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Integrability of Multivariable Continuous Functions[1]

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Summary. In this article, we prove the integrability of continuous functions on n-dimensional real normed spaces, using the Mizar formalism. Generalizing selected theorems from the Mizar Mathematical Library, we prove the integrability of continuous real n-variable functions and then, using the correspondence between product-type and tuple-type spaces, we demonstrate the integrability of continuous functions on the desired multidimensional spaces.

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Introduction

This paper extends the formalization of measure theory in Mizar [15], [13], [16] by developing foundational results for the integrability on continuous functions on n-dimensional real normed spaces [2]. Related formalizations of this area have also been carried out in Isabelle/HOL [14] and Coq [3]. The authors have previously formalized several theorems on the integrability of continuous functions of two and three real variables [9, 10], as well as the consistency between two constructions of higher-dimensional spaces (less important from informal

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point of view [1]) – product-type and tuple-type – within simple real spaces [7]; this article is a natural continuation of [8], where this correspondence is further explored [17]. First, a partial strenghtening of theorems from previous articles [6], [9], [8] is given (Sect. 1). These are mainly generalizations considering the case of empty sets.

In Section 2, we prove the integrability of continuous real n-variable functions. Although we are dealing with functions on n-dimensional real normed spaces of the direct product type, the essence of the proof is, of course, the proof of the integrability on n-dimensional real number spaces of the direct product type. In Section 3, we prove the integrability of continuous functions on tuple-type n-dimensional real normed spaces, based on the results of the previous section. Finally, the results obtained in this article can be generalized slightly, but since the Riemann integral is defined on a non-empty closed interval within the Mizar Mathematical Library [11], many articles need to be modified to achieve this goal; we can foresee a revision of the MML [12], which will incorporate some of our enhancements in order to have more general form of the original theorems listed in Sect. 1.

1. Preliminaries

Now we state the propositions:

(1) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\mathbb{R}$. If $\operatorname{dom} f = \emptyset$, then f is integrable on M.

(2) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. If $\operatorname{dom} f = \emptyset$, then f is integrable on M. The theorem is a consequence of (1).

(3) Generalized [6]:3:
Let us consider non zero natural numbers n, i, j, k, an n-element finite sequence X, a j-element finite sequence X_1, and a k-element finite sequence X_2. Suppose $i \leqslant j \leqslant k$ and $X_1 = X{\restriction}j$ and $X_2 = X{\restriction}k$. Then $(\prod_{\rm FinS} X_1)(i) = (\prod_{\rm FinS} X_2)(i)$.
Proof: Define $\mathcal{P}$[non zero natural number] $\equiv$ if $\$_1 \leqslant j$, then $(\prod_{\rm FinS} X_1)(\$_1) = (\prod_{\rm FinS} X_2)(\$_1)$. $\mathcal{P}[1]$. For every non zero natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every non zero natural number m, $\mathcal{P}[m]$. □

(4) Generalized [6]:6:
Let us consider a non zero natural number n, an $(n+1)$-element finite sequence D, and an n-element finite sequence D_1. Suppose $D_1 = D{\restriction}n$. Then $\prod_{\rm FS} D = \prod_{\rm FS} D_1 \times D(n+1)$.

PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ if $\$_1 \leqslant n$, then $(\prod_{\text{FinS}} D)(\$_1) = (\prod_{\text{FinS}} D_1)(\$_1)$. $\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. $\square$

(5) GENERALIZED [9]:51:
Let us consider a subset I of $\mathbb{R}$, a closed interval subset J of $\mathbb{R}$, a partial function f from (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Suppose $I \times J = \operatorname{dom} f$ and f is continuous on $I \times J$ and $f = g$. Then

(i) Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|) \restriction I$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and

(ii) Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction I$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$.

(6) MISSING CASE OF [8]:6:
Let us consider a non zero natural number n, and an n-element finite sequence D. If D is not non-empty, then $\prod_{\text{FS}} D = \emptyset$.
PROOF: Consider i being an object such that $i \in \operatorname{dom} D$ and $D(i) = \emptyset$. Define $\mathcal{P}$[non zero natural number] $\equiv$ if $i \leqslant \$_1 \leqslant n$, then $(\prod_{\text{FinS}} D)(\$_1) = \emptyset$. $\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. $\square$

(7) GENERALIZED [8]:6:
Let us consider a non zero natural number n, an n-element finite sequence X, and an object x. Then $x \in \prod_{\text{FS}} X$ if and only if there exists an n-element finite sequence p_1 such that $p_1 \in \prod X$ and $x = \text{PtCarProd}(p_1)$. The theorem is a consequence of (6).

(8) GENERALIZED [8]:39:
Let us consider a non zero natural number n, and an n-element finite sequence D. Suppose for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i)$ is an element of L-Field. Then $\prod_{\text{FS}} D$ is an element of $\prod_{\text{Field}}$ L-Field(n). The theorem is a consequence of (6).

(9) GENERALIZED [8]:41:
Let us consider a non zero natural number n, and an n-element finite sequence D. Suppose for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i)$ is an interval. Then $\prod_{\text{FS}} D$ is an element of $\prod_{\text{Field}}$ L-Field(n). The theorem is a consequence of (8).

(10) GENERALIZED [8]:48:
Let us consider a non zero natural number n, and n-element finite sequences X, Y. Suppose for every natural number i such that $i \in \operatorname{Seg} n$ holds $X(i) \subseteq Y(i)$. Then $\prod_{\text{FS}} X \subseteq \prod_{\text{FS}} Y$. The theorem is a consequence of (6).

(11) GENERALIZED [8]:50:
Let us consider non zero natural numbers n, k, a non empty set X, and

an n-element finite sequence D. Suppose $k \in \operatorname{Seg} n$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i)$ is a subset of X. Then $(\prod_{\mathrm{FinS}} D)(k)$ is a subset of $\prod_{\mathrm{FS}}(\operatorname{Seg} k \longmapsto X)$.

PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ if $\$_1 \in \operatorname{Seg} n$, then $(\prod_{\mathrm{FinS}} D)$ $(\$_1)$ is a subset of $\prod_{\mathrm{FS}}(\operatorname{Seg} \$_1 \longmapsto X)$. $\mathcal{P}[1]$. For every non zero natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every non zero natural number k, $\mathcal{P}[k]$. $\square$

(12) GENERALIZED [8]:71:
Let us consider a non zero natural number n, and an n-element finite sequence D. Suppose for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i) \subseteq \mathbb{R}$. Then $\prod D = (\operatorname{CarProd}(\operatorname{Seg} n \longmapsto \mathbb{R}))^{\circ}(\prod_{\mathrm{FS}} D)$.

PROOF: Set $I = \operatorname{CarProd}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). $\prod_{\mathrm{FS}} D \subseteq$ the carrier of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). For every object x, $x \in \prod D$ iff $x \in I^{\circ}(\prod_{\mathrm{FS}} D)$. $\square$

2. Integrability of Continuous Real n-variable Functions

Now we state the propositions:

(13) (i) $\prod_{\mathrm{FS}}(\operatorname{Seg} 1 \longmapsto$ (the real normed space of $\mathbb{R}$)) = the real normed space of $\mathbb{R}$, and

(ii) $\operatorname{ElmFin}(\operatorname{Seg} 1 \longmapsto$ (the real normed space of $\mathbb{R}$), $1)$ = the real normed space of $\mathbb{R}$, and

(iii) $\prod_{\mathrm{FS}}(\operatorname{Seg} 2 \longmapsto$ (the real normed space of $\mathbb{R}$)) = (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$), and

(iv) $\operatorname{ElmFin}(\operatorname{Seg} 2 \longmapsto$ (the real normed space of $\mathbb{R}$), $2)$ = the real normed space of $\mathbb{R}$, and

(v) $\prod_{\mathrm{FS}}(\operatorname{Seg} 3 \longmapsto$ (the real normed space of $\mathbb{R}$)) = (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$) $\times$ (the real normed space of $\mathbb{R}$).

(14) Let us consider a non zero natural number n, a partial function f from $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element p of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$. Then $\operatorname{ProjPMap1}(g, p)$ is continuous.

PROOF: Set $P_1 = \operatorname{ProjPMap1}(g, p)$. For every real number y_0 such that $y_0 \in \operatorname{dom} P_1$ holds P_1 is continuous in y_0. $\square$

(15) Let us consider non empty sets X, Y, Z, a function T from X into Y, a partial function f from X to Z, and a partial function g from Y to Z. Suppose T is bijective and $g = f \cdot (T^{-1})$. Then

(i) $\operatorname{dom} g = T^{\circ} \operatorname{dom} f$, and

(ii) $\operatorname{dom} g = ({}^{\circ}T)(\operatorname{dom} f)$.

(16) Let us consider a non zero natural number n, a partial function f from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_2 from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, and an element q of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $P_2 = \operatorname{ProjPMap2}(g, q)$. Then P_2 is continuous on $\operatorname{dom} P_2$.

PROOF: For every point x_0 of $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $x_0 \in \operatorname{dom} P_2$ holds P_2 is continuous in x_0. $\square$

(17) Let us consider a non zero natural number n, a partial function g from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element q of $\mathbb{R}$. Then

(i) $(\operatorname{ProjPMap2}(g, q)) \cdot ((\operatorname{CarProd}(\operatorname{Seg} n \longmapsto \mathbb{R}))^{-1})$ is a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and

(ii) $\operatorname{dom}((\operatorname{ProjPMap2}(g, q)) \cdot ((\operatorname{CarProd}(\operatorname{Seg} n \longmapsto \mathbb{R}))^{-1})) = (\operatorname{CarProd}(\operatorname{Seg} n \longmapsto \mathbb{R}))^{\circ} \operatorname{dom}(\operatorname{ProjPMap2}(g, q))$.

(18) Let us consider a non zero natural number n, a partial function f from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element p of $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto \mathbb{R})$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$. Then $\operatorname{ProjPMap1}(|g|, p)$ is continuous. The theorem is a consequence of (14).

(19) Let us consider a non zero natural number n, a partial function f from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_2 from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, and an element q of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$ and $f = g$ and $P_2 = \operatorname{ProjPMap2}(|g|, q)$. Then P_2 is continuous on $\operatorname{dom} P_2$.

PROOF: Reconsider $P_1 = \operatorname{ProjPMap2}(g, q)$ as a partial function from $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$. P_1 is continuous on $\operatorname{dom} P_1$. For every point x_0 of $\prod_{\rm FS}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $x_0 \in \operatorname{dom} P_2$ holds P_2 is continuous in x_0. $\square$

(20) Let us consider a non zero natural number n, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element q of $\mathbb{R}$. Then

(i) $(\mathrm{ProjPMap2}(|g|, q))\cdot((\mathrm{CarProd}(\mathrm{Seg}\,n \longmapsto \mathbb{R}))^{-1})$ is a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and

(ii) $\mathrm{dom}((\mathrm{ProjPMap2}(|g|, q)) \cdot ((\mathrm{CarProd}(\mathrm{Seg}\,n \longmapsto \mathbb{R}))^{-1})) = (\mathrm{CarProd}(\mathrm{Seg}\,n \longmapsto \mathbb{R}))^{\circ}\,\mathrm{dom}(\mathrm{ProjPMap2}(|g|, q))$.

(21) Let us consider a non zero natural number n, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element p of $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto \mathbb{R})$. Suppose f is uniformly continuous on $\mathrm{dom}\,f$ and $f = g$. Then $\mathrm{ProjPMap1}(g, p)$ is uniformly continuous.

PROOF: Set $P_1 = \mathrm{ProjPMap1}(g, p)$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every real numbers y_1, y_2 such that $y_1, y_2 \in \mathrm{dom}\,P_1$ and $|y_1 - y_2| < s$ holds $|P_1(y_1) - P_1(y_2)| < r$. □

(22) Let us consider a non zero natural number n, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function P_2 from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, and an element s of $\mathbb{R}$. Suppose f is uniformly continuous on $\mathrm{dom}\,f$ and $f = g$ and $P_2 = \mathrm{ProjPMap2}(g, s)$. Then P_2 is uniformly continuous on $\mathrm{dom}\,P_2$.

PROOF: For every real number r such that $0 < r$ there exists a real number s_0 such that $0 < s_0$ and for every points x_1, x_2 of $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $x_1, x_2 \in \mathrm{dom}\,P_2$ and $\|x_1 - x_2\| < s_0$ holds $\|P_{2/x_1} - P_{2/x_2}\| < r$. □

(23) Let us consider a non zero natural number n, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function $\hat{f}$ from $\mathbb{R}$ to $\mathbb{R}$, and an element x of $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto \mathbb{R})$. Suppose f is continuous on $\mathrm{dom}\,f$ and $f = g$ and $\hat{f} = \mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), x)$. Then $\hat{f}$ is continuous. The theorem is a consequence of (14).

(24) Let us consider a non zero natural number n, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function $\bar{f}$ from $\prod_{\rm FS}(\mathrm{Seg}\,n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, and an element y of $\mathbb{R}$. Suppose

f is continuous on dom f and $f = g$ and $\bar{f} = \mathrm{ProjPMap2}(\overline{\mathbb{R}}(g), y)$. Then $\bar{f}$ is continuous on dom $\bar{f}$. The theorem is a consequence of (16).

(25) Let us consider a non zero natural number n, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function $\hat{f}$ from $\mathbb{R}$ to $\mathbb{R}$, and an element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. Suppose f is continuous on dom f and $f = g$ and $\hat{f} = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x)$. Then $\hat{f}$ is continuous. The theorem is a consequence of (18).

(26) Let us consider a non zero natural number n, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a partial function $\bar{f}$ from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, and an element y of $\mathbb{R}$. Suppose f is continuous on dom f and $f = g$ and $\bar{f} = \mathrm{ProjPMap2}(|\overline{\mathbb{R}}(g)|, y)$. Then $\bar{f}$ is continuous on dom $\bar{f}$. The theorem is a consequence of (19).

Let us consider a non zero natural number n, a subset I of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a non empty, closed interval subset J of $\mathbb{R}$, an element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $\hat{f}$ from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(27) Suppose $x \in I$ and dom $f = I \times J$ and f is continuous on $I \times J$ and $f = g$ and $\hat{f} = \mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), x)$. Then

(i) $\hat{f}\restriction J$ is bounded, and

(ii) $\hat{f}$ is integrable on J.

The theorem is a consequence of (14).

(28) Suppose $x \in I$ and dom $f = I \times J$ and f is continuous on $I \times J$ and $f = g$ and $\hat{f} = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x)$. Then

(i) $\hat{f}\restriction J$ is bounded, and

(ii) $\hat{f}$ is integrable on J.

The theorem is a consequence of (25).

(29) Let us consider a non zero natural number n, a subset I of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a closed interval subset J of $\mathbb{R}$, an element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $\hat{f}$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose dom $f = I \times J$ and f is continuous on $I \times J$ and $f = g$ and

$\hat{f} = \mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), x)$. Then $\hat{f}$ is integrable on L-Meas. The theorem is a consequence of (27) and (1).

(30) Let us consider a non zero natural number n, a subset I of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a non empty, closed interval subset J of $\mathbb{R}$, an element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $\hat{f}$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $\mathrm{dom}\, f = I \times J$ and f is continuous on $I \times J$ and $f = g$ and $\hat{f} = \mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), x)$. Then

(i) $\int_J \hat{f}(x)dx = \int \hat{f}\, \mathrm{d}\,$L-Meas, and

(ii) $\int_J \hat{f}(x)dx = \int \mathrm{ProjPMap1}(\overline{\mathbb{R}}(g), x)\, \mathrm{d}\,$L-Meas, and

(iii) $\int_J \hat{f}(x)dx = (\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g)))(x)$.

The theorem is a consequence of (27).

(31) Let us consider a non zero natural number n, a subset I of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a closed interval subset J of $\mathbb{R}$, an element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $\hat{f}$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $\mathrm{dom}\, f = I \times J$ and f is continuous on $I \times J$ and $f = g$ and $\hat{f} = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x)$. Then $\hat{f}$ is integrable on L-Meas. The theorem is a consequence of (25) and (1).

(32) Let us consider a non zero natural number n, a subset I of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a non empty, closed interval subset J of $\mathbb{R}$, an element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $\hat{f}$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose $x \in I$ and $\mathrm{dom}\, f = I \times J$ and f is continuous on $I \times J$ and $f = g$ and $\hat{f} = \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x)$. Then

(i) $\int_J \hat{f}(x)dx = \int \hat{f}\, \mathrm{d}\,$L-Meas, and

(ii) $\int_J \hat{f}(x)dx = \int \mathrm{ProjPMap1}(|\overline{\mathbb{R}}(g)|, x)\, \mathrm{d}\,$L-Meas, and

(iii) $\int_J \hat{f}(x)dx = (\mathrm{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(x)$.

The theorem is a consequence of (28).

(33) Let us consider a non zero natural number n, a subset I of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a subset J of $\mathbb{R}$, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of $\sigma(\mathrm{MeasRect}(\prod_{\rm Field} \text{L-Field}(n), \text{L-Field}))$. Suppose $I \times J = \mathrm{dom}\, f$ and f is continuous on $I \times J$ and $f = g$ and $E = I \times J$. Then g is E-measurable.

PROOF: For every real number r, $E \cap \text{LE-dom}(g, r) \in \sigma(\mathrm{MeasRect}(\prod_{\rm Field} \text{L-Field}(n), \text{L-Field}))$. $\square$

(34) Let us consider a non zero natural number n, a subset I of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, a closed interval subset J of $\mathbb{R}$, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $I \times J = \mathrm{dom}\, f$ and f is continuous on $I \times J$ and $f = g$. Then

(i) Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|$) is a partial function from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$ to $\mathbb{R}$, and

(ii) Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is a partial function from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$ to $\mathbb{R}$.

The theorem is a consequence of (32) and (30).

(35) Let us consider a non zero natural number n, an n-element finite sequence D, a closed interval subset J of $\mathbb{R}$, and a subset E of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$). Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $E = \prod_{\rm FS} D \times J$. Then E is compact. The theorem is a consequence of (6).

(36) Let us consider a non zero natural number n, a set E, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $f = g$ and $E \subseteq \mathrm{dom}\, f$.

Then f is uniformly continuous on E if and only if for every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every points x_1, x_2 of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) and for every real numbers y_1, y_2 such that $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in E$ and $\|x_2 - x_1\| < r$ and $|y_2 - y_1| < r$ holds $|g(\langle x_2, y_2 \rangle) - g(\langle x_1, y_1 \rangle)| < e$.

PROOF: For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every points z_1, z_2 of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) such that z_1, $z_2 \in E$ and $\|z_1 - z_2\| < r$ holds $\|f_{/z_1} - f_{/z_2}\| < e$. $\square$

(37) Let us consider a non zero natural number n, an n-element finite sequence D, a closed interval subset J of $\mathbb{R}$, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$.

Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $f = g$ and f is continuous on $\prod_{\mathrm{FS}} D \times J$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every points x_1, x_2 of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) and for every real numbers y_1, y_2 such that $\langle x_1, y_1 \rangle$, $\langle x_2, y_2 \rangle \in \prod_{\mathrm{FS}} D \times J$ and $\|x_2 - x_1\| < r$ and $|y_2 - y_1| < r$ holds $|g(\langle x_2, y_2 \rangle) - g(\langle x_1, y_1 \rangle)| < e$.

PROOF: $\prod_{\mathrm{FS}} D$ is a subset of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. There exists a subset I of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $I = \prod_{\mathrm{FS}} D$ and I is compact. Consider I being a subset of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $I = \prod_{\mathrm{FS}} D$ and I is compact. Reconsider $J_1 = J$ as a subset of the real normed space of $\mathbb{R}$. Reconsider $E = \prod_{\mathrm{FS}} D \times J_1$ as a subset of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$). E is compact. $\square$

(38) Let us consider a set X, a real normed space S, a partial function f from S to the real normed space of $\mathbb{R}$, and a partial function g from X to $\mathbb{R}$. If $f = g$, then $\|f\| = |g|$.

(39) Let us consider a non zero natural number n, an n-element finite sequence D, a closed interval subset J of $\mathbb{R}$, a partial function f from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$.

Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and f is continuous on $\prod_{\mathrm{FS}} D \times J$ and $f = g$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every points x_1, x_2 of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) and for every real numbers y_1, y_2 such that $\langle x_1, y_1 \rangle$, $\langle x_2,$

$y_2\rangle \in \prod_{\rm FS} D \times J$ and $\|x_2 - x_1\| < r$ and $|y_2 - y_1| < r$ holds $||g|(\langle x_2, y_2\rangle) - |g|(\langle x_1, y_1\rangle)| < e$.

The theorem is a consequence of (38) and (37).

Let us consider a non zero natural number n, an n-element finite sequence D, a subset I of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), a closed interval subset J of $\mathbb{R}$, a partial function f from $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_2 from $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$. Now we state the propositions:

(40) Suppose for every natural number i such that $i \in {\rm Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $I = \prod_{\rm FS} D$ and $I \times J = {\rm dom}\, f$ and f is continuous on $I \times J$ and $f = g$ and $G_2 = $ Integral2(L-Meas, $|\overline{\mathbb{R}}(g)|) \restriction I$. Then G_2 is continuous on I.

PROOF: Consider c, d being real numbers such that $J = [c, d]$. Set $R_0 = \overline{\mathbb{R}}(g)$. For every element x of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R})$ and for every element y of $\mathbb{R}$ such that $x \in I$ and $y \in J$ holds $({\rm ProjPMap1}(|R_0|, x))(y) = |R_0|(x, y)$ and $|R_0|(x, y) = |g(\langle x, y\rangle)|$ and $|R_0|(x, y) = |g|(\langle x, y\rangle)$.

For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements x_1, x_2 of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R})$ and for every points x_4, x_5 of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $x_4 = x_1$ and $x_5 = x_2$ and $\|x_5 - x_4\| < r$ and x_1, $x_2 \in I$ for every element y of $\mathbb{R}$ such that $y \in J$ holds $|({\rm ProjPMap1}(|R_0|, x_2))(y) - ({\rm ProjPMap1}(|R_0|, x_1))(y)| < e$. $\prod_{\rm FS} D$ is a subset of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R})$. $\square$

(41) Suppose for every natural number i such that $i \in {\rm Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $I = \prod_{\rm FS} D$ and $I \times J = {\rm dom}\, f$ and f is continuous on $I \times J$ and $f = g$ and $G_2 = $ Integral2(L-Meas, $\overline{\mathbb{R}}(g)) \restriction I$. Then G_2 is continuous on I.

PROOF: Consider c, d being real numbers such that $J = [c, d]$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every points x_1, x_2 of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $\|x_2 - x_1\| < r$ and x_1, $x_2 \in I$ for every real number y such that $y \in J$ holds $|g(\langle x_2, y\rangle) - g(\langle x_1, y\rangle)| < e$. Set $R_0 = \overline{\mathbb{R}}(g)$.

For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements x_1, x_2 of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R})$ and for every points x_4, x_5 of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $x_4 = x_1$ and $x_5 = x_2$ and $\|x_5 - x_4\| < r$ and x_1, $x_2 \in I$ for every element y of $\mathbb{R}$ such that $y \in J$ holds $|({\rm ProjPMap1}(R_0, x_2))(y) - ({\rm ProjPMap1}(R_0, x_1))(y)| < e$. $\prod_{\rm FS} D$ is a subset of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R})$. $\square$

(42) Let us consider a non zero natural number n, an n-element finite sequence D, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, and a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$ to $\mathbb{R}$.

Suppose $f = g$ and g is continuous on $\prod_{\rm FS} D$ and $\prod_{\rm FS} D \subseteq \mathrm{dom}\, g$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$. Let us consider a real number e. Suppose $0 < e$. Then there exists a real number r such that

(i) $0 < r$, and

(ii) for every n-element finite sequences x, y of elements of $\mathbb{R}$ such that PtCarProd(x), PtCarProd(y) $\in \prod_{\rm FS} D$ and for every natural number i such that $i \in \mathrm{dom}\, D$ there exist real numbers x_3, y_3 such that $x_3 = x(i)$ and $y_3 = y(i)$ and $|x_3 - y_3| < r$ holds $|f(\mathrm{PtCarProd}(x)) - f(\mathrm{PtCarProd}(y))| < e$.

PROOF: Set $S = \mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$). $\prod_{\rm FS} D$ is a subset of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. There exists a subset I of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $I = \prod_{\rm FS} D$ and I is compact. Consider E being a subset of $\prod_{\rm FS} S$ such that $E = \prod_{\rm FS} D$ and E is compact. Consider r_0 being a real number such that $0 < r_0$ and for every points z_1, z_2 of $\prod_{\rm FS} S$ such that $z_1, z_2 \in E$ and $\|z_1 - z_2\| < r_0$ holds $\|g_{/z_1} - g_{/z_2}\| < e$. Set $r_1 = \frac{r_0}{2}$. Set $r = \frac{r_1}{n}$. Reconsider $z_1 = \mathrm{PtCarProd}(x)$, $z_2 = \mathrm{PtCarProd}(y)$ as a point of $\prod_{\rm FS} S$. Reconsider $m = n - 1$ as a natural number.

Consider p_2 being an $(m+1)$-element finite sequence such that $z_1 - z_2 = \mathrm{PtCarProd}(p_2)$. Consider n_1 being an element of $\mathcal{R}^{m+1}$ such that for every non zero natural number i such that $i \leqslant m + 1$ there exists a point p_3 of ElmFin(S, i) such that $p_3 = p_2(i)$ and $n_1(i) = \|p_3\|$ and $\|z_1 - z_2\| = |n_1|$. For every natural number i such that $i \in \mathrm{dom}\, n_1$ holds $0 \leqslant n_1(i) \leqslant r$. □

(43) Let us consider a non zero natural number n, an n-element finite sequence D, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$ to $\mathbb{R}$. Suppose $f = g$ and for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and f is continuous on $\prod_{\rm FS} D$ and $\mathrm{dom}\, f = \prod_{\rm FS} D$. Then g is integrable on $\mathrm{Measure}_{\rm Prod}(\text{L-Meas}(n))$.
PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every $\$_1$-element finite sequence D for every partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, \$_1 \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$ for every partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, \$_1 \longmapsto \mathbb{R})$ to $\mathbb{R}$ such that $f = g$ and for every natural number i such that $i \in \mathrm{Seg}\, \$_1$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and f is continuous on $\prod_{\rm FS} D$ and $\mathrm{dom}\, f = \prod_{\rm FS} D$ holds g is integrable

on $\text{Measure}_{\text{Prod}}(\text{L-Meas}(\$_1))$.

$\mathcal{P}[1]$ by (13), [7, (37)], [4, (72), (75)]. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [8, (38)], (9), (42), [8, (66)]. For every non zero natural number n, $\mathcal{P}[n]$. □

(44) Let us consider a non zero natural number n, an n-element finite sequence D, a subset J of $\mathbb{R}$, an element y of $\mathbb{R}$, a partial function f from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \text{(the real normed space of } \mathbb{R})) \times \text{(the real normed space of } \mathbb{R})$ to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose for every natural number i such that $i \in \text{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $y \in J$ and $\text{dom}\, f = \prod_{\text{FS}} D \times J$ and f is continuous on $\prod_{\text{FS}} D \times J$ and $f = g$. Then

(i) $\text{dom}(\text{ProjPMap2}(\overline{\mathbb{R}}(g), y)) = \prod_{\text{FS}} D$, and

(ii) $\text{ProjPMap2}(\overline{\mathbb{R}}(g), y)$ is integrable on $\text{Measure}_{\text{Prod}}(\text{L-Meas}(n))$.

The theorem is a consequence of (11), (24), and (43).

(45) Let us consider a non zero natural number n, an n-element finite sequence D, a subset J of $\mathbb{R}$, an element y of $\mathbb{R}$, a partial function f from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \text{(the real normed space of } \mathbb{R})) \times \text{(the real normed space of } \mathbb{R})$ to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $\bar{f}$ from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R})$ to $\mathbb{R}$. Suppose for every natural number i such that $i \in \text{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $y \in J$ and $\text{dom}\, f = \prod_{\text{FS}} D \times J$ and f is continuous on $\prod_{\text{FS}} D \times J$ and $f = g$ and $\bar{f} = \text{ProjPMap2}(\overline{\mathbb{R}}(g), y)$. Then

(i) $\bar{f}$ is integrable on $\text{Measure}_{\text{Prod}}(\text{L-Meas}(n))$, and

(ii) $\int \text{ProjPMap2}(\overline{\mathbb{R}}(g), y)\, \text{d}\, \text{Measure}_{\text{Prod}}(\text{L-Meas}(n)) = (\text{Integral1}(\text{Measure}_{\text{Prod}}(\text{L-Meas}(n)), \overline{\mathbb{R}}(g)))(y)$.

The theorem is a consequence of (44).

(46) Let us consider a non zero natural number n, an n-element finite sequence D, a subset J of $\mathbb{R}$, an element y of $\mathbb{R}$, a partial function f from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \text{(the real normed space of } \mathbb{R})) \times \text{(the real normed space of } \mathbb{R})$ to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose for every natural number i such that $i \in \text{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $y \in J$ and $\text{dom}\, f = \prod_{\text{FS}} D \times J$ and f is continuous on $\prod_{\text{FS}} D \times J$ and $f = g$. Then

(i) $\text{dom}(\text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, y)) = \prod_{\text{FS}} D$, and

(ii) $\text{ProjPMap2}(|\overline{\mathbb{R}}(g)|, y)$ is integrable on $\text{Measure}_{\text{Prod}}(\text{L-Meas}(n))$.

The theorem is a consequence of (11), (26), and (43).

(47) Let us consider a non zero natural number n, an n-element finite sequence D, a subset J of $\mathbb{R}$, an element y of $\mathbb{R}$, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and an element E of $\prod_{\rm Field}$ L-Field(n). Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $y \in J$ and $\mathrm{dom}\, f = \prod_{\rm FS} D \times J$ and f is continuous on $\prod_{\rm FS} D \times J$ and $f = g$ and $E = \prod_{\rm FS} D$. Then ProjPMap2$(|\overline{\mathbb{R}}(g)|, y)$ is E-measurable. The theorem is a consequence of (46).

(48) Let us consider a non zero natural number n, an n-element finite sequence D, a subset J of $\mathbb{R}$, an element y of $\mathbb{R}$, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function $\bar{f}$ from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$ to $\mathbb{R}$. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $y \in J$ and $\mathrm{dom}\, f = \prod_{\rm FS} D \times J$ and f is continuous on $\prod_{\rm FS} D \times J$ and $f = g$ and $\bar{f} =$ ProjPMap2$(|\overline{\mathbb{R}}(g)|, y)$. Then

(i) $\bar{f}$ is integrable on Measure$_{\rm Prod}$(L-Meas(n)), and

(ii) $\int$ ProjPMap2$(|\overline{\mathbb{R}}(g)|, y)\, \mathrm{d}\,$Measure$_{\rm Prod}$(L-Meas$(n)$) $=$ (Integral1(Measure$_{\rm Prod}$(L-Meas(n)), $|\overline{\mathbb{R}}(g)|$))$(y)$.

The theorem is a consequence of (46).

(49) Let us consider a non zero natural number n, an n-element finite sequence D, a subset I of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), and an interval J. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is an interval and $I = \prod_{\rm FS} D$. Then

(i) $I \times J$ is a subset of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$), and

(ii) $I \times J \in \sigma$(MeasRect($\prod_{\rm Field}$ L-Field(n), L-Field)).

The theorem is a consequence of (9).

(50) Let us consider a non zero natural number n, an n-element finite sequence D, a subset J of $\mathbb{R}$, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $\prod_{\rm FS} D \times J = \mathrm{dom}\, f$ and f is continuous on $\prod_{\rm FS} D \times J$ and $f = g$. Then

(i) Integral1(Measure$_{\rm Prod}$(L-Meas(n)), $|\overline{\mathbb{R}}(g)|$)$\restriction J$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$, and

(ii) Integral1($\text{Measure}_{\text{Prod}}$(L-Meas($n$)), $\overline{\mathbb{R}}(g)$)$\restriction J$ is a partial function from $\mathbb{R}$ to $\mathbb{R}$.

The theorem is a consequence of (48) and (45).

(51) Let us consider a non zero natural number n, an element E_1 of $\prod_{\text{Field}}$ L-Field(n), and an element E_2 of L-Field. Then

(i) $E_1 \times E_2 \in \sigma$(MeasRect($\prod_{\text{Field}}$ L-Field(n), L-Field)), and

(ii) ($\text{Measure}_{\text{Prod}}$(L-Meas($n+1$)))($E_1 \times E_2$) = ($\text{Measure}_{\text{Prod}}$(L-Meas($n$)))($E_1$) $\cdot$ (L-Meas)(E_2).

(52) Let us consider a non zero natural number n, and an n-element finite sequence D. Suppose for every natural number i such that $i \in \text{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$. Then there exists a real number r and there exists an element E of $\prod_{\text{Field}}$ L-Field(n) such that $E = \prod_{\text{FS}} D$ and ($\text{Measure}_{\text{Prod}}$(L-Meas($n$)))($E$) $= r$ and $0 \leqslant r$.

PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every $\$_1$-element finite sequence D such that for every natural number i such that $i \in \text{Seg}\, \$_1$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ there exists a real number r and there exists an element E of $\prod_{\text{Field}}$ L-Field($\$_1$) such that $E = \prod_{\text{FS}} D$ and ($\text{Measure}_{\text{Prod}}$(L-Meas($\$_1$)))($E$) $= r$ and $0 \leqslant r$. $\mathcal{P}[1]$ by [7, (41)], [5, (5)], [7, (45)]. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. $\square$

Let us consider a non zero natural number n, an n-element finite sequence D, a closed interval subset J of $\mathbb{R}$, a partial function f from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_1 from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(53) Suppose for every natural number i such that $i \in \text{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $\prod_{\text{FS}} D \times J = \text{dom}\, f$ and f is continuous on $\prod_{\text{FS}} D \times J$ and $f = g$ and $G_1 =$ Integral1($\text{Measure}_{\text{Prod}}$(L-Meas($n$)), $|\overline{\mathbb{R}}(g)|$)$\restriction J$. Then G_1 is continuous.

PROOF: For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers y_1, y_2 such that $|y_2 - y_1| < r$ and $y_1, y_2 \in J$ for every point x of $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $x \in \prod_{\text{FS}} D$ holds $||g|(\langle x, y_2\rangle) - |g|(\langle x, y_1\rangle)| < e$. Set $R_0 = \overline{\mathbb{R}}(g)$. For every element x of $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R})$ and for every element y of $\mathbb{R}$ such that $x \in \prod_{\text{FS}} D$ and $y \in J$ holds (ProjPMap2($|R_0|$, y))(x) $=$ $|R_0|(x, y)$ and $|R_0|(x, y) = |g(\langle x, y\rangle)|$ and $|R_0|(x, y) = |g|(\langle x, y\rangle)$.

For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements y_1, y_2 of $\mathbb{R}$ such that $|y_2 - y_1| < r$

and y_1, $y_2 \in J$ for every element x of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R})$ such that $x \in \prod_{\rm FS} D$ holds $|({\rm ProjPMap2}(|R_0|, y_2))(x) - ({\rm ProjPMap2}(|R_0|, y_1))(x)| < e$. For every real numbers y_0, r such that $y_0 \in J$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number y_1 such that $y_1 \in J$ and $|y_1 - y_0| < s$ holds $|G_1(y_1) - G_1(y_0)| < r$. □

(54) Suppose for every natural number i such that $i \in {\rm Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $\prod_{\rm FS} D \times J = {\rm dom}\, f$ and f is continuous on $\prod_{\rm FS} D \times J$ and $f = g$ and $G_1 = {\rm Integral1}({\rm Measure}_{\rm Prod}(\text{L-Meas}(n)), \overline{\mathbb{R}}(g)) \restriction J$. Then G_1 is continuous.

PROOF: Set $I = \prod_{\rm FS} D$. For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every real numbers y_1, y_2 such that $|y_2 - y_1| < r$ and y_1, $y_2 \in J$ for every point x of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $x \in I$ holds $|g(\langle x, y_2\rangle) - g(\langle x, y_1\rangle)| < e$. Set $R_0 = \overline{\mathbb{R}}(g)$.

For every real number e such that $0 < e$ there exists a real number r such that $0 < r$ and for every elements y_1, y_2 of $\mathbb{R}$ such that $|y_2 - y_1| < r$ and y_1, $y_2 \in J$ for every element x of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R})$ such that $x \in I$ holds $|({\rm ProjPMap2}(R_0, y_2))(x) - ({\rm ProjPMap2}(R_0, y_1))(x)| < e$. For every real numbers y_0, r such that $y_0 \in J$ and $0 < r$ there exists a real number s such that $0 < s$ and for every real number y_1 such that $y_1 \in J$ and $|y_1 - y_0| < s$ holds $|G_1(y_1) - G_1(y_0)| < r$. □

(55) Let us consider a non zero natural number n, an n-element finite sequence D, a subset I of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), a closed interval subset J of $\mathbb{R}$, a partial function f from $\prod_{\rm FS}({\rm Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, and a partial function g from $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose for every natural number i such that $i \in {\rm Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $I = \prod_{\rm FS} D$ and $I \times J = {\rm dom}\, f$ and f is continuous on $I \times J$ and $f = g$. Then

(i) g is integrable on ${\rm ProdMeas}({\rm Measure}_{\rm Prod}(\text{L-Meas}(n)), \text{L-Meas})$, and

(ii) for every element x of $\prod_{\rm FS}({\rm Seg}\, n \longmapsto \mathbb{R})$, $({\rm Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g)|))(x) < +\infty$, and

(iii) for every element y of $\mathbb{R}$, $({\rm Integral1}({\rm Measure}_{\rm Prod}(\text{L-Meas}(n)), |\overline{\mathbb{R}}(g)|))(y) < +\infty$, and

(iv) for every element U of $\prod_{\rm Field} \text{L-Field}(n)$, ${\rm Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g))$ is U-measurable, and

(v) for every element V of L-Field, ${\rm Integral1}({\rm Measure}_{\rm Prod}(\text{L-Meas}(n)), \overline{\mathbb{R}}(g))$ is V-measurable, and

(vi) Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) is integrable on $\text{Measure}_{\text{Prod}}$(L-Meas($n$)), and

(vii) Integral1($\text{Measure}_{\text{Prod}}$(L-Meas($n$)), $\overline{\mathbb{R}}(g)$) is integrable on L-Meas, and

(viii) $\int g \, \mathrm{d}$ ProdMeas($\text{Measure}_{\text{Prod}}$(L-Meas($n$)), L-Meas) = $\int$ Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) d $\text{Measure}_{\text{Prod}}$(L-Meas($n$)), and

(ix) $\int g \, \mathrm{d}$ ProdMeas($\text{Measure}_{\text{Prod}}$(L-Meas($n$)), L-Meas) = $\int$ Integral1($\text{Measure}_{\text{Prod}}$(L-Meas($n$)), $\overline{\mathbb{R}}(g)$) d L-Meas.

(56) Let us consider a non zero natural number n, an $(n+1)$-element finite sequence D, a partial function f from $\prod_{\text{FS}}(\text{Seg}(n+1) \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\text{FS}}(\text{Seg}(n+1) \longmapsto \mathbb{R})$ to $\mathbb{R}$, and a partial function g_0 from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose $f = g$ and $g_0 = g$ and for every natural number i such that $i \in \text{Seg}(n+1)$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and f is continuous on $\prod_{\text{FS}} D$ and $\text{dom}\, f = \prod_{\text{FS}} D$. Then

(i) Integral2(L-Meas, $\overline{\mathbb{R}}(g_0)$) is integrable on $\text{Measure}_{\text{Prod}}$(L-Meas($n$)), and

(ii) $\int g \, \mathrm{d}\, \text{Measure}_{\text{Prod}}$(L-Meas($n+1$)) = $\int g_0 \, \mathrm{d}$ ProdMeas($\text{Measure}_{\text{Prod}}$(L-Meas($n$)), L-Meas).

PROOF: Reconsider $D_3 = D \restriction n$ as an n-element finite sequence. For every natural number i such that $i \in \text{Seg}\, n$ holds $D_3(i)$ is a closed interval subset of $\mathbb{R}$. $\prod_{\text{FS}} D_3$ is a subset of $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R})$. Reconsider $D_0 = \prod_{\text{FS}} D_3$ as a subset of $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)). Reconsider $D_1 = D(n+1)$ as a closed interval subset of $\mathbb{R}$. $\prod_{\text{FS}} D = D_0 \times D_1$. □

(57) Let us consider a non zero natural number n, an n-element finite sequence D, a subset I of $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), a closed interval subset J of $\mathbb{R}$, a partial function f from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_2 from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R})$ to $\mathbb{R}$.

Suppose for every natural number i such that $i \in \text{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $I = \prod_{\text{FS}} D$ and $I \times J = \text{dom}\, f$ and f is continuous on $I \times J$ and $f = g$ and G_2 = Integral2(L-Meas, $\overline{\mathbb{R}}(g)$) $\restriction \prod_{\text{FS}} D$. Then $\int \overline{\mathbb{R}}(g) \, \mathrm{d}$ ProdMeas($\text{Measure}_{\text{Prod}}$(L-Meas($n$)), L-Meas) = $\int G_2 \, \mathrm{d}\, \text{Measure}_{\text{Prod}}$(L-Meas($n$)).

PROOF: Set $R_0 = \overline{\mathbb{R}}(g)$. Set R_2 = Integral2(L-Meas, R_0). Reconsider $I_0 = \prod_{\text{FS}} D$ as a subset of $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R})$. Set $N_1 = (\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R})) \setminus I_0$. Reconsider $F_0 = R_2 \restriction I_0$, $F_1 = R_2 \restriction N_1$ as a partial function from $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R})$ to $\overline{\mathbb{R}}$. I_0 is an element of $\prod_{\text{Field}}$ L-Field(n). For every element x of $\prod_{\text{FS}}(\text{Seg}\, n \longmapsto \mathbb{R})$ such that $x \in \text{dom}\, F_1$ holds $F_1(x) = 0$. □

(58) Let us consider a non zero natural number n, an n-element finite sequence D, a subset I of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), a non empty, closed interval subset J of $\mathbb{R}$, a partial function f from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, and a partial function G_1 from $\mathbb{R}$ to $\mathbb{R}$.

Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and $I = \prod_{\rm FS} D$ and $I \times J = \mathrm{dom}\, f$ and f is continuous on $I \times J$ and $f = g$ and $G_1 = \mathrm{Integral1}(\mathrm{Measure}_{\rm Prod}(\text{L-Meas}(n)), \overline{\mathbb{R}}(g)) \restriction J$. Then $\int \overline{\mathbb{R}}(g)\, \mathrm{d}\, \mathrm{ProdMeas}(\mathrm{Measure}_{\rm Prod}(\text{L-Meas}(n)), \text{L-Meas}) = \int_J G_1(x) dx$.

Proof: Set $R_0 = \overline{\mathbb{R}}(g)$. Set $N_2 = \mathbb{R} \setminus J$. Set $R_1 = \mathrm{Integral1}(\mathrm{Measure}_{\rm Prod}(\text{L-Meas}(n)), R_0)$. Reconsider $F_0 = R_1 \restriction J$, $F_1 = R_1 \restriction N_2$ as a partial function from $\mathbb{R}$ to $\overline{\mathbb{R}}$. $G_1 \restriction J$ is bounded and G_1 is integrable on J. $\prod_{\rm FS} D$ is a subset of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. For every element y of $\mathbb{R}$ such that $y \in \mathrm{dom}\, F_1$ holds $F_1(y) = 0$. $\square$

3. Integrability of Continuous Functions on n-dimensional Real Normed Spaces

Now we state the propositions:

(59) Let us consider a non zero natural number n, and a partial function f from $\prod(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$. Then $f \cdot (\mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$))) is a partial function from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$.

(60) Let us consider a non zero natural number n, an n-element finite sequence D, and a partial function f from $\prod(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a subset of $\mathbb{R}$ and f is continuous on $\mathrm{dom}\, f$ and $\mathrm{dom}\, f = \prod D$. Then $f \cdot (\mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$))) is continuous on $\prod_{\rm FS} D$.

Proof: Set $I = \mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)). Consider C being a non-empty, n-element finite sequence such that $C = \overline{\mathrm{Seg}\, n \longmapsto \alpha}$ and $\prod_{\rm FS} C =$ the carrier of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) and $I = \mathrm{CarProd}(C)$ and I is bijective, where α is the real normed space of $\mathbb{R}$. Reconsider $F = f \cdot I$ as a partial function from

$\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$. $\prod_{\rm FS} D$ is a subset of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$.

For every object x, $x \in \prod D$ iff $x \in I^{\circ}(\prod_{\rm FS} D)$. For every point x_0 of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) and for every real number r such that $x_0 \in \prod_{\rm FS} D$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $x_1 \in \prod_{\rm FS} D$ and $\|x_1 - x_0\| < s$ holds $\|F_{/x_1} - F_{/x_0}\| < r$. □

(61) Let us consider a non zero natural number n. Then

(i) $\langle \mathcal{E}^n, \|\cdot\| \rangle = \prod(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), and

(ii) $\mathcal{R}^n =$ the carrier of $\prod(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), and

(iii) $\mathcal{R}^n = \prod(\mathrm{Seg}\, n \longmapsto \mathbb{R})$.

(62) Let us consider a non zero natural number n, an n-element finite sequence D, and a partial function f from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to the real normed space of $\mathbb{R}$. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a subset of $\mathbb{R}$ and f is continuous on $\mathrm{dom}\, f$ and $\mathrm{dom}\, f = \prod D$. Then $f \cdot (\mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$))) is continuous on $\prod_{\rm FS} D$. The theorem is a consequence of (61) and (60).

(63) Let us consider a non zero natural number n, an n-element finite sequence D, a partial function f from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to the real normed space of $\mathbb{R}$, a partial function g from $\mathcal{R}^n$ to $\mathbb{R}$, and a partial function G from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$ to $\mathbb{R}$. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and f is continuous on $\prod D$ and $\mathrm{dom}\, f = \prod D$ and $g = f$ and $G = f \cdot (\mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto \mathbb{R}))$. Then

(i) G is integrable on $\mathrm{Measure}_{\rm Prod}(\text{L-Meas}(n))$, and

(ii) g is integrable on $\text{XL-Meas}(n)$, and

(iii) $\int g \, \mathrm{d}\, \text{XL-Meas}(n) = \int G \, \mathrm{d}\, \mathrm{Measure}_{\rm Prod}(\text{L-Meas}(n))$.

PROOF: Set $I = \mathrm{CarProd}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)). Consider C being a non-empty, n-element finite sequence such that $C = \overline{\mathrm{Seg}\, n \longmapsto \alpha}$ and $\prod_{\rm FS} C =$ the carrier of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) and $I = \mathrm{CarProd}(C)$ and I is bijective, where α is the real normed space of $\mathbb{R}$. $\langle \mathcal{E}^n, \|\cdot\| \rangle = \prod(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)). The carrier of $\langle \mathcal{E}^n, \|\cdot\| \rangle = \mathcal{R}^n$. Reconsider $F = f \cdot I$ as a partial function from $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$. F is continuous on $\prod_{\rm FS} D$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i) \subseteq \mathbb{R}$. $\prod_{\rm FS} D \subseteq \prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. $\prod D = I^{\circ}(\prod_{\rm FS} D)$. □

(64) Let us consider a non zero natural number n, an $(n+1)$-element finite sequence D, a partial function f from $\langle \mathcal{E}^{n+1}, \| \cdot \| \rangle$ to the real normed space of $\mathbb{R}$, a partial function G from $\prod_{\mathrm{FS}}(\mathrm{Seg}(n+1) \longmapsto \mathbb{R})$ to $\mathbb{R}$, and a partial function g_0 from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$. Suppose for every natural number i such that $i \in \mathrm{Seg}(n+1)$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ and f is continuous on $\prod D$ and $\mathrm{dom}\, f = \prod D$ and $G = f \cdot (\mathrm{CarProd}(\mathrm{Seg}(n+1) \longmapsto \mathbb{R}))$ and $g_0 = G$. Then

(i) for every element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, $(\mathrm{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g_0)|))(x) < +\infty$, and

(ii) for every element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, $\mathrm{ProjPMap1}(\overline{\mathbb{R}}(g_0), x)$ is integrable on L-Meas, and

(iii) for every element U of $\prod_{\mathrm{Field}} \text{L-Field}(n)$, $\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g_0))$ is U-measurable, and

(iv) $\mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g_0))$ is integrable on $\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n))$, and

(v) $\int G \, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n+1)) = \int \mathrm{Integral2}(\text{L-Meas}, \overline{\mathbb{R}}(g_0)) \, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n))$.

PROOF: Set $I = \mathrm{CarProd}(\mathrm{Seg}(n+1) \longmapsto$ (the real normed space of $\mathbb{R}$)). For every natural number i such that $i \in \mathrm{Seg}(n+1)$ holds $D(i) \subseteq \mathbb{R}$. For every natural number i such that $i \in \mathrm{Seg}(n+1)$ holds $D(i) \subseteq (\mathrm{Seg}(n+1) \longmapsto \mathbb{R})(i)$. $\prod_{\mathrm{FS}} D \subseteq \prod_{\mathrm{FS}}(\mathrm{Seg}(n+1) \longmapsto \mathbb{R})$. Reconsider $D_1 = D{\restriction}n$ as an n-element finite sequence. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D_1(i) \subseteq (\mathrm{Seg}\, n \longmapsto \mathbb{R})(i)$. $\prod_{\mathrm{FS}} D_1$ is a subset of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. Reconsider $I_1 = \prod_{\mathrm{FS}} D_1$ as a subset of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)).

Reconsider $J_1 = D(n+1)$ as a closed interval subset of $\mathbb{R}$. Reconsider $f_0 = f \cdot I$ as a partial function from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. $f \cdot I$ is continuous on $\prod_{\mathrm{FS}} D$. $\prod_{\mathrm{FS}} D = \prod_{\mathrm{FS}} D_1 \times D(n+1)$. $\mathrm{dom}(f \cdot I) = I^{-1}(I^{\circ}(\prod_{\mathrm{FS}} D))$. $\mathrm{dom}\, f_0 = I_1 \times J_1$. For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D_1(i)$ is a closed interval subset of $\mathbb{R}$. For every element x of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, $\mathrm{ProjPMap1}(\overline{\mathbb{R}}(g_0), x)$ is integrable on L-Meas. $\int G \, \mathrm{d}\, \mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n+1)) = \int g_0 \, \mathrm{d}\, \mathrm{ProdMeas}(\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n)), \text{L-Meas})$. □

(65) Let us consider a non zero natural number n, an $(n+1)$-element finite sequence D, an n-element finite sequence D_1, a partial function f from $\langle \mathcal{E}^{n+1}, \| \cdot \| \rangle$ to the real normed space of $\mathbb{R}$, a partial function G from $\prod_{\mathrm{FS}}(\mathrm{Seg}(n+1) \longmapsto \mathbb{R})$ to $\mathbb{R}$, a partial function g_0 from $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$

$\mathbb{R}) \times \mathbb{R}$ to $\mathbb{R}$, a non empty, closed interval subset D_2 of $\mathbb{R}$, an element x of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$, and a partial function P_3 from $\mathbb{R}$ to $\mathbb{R}$. Suppose $D_1 = D{\restriction}n$ and for every natural number i such that $i \in \operatorname{Seg}(n+1)$ holds $D(i)$ is a subset of $\mathbb{R}$ and f is continuous on $\prod D$ and $\operatorname{dom} f = \prod D$ and $G = f \cdot (\operatorname{CarProd}(\operatorname{Seg}(n+1) \longmapsto \mathbb{R}))$ and $g_0 = G$ and $D_2 = D(n+1)$ and $x \in \prod_{\mathrm{FS}} D_1$ and $P_3 = \operatorname{ProjPMap1}(|\overline{\mathbb{R}}(g_0)|, x)$. Then

(i) $\operatorname{dom}(\operatorname{ProjPMap1}(|\overline{\mathbb{R}}(g_0)|, x)) = D(n+1)$, and

(ii) $P_3{\restriction}D_2$ is continuous and bounded, and

(iii) P_3 is integrable on D_2, and

(iv) $\operatorname{ProjPMap1}(|\overline{\mathbb{R}}(g_0)|, x)$ is integrable on L-Meas, and

(v) $\int \operatorname{ProjPMap1}(|\overline{\mathbb{R}}(g_0)|, x)\, \mathrm{d}\, \text{L-Meas} = \int\limits_{D_2} P_3(x)dx$, and

(vi) $(\operatorname{Integral2}(\text{L-Meas}, |\overline{\mathbb{R}}(g_0)|))(x) = \int\limits_{D_2} P_3(x)dx$.

Proof: Set $I = \operatorname{CarProd}(\operatorname{Seg}(n+1) \longmapsto$ (the real normed space of $\mathbb{R}$)). For every natural number i such that $i \in \operatorname{Seg}(n+1)$ holds $D(i) \subseteq \mathbb{R}$. For every natural number i such that $i \in \operatorname{Seg}(n+1)$ holds $D(i) \subseteq (\operatorname{Seg}(n+1) \longmapsto \mathbb{R})(i)$. $\prod_{\mathrm{FS}} D \subseteq \prod_{\mathrm{FS}}(\operatorname{Seg}(n+1) \longmapsto \mathbb{R})$. $\operatorname{dom}(f \cdot I) = I^{-1}(I^{\circ}(\prod_{\mathrm{FS}} D))$. $f \cdot I$ is continuous on $\prod_{\mathrm{FS}} D$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $D_1(i)$ is a subset of $\mathbb{R}$. $\prod_{\mathrm{FS}} D_1$ is a subset of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$. Reconsider $I_1 = \prod_{\mathrm{FS}} D_1$ as a subset of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). Reconsider $f_0 = f \cdot I$ as a partial function from $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$) to the real normed space of $\mathbb{R}$. $\operatorname{dom} f_0 = I_1 \times D_2$. P_3 is continuous. P_3 is integrable on L-Meas. □

References

[1] Charalambos D. Aliprantis and Kim C. Border. *Infinite dimensional analysis.* Springer-Verlag, Berlin, Heidelberg, 2006.

[2] Vladimir Igorevich Bogachev and Maria Aparecida Soares Ruas. *Measure theory*, volume 1. Springer, 2007.

[3] Sylvie Boldo, Catherine Lelay, and Guillaume Melquiond. Improving real analysis in Coq: A user-friendly approach to integrals and derivatives. In Chris Hawblitzel and Dale Miller, editors, *Certified Programs and Proofs – Second International Conference, CPP 2012, Kyoto, Japan, December 13–15, 2012. Proceedings*, volume 7679 of *Lecture Notes in Computer Science*, pages 289–304. Springer, 2012. doi:10.1007/978-3-642-35308-6_22.

[4] Noboru Endou. Reconstruction of the one-dimensional Lebesgue measure. *Formalized Mathematics*, 28(**1**):93–104, 2020. doi:10.2478/forma-2020-0008.

[5] Noboru Endou. Relationship between the Riemann and Lebesgue integrals. *Formalized Mathematics*, 29(**4**):185–199, 2021. doi:10.2478/forma-2021-0018.

[6] Noboru Endou and Yasunari Shidama. Multidimensional measure space and integration. *Formalized Mathematics*, 31(1):181–192, 2023. doi:10.2478/forma-2023-0017.

[7] Noboru Endou and Yasunari Shidama. Universality of measure space. *Formalized Mathematics*, 32(1):149–163, 2024. doi:10.2478/forma-2024-0012.

[8] Noboru Endou and Yasunari Shidama. Measure for product space of real normed spaces. *Formalized Mathematics*, 33(1):117–135, 2025. doi:10.2478/forma-2025-0010.

[9] Noboru Endou and Yasunari Shidama. Integral of continuous functions of two variables. *Formalized Mathematics*, 31(1):309–324, 2023. doi:10.2478/forma-2023-0025.

[10] Noboru Endou and Yasunari Shidama. Integral of continuous three variable functions. *Formalized Mathematics*, 32(1):9–31, 2024. doi:10.2478/forma-2024-0002.

[11] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definition of integrability for partial functions from $\mathbb{R}$ to $\mathbb{R}$ and integrability for continuous functions. *Formalized Mathematics*, 9(**2**):281–284, 2001.

[12] Adam Grabowski and Christoph Schwarzweller. Revisions as an essential tool to maintain mathematical repositories. In M. Kauers, M. Kerber, R. Miner, and W. Windsteiger, editors, *Towards Mechanized Mathematical Assistants. Lecture Notes in Computer Science*, volume 4573, pages 235–249. Springer: Berlin, Heidelberg, 2007.

[13] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Four decades of Mizar. *Journal of Automated Reasoning*, 55(3):191–198, 2015. doi:10.1007/s10817-015-9345-1.

[14] Johannes Hölzl and Armin Heller. Three chapters of measure theory in Isabelle/HOL. In Marko C. J. D. van Eekelen, Herman Geuvers, Julien Schmaltz, and Freek Wiedijk, editors, *Interactive Theorem Proving (ITP 2011)*, volume 6898 of *LNCS*, pages 135–151, 2011.

[15] M.M. Rao. *Measure Theory and Integration.* Marcel Dekker, 2nd edition, 2004.

[16] Colin Rothgang, Artur Korniłowicz, and Florian Rabe. A new export of the Mizar Mathematical Library. In Fairouz Kamareddine and Claudio Sacerdoti Coen, editors, *Intelligent Computer Mathematics*, pages 205–210, Cham, 2021. Springer International Publishing. doi:10.1007/978-3-030-81097-9_17.

[17] Walter Rudin. *Functional Analysis.* New York, McGraw-Hill, 2nd edition, 1991.

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