FORMALIZED MATHEMATICS
Vol. 33, No. 1, pp. 95–102, 2025
DOI: 10.2478/forma-2025-0008
e-ISSN: 1898–9934

A Formal Proof of Stirling's Formula

Yasushige Watase
Faculty of Data Science
University of Rissho
Magechi Kumagaya, Japan

Summary. In this article, we formalized the proof of the Stirling's formula, which is considered an essential item in the field of statistics, as shown below:

$$\lim_{n \to \infty} \frac{n!}{n^{n+\frac{1}{2}} e^{-n}} = \sqrt{2\pi}$$

using the Mizar formalism.

MSC: 33B99 68V20 68V35

Keywords: Wallis product; Stirling formula

MML identifier: STIRLIN1, version: 8.1.15 5.95.1495

Introduction

In this article, using the Mizar formalism [5], we prove Stirling's formula, considered challenging for automated proof-asistants as 90th item in Freek Wiedijk's "Top 100 Mathematical Theorems" [13] (for another recently encoded theorem from this collection, see [11]). The proof of Stirling's formula has been implemented over the past decade within many theorem provers, such as HOL Light [6], Rocq [1], and also Isabelle [3] (Metamath [8] and Lean [9] versions are also available). In our approach, instead of using the gamma function $\Gamma(z)$ (as in most of the aforementioned formal developments), we demonstrated Wallis product formula by elementary means [12], based on the work of Prof. N. Kurokawa [7] (see also [2]). The proof is divided into two parts.

In the first part, we formalize the following lemma which is essential to compute the integral using the Riemann sum over an equal partition.

Lemma 1 (Th. 10) Let $f(x)$ be a $\mathcal{C}^1$ function on $[0,1]$ (i.e., $f'(x)$ exists and is continuous). Then the following holds:

$$\lim_{n\to\infty}\left\{\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_0^1 f(x)\,dx\right\} = \frac{f(1)-f(0)}{2}.$$

The class of continuously differentiable functions $\mathcal{C}^1$ on the open half-interval $]-1,\infty[$ is formulated as Def. 3.

In the second part, we apply the lemma to $\log(1+x)$, and obtained the result:

$$\sum_{k=1}^{n}\left\{\log\left(1+\frac{k}{n}\right) - n\int_0^1 \log(1+x)\,dx\right\} = \log\left(\frac{(2n)!}{n!}\left(\frac{e}{4n}\right)^n\right).$$

From Lemma 1, the left-hand side limit is evaluated as $\frac{\log 2}{2}$ (Th. 17), and thus:

$$\lim_{n\to\infty}\frac{(2n)!}{n!}\left(\frac{e}{4n}\right)^n = \sqrt{2} \qquad \text{(Th. 23).}$$

Considering the ratio between $n!$ and $n^{n+\frac{1}{2}}e^{-n}$, it may be transformed as follows:

$$\frac{n!}{n^{n+\frac{1}{2}}\mathrm{e}^{-n}} = \frac{(2n)!}{n!}\left(\frac{e}{4n}\right)^n \bigg/ \frac{\sqrt{n}(2n)!}{4^n(n!)^2}.$$

The limit left hand side can be calculated, and the denominator on the right-hand side equals the square root of the Wallis product sequence. It is known that this limit equals $\frac{1}{\sqrt{\pi}}$ (see theorem 51 in [12]), so the final limit is $\sqrt{2\pi}$ (Th. 24).

1. Lemma on the Riemann Sum over an Equal Partition

Now we state the proposition:

(1) Let us consider a natural number n, and a natural number k. Suppose $k \in \mathrm{Seg}(n+1)$. Then $\mathrm{divset}(\mathrm{EqDiv}([0,1], n+1), k) = [\frac{k-'1}{n+1}, \frac{k}{n+1}]$.

The functor D[01] yielding a sequence of divs$[0,1]$ is defined by

(Def. 1) for every element n of $\mathbb{N}$, $it(n) = \mathrm{EqDiv}([0,1], n+1)$.

Now we state the propositions:

(2) Let us consider a natural number n. Then $\delta_{\mathrm{EqDiv}([0,1],n+1)} = \frac{1}{n+1}$.

Proof: Set $A = [0,1]$. Set $D = \mathrm{EqDiv}([0,1], n+1)$. For every natural number i such that $i \in \mathrm{dom}\, D$ holds $(\mathrm{upper\_volume}(\chi_{A,A}, D))(i) = \frac{1}{n+1}$ by [4, (15)]. $\mathrm{rng}\,\mathrm{upper\_volume}(\chi_{A,A}, D) = \{\frac{1}{n+1}\}$ by [14, (104)]. □

(3) $\delta_{\mathrm{D[01]}}$ is a 0-convergent, non-zero sequence of real numbers. The theorem is a consequence of (2).

Let r be a real number. Let us note that AffineMap$(0, r)$ is constant.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$ and A be a non empty, closed interval subset of $\mathbb{R}$. We say that f is of class $\mathcal{C}^1$ on A if and only if

(Def. 2) f is differentiable on $]-1, +\infty[$ and $f'_{\restriction]-1,+\infty[}$ is continuous and $A \subseteq\,]-1, +\infty[\, \subseteq \operatorname{dom} f$.

Assume f is of class $\mathcal{C}^1$ on A. The functor $@'(f, A)$ yielding a function from A into $\mathbb{R}$ is defined by the term

(Def. 3) $f'_{\restriction]-1,+\infty[} \restriction A$.

Assume $A \subseteq \operatorname{dom} f$. The functor ${}^{@}(f, A)$ yielding a function from A into $\mathbb{R}$ is defined by the term

(Def. 4) $f \restriction A$.

From now on Z denotes an open subset of $\mathbb{R}$. Now we state the proposition:

(4) Let us consider a natural number n, a natural number k, a real number x_0, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $k \in \operatorname{Seg}(n + 1)$ and $x_0 \in [\frac{k -' 1}{n+1}, \frac{k}{n+1}]$ and f is of class $\mathcal{C}^1$ on $[0, 1]$. Then $\int\limits_{[x_0, \frac{k}{n+1}]} f'_{\restriction]-1,+\infty[}(x)dx = f(\sup[x_0, \dfrac{k}{n+1}]) - f(\inf[x_0, \dfrac{k}{n+1}])$.

Let n be a natural number and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\text{UBnd}_{\text{rng\_d}}(f, n)$ yielding a finite sequence of elements of $\mathbb{R}_{\text{F}}$ is defined by

(Def. 5) len $it = n + 1$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = \sup \operatorname{rng}(f'_{\restriction]-1,+\infty[} \restriction [\frac{i -' 1}{n+1}, \frac{i}{n+1}])$.

The functor $\text{LBnd}_{\text{rng\_d}}(f, n)$ yielding a finite sequence of elements of $\mathbb{R}_{\text{F}}$ is defined by

(Def. 6) len $it = n + 1$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = \inf \operatorname{rng}(f'_{\restriction]-1,+\infty[} \restriction [\frac{i -' 1}{n+1}, \frac{i}{n+1}])$.

The functor $\Sigma_{\text{UBnd}_{\text{rng\_d}}} f$ yielding a sequence of $\mathbb{R}$ is defined by

(Def. 7) for every natural number i, $it(i) = \frac{\sum \text{UBnd}_{\text{rng\_d}}(f, i)}{i+1}$.

The functor $\Sigma_{\text{LBnd}_{\text{rng\_d}}} f$ yielding a sequence of $\mathbb{R}$ is defined by

(Def. 8) for every natural number i, $it(i) = \frac{\sum \text{LBnd}_{\text{rng\_d}}(f, i)}{i+1}$.

Let us consider a natural number n, a natural number k, a real number x_0, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(5) Suppose $k \in \operatorname{Seg}(n+1)$ and $x_0 \in [\frac{k -' 1}{n+1}, \frac{k}{n+1}]$ and f is of class $\mathcal{C}^1$ on $[0, 1]$. Then

(i) $\int\limits_{x_0}^{\frac{k}{n+1}} (\mathrm{AffineMap}(0, (\mathrm{UBnd}_{\mathrm{rng\_d}}(f,n))(k)))(x)dx = (\mathrm{UBnd}_{\mathrm{rng\_d}}(f,n))(k) \cdot (\frac{k}{n+1} - x_0)$, and

(ii) $\int\limits_{x_0}^{\frac{k}{n+1}} (\mathrm{AffineMap}(0, (\mathrm{LBnd}_{\mathrm{rng\_d}}(f,n))(k)))(x)dx = (\mathrm{LBnd}_{\mathrm{rng\_d}}(f,n))(k) \cdot (\frac{k}{n+1} - x_0)$.

PROOF: For every real number t such that $t \in [x_0, \frac{k}{n+1}]$ holds $(\mathrm{AffineMap}(0, (\mathrm{UBnd}_{\mathrm{rng\_d}}(f,n))(k)))(t) = (\mathrm{UBnd}_{\mathrm{rng\_d}}(f,n))(k)$. For every real number t such that $t \in [x_0, \frac{k}{n+1}]$ holds $(\mathrm{AffineMap}(0, (\mathrm{LBnd}_{\mathrm{rng\_d}}(f,n))(k)))(t) = (\mathrm{LBnd}_{\mathrm{rng\_d}}(f,n))(k)$. □

(6) Suppose $k \in \mathrm{Seg}(n+1)$ and $x_0 \in [\frac{k-'1}{n+1}, \frac{k}{n+1}]$ and f is of class $\mathcal{C}^1$ on $[0,1]$. Then $\int\limits_{x_0}^{\frac{k}{n+1}} f'_{\restriction]-1,+\infty[}(x)dx \leqslant \int\limits_{x_0}^{\frac{k}{n+1}} (\mathrm{AffineMap}(0, (\mathrm{UBnd}_{\mathrm{rng\_d}}(f,n))(k)))(x)dx$.

(7) Suppose $k \in \mathrm{Seg}(n+1)$ and $x_0 \in [\frac{k-'1}{n+1}, \frac{k}{n+1}]$ and f is of class $\mathcal{C}^1$ on $[0,1]$. Then $\int\limits_{x_0}^{\frac{k}{n+1}} (\mathrm{AffineMap}(0, (\mathrm{LBnd}_{\mathrm{rng\_d}}(f,n))(k)))(x)dx \leqslant \int\limits_{x_0}^{\frac{k}{n+1}} f'_{\restriction]-1,+\infty[}(x)dx$.

(8) Suppose $k \in \mathrm{Seg}(n+1)$ and $x_0 \in [\frac{k-'1}{n+1}, \frac{k}{n+1}]$ and f is of class $\mathcal{C}^1$ on $[0,1]$. Then

(i) $f(\frac{k}{n+1}) - f(x_0) \leqslant (\mathrm{UBnd}_{\mathrm{rng\_d}}(f,n))(k) \cdot (\frac{k}{n+1} - x_0)$, and

(ii) $(\mathrm{LBnd}_{\mathrm{rng\_d}}(f,n))(k) \cdot (\frac{k}{n+1} - x_0) \leqslant f(\frac{k}{n+1}) - f(x_0)$.

The theorem is a consequence of (4), (5), (7), and (6).

Let n be a natural number and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\mathcal{F}(f,n)$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 9) dom $it = \mathrm{Seg}(n+1)$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = \int\limits_{\frac{i-'1}{n+1}}^{\frac{i}{n+1}} (\mathrm{AffineMap}(0, f(\frac{i}{n+1})))(x)dx$.

The functor $\mathcal{G}(f,n)$ yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 10) dom $it = \mathrm{Seg}(n+1)$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = \int\limits_{\frac{i-'1}{n+1}}^{\frac{i}{n+1}} (-f)(x)dx$.

The functor step(f, n) yielding a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 11) dom $it = \mathrm{Seg}(n+1)$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = f(\frac{i}{n+1})$.

Now we state the proposition:

(9) Let us consider a natural number n, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Then $\int\limits_{0}^{0} (-f)(x)dx = 0$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor SumStep f yielding a sequence of real numbers is defined by

(Def. 12) for every natural number n, $it(n) = \sum \mathrm{step}(f, n)$.

The functor nIntegral f yielding a sequence of real numbers is defined by

(Def. 13) for every natural number n, $it(n) = (n+1) \cdot (\int\limits_{0}^{1} (-f)(x)dx)$.

The functor WSeq f yielding a sequence of real numbers is defined by the term

(Def. 14) SumStep f + nIntegral f.

Now we state the proposition:

(10) Let us consider a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose f is of class $\mathcal{C}^1$ on $[0, 1]$. Then $\lim \mathrm{WSeq}\, f = \frac{1}{2} \cdot (\mathrm{integral}\, f'_{\restriction]-1,+\infty[} \restriction [0, 1])$.

2. Apply the Lemma to log$(1 + x)$

Now we state the propositions:

(11) (The function ln) $\cdot$ ((AffineMap$(1, 1)) \restriction]-1, +\infty[$) is differentiable on $]-1, +\infty[$.

PROOF: Set $Z =]-1, +\infty[$. For every real number x_0 such that $x_0 \in Z$ holds (the function ln) $\cdot$ ((AffineMap$(1, 1)) \restriction]-1, +\infty[$) is differentiable in x_0 by [10, (20)]. □

(12) Let us consider an open subset Z of $\mathbb{R}$. Suppose $Z \subseteq$ dom((the function ln) $\cdot$ ((AffineMap$(1, 1)) \restriction]-1, +\infty[$)). Then

(i) (the function ln) $\cdot$ ((AffineMap(1, 1))$\restriction$]$-1, +\infty$[) is differentiable on Z, and

(ii) for every real number x such that $x \in Z$ holds ((the function ln) $\cdot$ ((AffineMap(1, 1))$\restriction$]$-1, +\infty$[))$'_{\restriction Z}(x) = \frac{1}{1+x}$.

PROOF: Set f = (AffineMap(1, 1))$\restriction$]$-1, +\infty$[. For every real number x such that $x \in Z$ holds $f(x) = 1 + x$ and $f(x) > 0$. $\square$

(13) ((The function ln) $\cdot$ ((AffineMap(1, 1))$\restriction$]$-1, +\infty$[))$'_{\restriction]-1,+\infty[}$ = $\frac{1}{\text{AffineMap}(1,1)}\restriction$]$-1, +\infty$[. The theorem is a consequence of (12).

In the sequel x, x_0, x_1, x_2 denote real numbers. Now we state the propositions:

(14) ((AffineMap(1, 1)) $\cdot$ (the function ln) $\cdot$ ((AffineMap(1, 1))$\restriction$]$-1, +\infty$[) + -1 $\cdot$ (AffineMap(1, 0)))$'_{\restriction]-1,+\infty[}$ =
(the function ln) $\cdot$ ((AffineMap(1, 1))$\restriction$]$-1, +\infty$[).

PROOF: Set f_1 = AffineMap(1, 1). Set f_2 = (the function ln) $\cdot$ ((AffineMap (1, 1))$\restriction$]$-1, +\infty$[). f_2 is differentiable on]$-1, +\infty$[. $f_1 \cdot (\frac{1}{f_1}\restriction$]$-1, +\infty$[) =]$-1, +\infty$[$\longmapsto$ 1. (]$-1, +\infty$[$\longmapsto$ 1) $\cdot$ f_2 + (]$-1, +\infty$[$\longmapsto$ 1) + (-1) $\cdot$ (]$-1, +\infty$[$\longmapsto$ 1) = f_2. $\square$

(15) $\int_0^1$(the function ln) $\cdot$ ((AffineMap(1, 1))$\restriction$]$-1, +\infty$[)$(x)dx$ = $\log_e \frac{4}{e}$. The theorem is a consequence of (12) and (14).

(16) $\int_{[0,1]}$((the function ln) $\cdot$ ((AffineMap(1, 1))$\restriction$]$-1, +\infty$[))$'_{\restriction]-1,+\infty[}(x)dx$ =
(the function ln)(2).

(17) lim WSeq(the function ln) $\cdot$ ((AffineMap(1, 1))$\restriction$]$-1, +\infty$[) =
$\frac{1}{2}$ $\cdot$ (the function ln)(2). The theorem is a consequence of (10) and (16).

Let m be a non zero natural number and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor Step(f, m) yielding a sequence of real numbers is defined by

(Def. 15) for every natural number i, $it(i) = f(\frac{i}{m})$.

Let us consider a non zero natural number m and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(18) XFS2FS(Step(f, m)$\restriction\mathbb{Z}_{m+1}$) = $\langle f(0)\rangle$ $\frown$ step($f, m -' 1$).

PROOF: Reconsider $m_1 = m + 1$ as a natural number. Reconsider o_1 = $m -' 1$ as a natural number. For every natural number x such that $x \in$ dom(Shift(Step(f, m)$\restriction\mathbb{Z}_{m_1}$, 1)) holds (Shift(Step($f, m$)$\restriction\mathbb{Z}_{m_1}$, 1))($x$) = (($\langle f(0)\rangle$ $\frown$ step(f, o_1))(x). $\square$

(19) If f is of class $\mathcal{C}^1$ on [0, 1], then $\sum$ XFS2FS(Step(f, m)$\restriction\mathbb{Z}_{m+1}$) = $f(0)$ + $\sum$ step($f, m -' 1$). The theorem is a consequence of (18).

Let us consider a non zero natural number m. Now we state the propositions:

(20) $\sum$ XFS2FS(Step((the function ln) $\cdot$ ((AffineMap$(1,1))\restriction]-1,+\infty[),m)$ $\restriction\mathbb{Z}_{m+1}) = \sum$ step((the function ln)$\cdot$((AffineMap$(1,1))\restriction]-1,+\infty[),m-'1)$. The theorem is a consequence of (19).

(21) (WSeq(the function ln) $\cdot$ ((AffineMap$(1,1))\restriction]-1,+\infty[))(m-'1) =$ (the function ln)$(\frac{(2\cdot m)!}{m!}\cdot(\frac{e}{4\cdot m}{}^{m}))$. The theorem is a consequence of (15).

The functor WStirl yielding a sequence of real numbers is defined by

(Def. 16) for every natural number n, $it(n) = \frac{(2\cdot(n+1))!}{(n+1)!}\cdot(\frac{e}{4\cdot(n+1)}{}^{n+1})$.

Now we state the propositions:

(22) Let us consider a natural number n. Then (WSeq(the function ln) $\cdot$ ((AffineMap$(1,1))\restriction]-1,+\infty[))(n)$ = (the function ln)((WStirl)(n)). The theorem is a consequence of (15).

(23) (i) lim WStirl $=\sqrt{2}$, and

(ii) WStirl is convergent.

The theorem is a consequence of (17).

The functor StirlSeq yielding a sequence of real numbers is defined by

(Def. 17) for every natural number n, $it(n) = \frac{n!}{n^{n+\frac{1}{2}}\cdot(e^{-n})}$.

Now we state the proposition:

(24) lim StirlSeq $=\sqrt{2\cdot\pi}$. The theorem is a consequence of (23).

REFERENCES

[1] Yves Bertot. A short presentation of Coq. In Otmane Aït Mohamed, César A. Muñoz, and Sofiène Tahar, editors, *Theorem Proving in Higher Order Logics (TPHOLs 2008)*, volume 5170 of *LNCS*, pages 12–16. Springer, 2008. doi:10.1007/978-3-540-71067-7_3.

[2] Richard Courant and Fritz John. *Introduction to Calculus and Analysis I*. John Wiley and Sons Inc., New York, 1st edition, 1965.

[3] Manuel Eberl. Stirling's formula. *Archive of Formal Proofs*, September 2016. `https://isa-afp.org/entries/Stirling_Formula.html`, Formal proof development.

[4] Noboru Endou. Relationship between the Riemann and Lebesgue integrals. *Formalized Mathematics*, 29(**4**):185–199, 2021. doi:10.2478/forma-2021-0018.

[5] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Four decades of Mizar. *Journal of Automated Reasoning*, 55(3):191–198, 2015. doi:10.1007/s10817-015-9345-1.

[6] John Harrison. Stirling's approximation. 2010. Available online at `https://github.com/jrh13/hol-light/blob/master/100/stirling.ml`.

[7] Nobushige Kurokawa. *Modern Trigonometric Function Theory*. Iwanami Shoten, Publishers (Tokyo), 1st edition, 2013. In Japanese.

[8] Norman D. Megill and David A. Wheeler. *Metamath: A Computer Language for Mathematical Proofs*. Lulu Press, Morrisville, North Carolina, 2019. `http://us.metamath.org/downloads/metamath.pdf`.

[9] Leonardo de Moura and Sebastian Ullrich. The Lean 4 theorem prover and programming language. In *Automated Deduction – CADE 28: 28th International Conference on Automated Deduction, Virtual Event, July 12–15, 2021, Proceedings*, pages 625–635, Berlin, Heidelberg, 2021. Springer-Verlag. doi:10.1007/978-3-030-79876-5_37.

[10] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(**2**):195–200, 2004.

[11] Yasushige Watase. Formal proof of transcendence of the number e. Part II. *Formalized Mathematics*, 32(1):121–131, 2024. doi:10.2478/forma-2024-0009.

[12] Yasushige Watase. Formalization of Wallis infinite product formula for π and the Wallis integral. *Formalized Mathematics*, 33(1):85–94, 2025. doi:10.2478/forma-2025-0007.

[13] Freek Wiedijk. Formalizing 100 theorems. Available online at `http://www.cs.ru.nl/~freek/100/`.

[14] Rafał Ziobro. Fermat's Little Theorem via divisibility of Newton's binomial. *Formalized Mathematics*, 23(**3**):215–229, 2015. doi:10.1515/forma-2015-0018.

Received June 1, 2025; Accepted September 17, 2025