0$, or

(ii) $a - b \bmod 4 = 0$.

The theorem is a consequence of (61) and (6).

(63) $\max(a + b \bmod 4, a - b \bmod 4) = 2$. The theorem is a consequence of (61).

(64) $\min(a + b \bmod 4, a - b \bmod 4) = 0$. The theorem is a consequence of (62).

(65) Let us consider natural numbers a, b. Suppose a and b are relatively prime. Then

(i) $a^4 + b^4 \bmod 5 = 1$, or

(ii) $a^4 + b^4 \bmod 5 = 2$.

The theorem is a consequence of (9).

(66) Let us consider integers a, b. Then $a \bmod b = b \cdot \operatorname{frac}(\frac{a}{b})$.

(67) Let us consider an integer a, and a non zero integer b. Then $\operatorname{frac}(b \cdot \operatorname{frac}(\frac{a}{b})) = 0$. The theorem is a consequence of (15).

Let a be a heavy, positive real number. One can verify that $\frac{1}{a}$ is light and positive. Now we state the proposition:

(68) Let us consider positive natural numbers b, c. Then $c \cdot \operatorname{frac}(\frac{1}{b+c}) < 1$.

7. Integer Division Revisited

Let a be an integer. One can check that $a \operatorname{div} 1$ reduces to a.

Let a be a non zero integer. Let us observe that $1 \cdot a \operatorname{div} a$ reduces to 1.

Let a be a heavy, positive integer. Observe that $1 \operatorname{div} a$ is zero.

Let b be an integer. Observe that $b \operatorname{div} a \cdot b$ is zero.

Observe that $b \bmod a \cdot b$ reduces to b.

Now we state the propositions:

(69) Let us consider integers a, b, c. Then $a \cdot b \bmod a \cdot c = a \cdot (b \bmod c)$.

(70) Let us consider non zero integers a, b, and an integer c. Then $(c \bmod a \cdot b) \bmod a = c \bmod a$. The theorem is a consequence of (66) and (15).

(71) Let us consider an integer a, a non zero integer b, and a non zero natural number n. Then $(a \bmod b^n) \bmod b = a \bmod b$. The theorem is a consequence of (70).

Let us consider a non zero natural number a and a natural number b. Now we state the propositions:

(72) If $b \bmod a < \lceil \frac{a}{2} \rceil$, then $\operatorname{frac}(\frac{b}{a}) < \frac{1}{2}$. The theorem is a consequence of (66).

(73) If $b \bmod a < \lceil \frac{a}{2} \rceil$, then $2 \cdot b \bmod a = 2 \cdot (b \bmod a)$. The theorem is a consequence of (6), (66), (72), and (15).

(74) Let us consider an odd, a square integer a. Then $a \bmod 8 = 1$.

Let a be square integer. One can check that $a \bmod 3$ is a square and $a \bmod 4$ is a square and $a \bmod 8$ is a square. Now we state the propositions:

(75) Let us consider an integer a. Then $a^4 \bmod 8$ is a trivial natural number. The theorem is a consequence of (60).

(76) Let us consider an odd integer a. Then $a^4 \bmod 8 = 1$. The theorem is a consequence of (60).

(77) Let us consider an integer i. Then

(i) $i^4 \bmod 5 = 0$, or

(ii) $i^4 \bmod 5 = 1$.

(78) Let us consider odd natural numbers a, b. Suppose a and b are relatively prime. Then

(i) $\frac{a^4+b^4}{2} \bmod 5 = 3$, or

(ii) $\frac{a^4+b^4}{2} \bmod 5 = 1$.

The theorem is a consequence of (65).

(79) Let us consider natural numbers a, b. Suppose a and b are relatively prime. Then $a^4 - b^4 \bmod 5$ is a square. The theorem is a consequence of (9).

(80) Let us consider integers a, b, and a natural number n. Then $a^n \bmod b = (a \bmod b)^n \bmod b$.

(81) Let us consider an integer a, and a non zero integer b. Then $a^2 \bmod b = (b-a)^2 \bmod b$. The theorem is a consequence of (80).

(82) Let us consider integers a, b, and an odd integer c. If $a + b \bmod c = a - b \bmod c$, then $c \mid b$. The theorem is a consequence of (6).

(83) Let us consider an integer k. Then there exist integers a, b such that $a^2 - b^2 = k$ if and only if $k \bmod 4 \neq 2$.

PROOF: If there exist integers a, b such that $a^2 - b^2 = k$, then $k \bmod 4 \neq 2$ by [15, (1)], [17, (53)], [16, (2)]. If $k \bmod 4 \neq 2$, then there exist integers a, b such that $a^2 - b^2 = k$ [6, (11)], [15, (1)]. □

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