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Formalization of Wallis Infinite Product Formula for π and the Wallis Integral

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Summary. In this article, we develop the proof of the so-called Wallis formula using the Mizar formalism. The purpose of this formalization is to complete the proof of Stirling's formula using elementary techniques of calculus; however, the formal encoding of the Wallis-type integral is also included.

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Introduction

In this article, we prove the Wallis product formula using the Mizar formalism [4]. This result constitutes the main step in the proof of Stirling's formula, which is considered challenging and is listed as the 90th item in Freek Wiedijk's "Top 100 Mathematical Theorems" [13]. Consequently, Stirling's formula has been formalized over the past decade in many computerized proof asistants, such as HOL Light [5], Rocq [1], and Isabelle [3] (with versions also available in Metamath [7] and Lean [8]). While some mechanized proofs implicitly use the gamma function $\Gamma(z)$, we focus instead on more elementary tools from mathematical analysis (similar philosophy motivated the approach in [12], another item from [13] formalized recently by the present author). In the article, we formulate and prove several auxiliary results, mainly concerning the Wallis integral, which are needed to carry out the proof of Stirling's formula smoothly.

Throughout the formalization, we follow the approach of [2], [6]. The actual encoding in Mizar consists of five main sections, which we outline below. After the preliminaries, in Section 2, we introduce two functions, `even_seq(n)` and odd_seq(n). These represent, respectively, the sequences of even and odd natural numbers of length n, and their products correspond to double factorials. We establish the properties of double factorials as theorems to be used in later proofs.

In Section 3, in order to formalize the fact that any positive integer power of a continuous function f is continuous, we introduce the transformation @ that treats f as an element of an $\mathbb{R}$-algebra. More precisely, let A be an arbitrary set, and let RAlgebra(A) denote the set of all real-valued functions on A forming an $\mathbb{R}$-algebra. When a function f from A to $\mathbb{R}$ is considered as an element of RAlgebra(A), we write @f. Additionally, when g is an element of RAlgebra(A), we define the inverse transformation, which regards g as a function from A to $\mathbb{R}$, using the same symbol @g.

In Section 4, we apply the result from Section 3 to the trigonometric function $\sin x$ and obtain the following recurrence relation from the integration by parts of $\sin^n x$ over 0 to $\frac{\pi}{2}$:

$$I(n) = \frac{n-1}{n} I(n-2), \quad \text{where} \quad I(n) = \int_0^{\frac{\pi}{2}} \sin^n x \, dx.$$

The aforementioned function $I(n)$, often denoted by W_n in the literature, is represented as WallisInt (Def. 5, actually just 'I') in the source. Using this recurrence relation, we obtain the result of the Wallis integral:

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \begin{cases} \dfrac{(2m)!!}{(2m+1)!!} & \text{if } n = 2m+1, \\ \dfrac{(2m-1)!!}{(2m)!!} \cdot \dfrac{\pi}{2} & \text{if } n = 2m. \end{cases}$$

This formula was used in Section 5 to derive the Wallis product formula (for another expansion of number π in Mizar, see [9]). Here, for any $0 < \theta < \frac{\pi}{2}$, $\sin^{2n-1}\theta > \sin^{2n}\theta > \sin^{2n+1}\theta > 0$ holds, it follows that

$$\frac{I(2n-1)}{I(2n+1)} > \frac{I(2n)}{I(2n+1)} > 1.$$

From the recurrence relation and

$$\frac{I(2n)}{I(2n+1)} = \left(\frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\right)^2 \left(n - \frac{1}{2}\right)\pi,$$

which leads to the inequality

$$1 + \frac{1}{2n} > \left(\frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\right)^2 \left(n - \frac{1}{2}\right)\pi > 1.$$

We denote this middle term as Wallis_Seq.n in the actual Mizar code (see Def. 6) and derive the limit formula by computing its limit value.

1. Preliminaries

Now we state the propositions:

(1) Let us consider a natural number n, and a real number x. If $0 < x < \frac{\pi}{2}$, then $0 <$ (the function sin)$(x) < 1$.

(2) Let us consider natural numbers n, m, and a real number x. If $n < m$ and $0 < x < 1$, then $x^m < x^n$.

(3) Let us consider a non zero natural number n, and a real number x. Suppose $0 < x < \frac{\pi}{2}$. Then

(i) (the function sin)$(x)^{2 \cdot n} <$ (the function sin)$(x)^{2 \cdot n -' 1}$, and

(ii) (the function sin)$(x)^{2 \cdot n+1} <$ (the function sin)$(x)^{2 \cdot n}$.

The theorem is a consequence of (1) and (2).

2. Some Results on Double Factorial

Let m be a natural number. The functor even-seq(m) yielding a finite sequence of elements of $\mathbb{Z}$ is defined by

(Def. 1) len $it = m$ and for every natural number i such that $i \in \operatorname{dom} it$ holds $it(i) = 2 \cdot i$.

Let m be a non zero natural number. The functor odd-seq(m) yielding a finite sequence of elements of $\mathbb{Z}$ is defined by the term

(Def. 2) even-seq(m) + (dom(even-seq(m)) $\longmapsto -1$).

Now we state the propositions:

(4) Let us consider a non zero natural number n. Then

(i) dom(even-seq(n)) = Seg n, and

(ii) dom(odd-seq(n)) = Seg n, and

(iii) len odd-seq$(n) = n$, and

(iv) len even-seq$(n) = n$.

(5) Let us consider a non zero natural number n, and a natural number i. Suppose $i \in$ dom(odd-seq(n)). Then (odd-seq(n))$(i) = 2 \cdot i - 1$. The theorem is a consequence of (4).

Let n be a non zero natural number. Note that even-seq(n) is positive yielding and even-seq(n) is non empty and odd-seq(n) is positive yielding and odd-seq(n) is non empty and even-seq(n) is increasing and odd-seq(n) is increasing. Let us consider a non zero natural number n. Now we state the propositions:

(6) $\text{even-seq}(n+1) = \text{even-seq}(n) \frown \langle 2 \cdot (n+1) \rangle$.

PROOF: Set $p = \text{even-seq}(n+1)$. Set $q = \text{even-seq}(n)$. Set $x = \langle 2 \cdot (n+1) \rangle$. $\operatorname{dom} q = \operatorname{Seg} n$. For every natural number k such that $1 \leqslant k \leqslant \operatorname{len} p$ holds $p(k) = (q \frown x)(k)$. □

(7) $\prod \text{even-seq}(n+1) = 2 \cdot (n+1) \cdot (\prod \text{even-seq}(n))$. The theorem is a consequence of (6).

(8) (i) $\text{even-seq}(1) = \langle 2 \rangle$, and

(ii) $\text{even-seq}(2) = \langle 2, 4 \rangle$, and

(iii) $\text{even-seq}(3) = \langle 2, 4, 6 \rangle$.

The theorem is a consequence of (6).

Let us consider a non zero natural number n. Now we state the propositions:

(9) $\text{odd-seq}(n+1) \restriction \operatorname{Seg} n = \text{odd-seq}(n)$. The theorem is a consequence of (6).

(10) $\text{odd-seq}(n+1) = \text{odd-seq}(n) \frown \langle 2 \cdot n + 1 \rangle$. The theorem is a consequence of (4) and (9).

(11) (i) $\text{odd-seq}(1) = \langle 1 \rangle$, and

(ii) $\text{odd-seq}(2) = \langle 1, 3 \rangle$, and

(iii) $\text{odd-seq}(3) = \langle 1, 3, 5 \rangle$.

The theorem is a consequence of (4) and (10).

Let us consider a non zero natural number n. Now we state the propositions:

(12) $\prod \text{even-seq}(n+1) = 2 \cdot (n+1) \cdot (\prod \text{even-seq}(n))$. The theorem is a consequence of (6).

(13) $\prod \text{odd-seq}(n+1) = (2 \cdot n + 1) \cdot (\prod \text{odd-seq}(n))$. The theorem is a consequence of (10).

Let us consider a non zero natural number k. Now we state the propositions:

(14) $\prod \text{even-seq}(k) = 2^k \cdot k!$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \prod \text{even-seq}(\$_1) = 2^{\$_1} \cdot \$_1!$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. $\mathcal{P}[1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(15) $\prod \text{odd-seq}(k) = \frac{(2 \cdot k)!}{2^k \cdot k!}$.

PROOF: Define $\mathcal{P}[\text{non zero natural number}] \equiv \prod \text{odd-seq}(\$_1) = \frac{(2 \cdot \$_1)!}{2^{\$_1} \cdot \$_1!}$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. $\mathcal{P}[1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

3. Treating a Function as Elements of $\mathbb{R}$-algebra

Let A be a set and f be an element of RAlgebra A. The functor ${}^{@}f$ yielding an element of $\mathbb{R}^A$ is defined by the term

(Def. 3) f.

Let f be a function from A into $\mathbb{R}$. The functor ${}^{@}f$ yielding an element of RAlgebra A is defined by the term

(Def. 4) f.

Let f be an element of RAlgebra A. Let us observe that ${}^{@@}f$ reduces to f. Let f be a function from A into $\mathbb{R}$. Observe that ${}^{@@}f$ reduces to f. Now we state the propositions:

(16) Let us consider a non empty set A, and elements f, g of RAlgebra A. Then

 (i) $f \cdot g = {}^{@}f \cdot {}^{@}g$, and

 (ii) ${}^{@}f \cdot g = {}^{@}f \cdot {}^{@}g$.

(17) Let us consider a non empty set A, an element a of $\mathbb{R}$, and an element f of RAlgebra A. Then $a \cdot {}^{@}f = a \cdot f$.

Let us observe that RAlgebra $\mathbb{R}$ is scalar unital. Now we state the propositions:

(18) RAlgebra $\mathbb{R}$ is a real linear space.

(19) Let us consider a non empty set A, and elements f, g of RAlgebra A. Then $+_{\mathbb{R}^A}(f, g) = +_{\mathbb{R}^A}({}^{@}f, {}^{@}g)$.

(20) Let us consider a non empty set A, and an element f of RAlgebra A. Then

 (i) $(-1) \cdot f = (-1) \cdot {}^{@}f$, and

 (ii) ${}^{@}(-1) \cdot f = -{}^{@}f$, and

 (iii) $0_{\mathrm{RAlgebra}\,A} = 0 \cdot {}^{@}f$.

(21) Let us consider a non empty set A, and elements f, g of RAlgebra A. Then $f + g = {}^{@}f + {}^{@}g$.

(22) Let us consider elements f, g of RAlgebra $\mathbb{R}$. Then $f - g = {}^{@}f - {}^{@}g$. The theorem is a consequence of (20).

(23) Let us consider a unital, associative, non empty multiplicative magma M, a natural number n, and an element a of M. Then $a^{n+1} = a^n \cdot a$.

(24) Let us consider a natural number n, and a function x from $\mathbb{R}$ into $\mathbb{R}$. If x is continuous, then ${}^{@}({}^{@}x)^n$ is continuous.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv {}^{@}({}^{@}x)^{\$_1}$ is continuous. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. □

(25) Let us consider a function x from $\mathbb{R}$ into $\mathbb{R}$, a natural number n, and a real number a. If $({}^{@@}x)(a) > 0$, then $({}^{@}({}^{@}x)^{n})(a) = ({}^{@@}x)(a)^{n}$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv ({}^{@}({}^{@}x)^{\$_1})(a) = ({}^{@@}x)(a)^{\$_1}$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. $\mathcal{P}[0]$. For every natural number n, $\mathcal{P}[n]$. □

(26) Let us consider a function x from $\mathbb{R}$ into $\mathbb{R}$, a non zero natural number n, and a real number a. If $({}^{@@}x)(a) = 0$, then $({}^{@}({}^{@}x)^{n})(a) = 0$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv ({}^{@}({}^{@}x)^{\$_1})(a) = 0$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(27) Let us consider an open subset Z of $\mathbb{R}$, a non zero natural number n, and a function x from $\mathbb{R}$ into $\mathbb{R}$. Suppose x is differentiable on Z. Then ${}^{@}({}^{@}x)^{n}$ is differentiable on Z.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv {}^{@}({}^{@}x)^{\$_1}$ is differentiable on Z. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(28) Let us consider a natural number n, a non empty, closed interval subset I of $\mathbb{R}$, and a function x from $\mathbb{R}$ into $\mathbb{R}$. Suppose x is continuous. Then

(i) ${}^{@}({}^{@}x)^{n} \restriction I$ is continuous, and

(ii) ${}^{@}({}^{@}x)^{n}$ is integrable on I, and

(iii) ${}^{@}({}^{@}x)^{n} \restriction I$ is bounded.

The theorem is a consequence of (24).

From now on r denotes a real number. Now we state the proposition:

(29) Let us consider a natural number n, a non empty, closed interval subset I of $\mathbb{R}$, and a function x from $\mathbb{R}$ into $\mathbb{R}$. Suppose x is continuous. Then

(i) $(r \cdot {}^{@}({}^{@}x)^{n}) \restriction I$ is continuous, and

(ii) $r \cdot {}^{@}({}^{@}x)^{n}$ is integrable on I, and

(iii) $(r \cdot {}^{@}({}^{@}x)^{n}) \restriction I$ is bounded.

The theorem is a consequence of (24).

Let us consider a non zero natural number n. Now we state the propositions:

(30) (i) $({}^{@}({}^{@}(\text{the function sin}))^{n})(0) = 0$, and

(ii) $({}^{@}({}^{@}(\text{the function sin}))^{n})(\frac{\pi}{2}) = 1$.

The theorem is a consequence of (25) and (26).

(31) $({}^{@}({}^{@}(\text{the function sin}))^{n+1})'_{\restriction \mathbb{R}} =$
$((n+1)\cdot {}^{@}({}^{@}(\text{the function sin}))^{n})\cdot {}^{@@}(\text{the function cos})$.
Proof: Set $Z = \mathbb{R}$. Set $x =$ the function sin. Define $\mathcal{P}[\text{natural number}] \equiv$ $({}^{@}({}^{@}x)^{\$_1+1})'_{\restriction Z} = ((\$_1+1)\cdot {}^{@}({}^{@}x)^{\$_1})\cdot {}^{@@}(\text{the function cos})$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. $\mathcal{P}[1]$ by [10, (8)], (23), (16), [14, (68)]. For every non zero natural number n, $\mathcal{P}[n]$. □

4. Recurrence Relation on $\int_0^{\frac{\pi}{2}} \sin^n x\, dx$ and the Wallis Integral

Now we state the propositions:

(32) $({}^{@}({}^{@}(\text{the function sin}))^{1})'_{\restriction \mathbb{R}}$ is differentiable on $\mathbb{R}$.

(33) Let us consider a non zero natural number n.
Then $({}^{@}({}^{@}(\text{the function sin}))^{n+1})'_{\restriction \mathbb{R}}$ is differentiable on $\mathbb{R}$.
Proof: Set $Z = \mathbb{R}$. Set $x =$ the function sin. ${}^{@}({}^{@}x)^{n}$ is differentiable on Z and ${}^{@@}x$ is differentiable on Z. $((n+1)\cdot {}^{@}({}^{@}(\text{the function sin}))^{n})\cdot {}^{@@}(\text{the function cos})$ is differentiable on Z by [14, (67)], [11, (6),(7)]. □

(34) Let us consider a natural number n. Suppose $n \geqslant 2$.
Then ${}^{@}({}^{@}(\text{the function sin}))^{n-'1}$ is an integral of $((n-'1)\cdot {}^{@}({}^{@}(\text{the function sin}))^{n-'2})\cdot {}^{@@}(\text{the function cos})$ on $\mathbb{R}$. The theorem is a consequence of (16), (27), and (31).

(35) $\int_0^{\frac{\pi}{2}} {}^{@}({}^{@}(\text{the function sin}))^{0}(x)dx = \frac{\pi}{2}$.

(36) $\int_0^{\frac{\pi}{2}} {}^{@}({}^{@}(\text{the function sin}))^{1}(x)dx = 1$.

(37) (The function sin) $\cdot$ (the function sin) + (the function cos) $\cdot$ (the function cos) $= 1_{\mathrm{RAlgebra}\,\mathbb{R}}$.
Proof: For every object o such that $o \in \mathrm{dom}(\mathbb{R} \longmapsto 1)$ holds $((\text{the function sin})\cdot(\text{the function sin}) + (\text{the function cos})\cdot(\text{the function cos}))(o) = (\mathbb{R} \longmapsto 1)(o)$. □

(38) (i) $({}^{@}(\text{the function sin}))^{2} + ({}^{@}(\text{the function cos}))^{2} = 1_{\mathrm{RAlgebra}\,\mathbb{R}}$, and

(ii) $({}^{@}(\text{the function sin}))^{2} = (\text{the function sin})\cdot(\text{the function sin})$, and

(iii) $({}^{@}(\text{the function cos}))^{2} = (\text{the function cos})\cdot(\text{the function cos})$, and

(iv) $({}^{@}(\text{the function sin}))^{2} = ({}^{@}(\text{the function sin}))\cdot({}^{@}(\text{the function sin}))$, and

(v) $({}^{@}(\text{the function cos}))^2 = ({}^{@}(\text{the function cos})) \cdot ({}^{@}(\text{the function cos}))$.

The theorem is a consequence of (23), (16), (21), and (37).

The functor WallisInt yielding a sequence of real numbers is defined by

(Def. 5) for every natural number m, $it(m) = \int_0^{\frac{\pi}{2}} {}^{@}({}^{@}(\text{the function sin}))^m(x)dx$.

5. The Wallis Product Formula

Now we state the proposition:

(39) Let us consider a natural number n. Suppose $n \geqslant 2$.
Then $\frac{n -' 1}{n} \cdot \text{WallisInt}(n -' 2) = \text{WallisInt}(n)$. The theorem is a consequence of (23), (16), (24), (34), (38), (22), (20), (29), (28), and (30).

Let us consider a non zero natural number n. Now we state the propositions:

(40) $\text{WallisInt}(2 \cdot n + 1) = \frac{\prod \text{even-seq}(n)}{\prod \text{odd-seq}(n+1)}$.

Proof: Define $\mathcal{P}[\text{non zero natural number}] \equiv \text{WallisInt}(2 \cdot \$_1 + 1) = \frac{\prod \text{even-seq}(\$_1)}{\prod \text{odd-seq}(\$_1+1)}$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. $\mathcal{P}[1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(41) $\text{WallisInt}(2 \cdot n) = \frac{\prod \text{odd-seq}(n)}{\prod \text{even-seq}(n)} \cdot \frac{\pi}{2}$.

Proof: Define $\mathcal{P}[\text{non zero natural number}] \equiv \text{WallisInt}(2 \cdot \$_1) = \frac{\prod \text{odd-seq}(\$_1)}{\prod \text{even-seq}(\$_1)} \cdot \frac{\pi}{2}$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. $\mathcal{P}[1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

6. The Wallis Product

Now we state the proposition:

(42) Let us consider a non zero natural number n, and a real number x. Suppose $0 < x < \frac{\pi}{2}$. Then

(i) $({}^{@}({}^{@}(\text{the function sin}))^{2 \cdot n})(x) < ({}^{@}({}^{@}(\text{the function sin}))^{2 \cdot n -' 1})(x)$, and

(ii) $({}^{@}({}^{@}(\text{the function sin}))^{2 \cdot n+1})(x) < ({}^{@}({}^{@}(\text{the function sin}))^{2 \cdot n})(x)$.

The theorem is a consequence of (1), (25), and (3).

Let us consider a non zero natural number n. Now we state the propositions:

(43) (i) $\text{WallisInt}(2 \cdot n) \leqslant \text{WallisInt}(2 \cdot n -' 1)$, and

(ii) $\mathrm{WallisInt}(2 \cdot n + 1) \leqslant \mathrm{WallisInt}(2 \cdot n)$.

PROOF: Set $I_0 = {}^{@}({}^{@}(\text{the function sin}))^{2 \cdot n -' 1}$.
Set $I_1 = {}^{@}({}^{@}(\text{the function sin}))^{2 \cdot n}$. Set $I_2 = {}^{@}({}^{@}(\text{the function sin}))^{2 \cdot n + 1}$.
$I_0 \restriction [0, \frac{\pi}{2}]$ is continuous and I_0 is integrable on $[0, \frac{\pi}{2}]$ and $I_0 \restriction [0, \frac{\pi}{2}]$ is bounded.
$I_1 \restriction [0, \frac{\pi}{2}]$ is continuous and I_1 is integrable on $[0, \frac{\pi}{2}]$ and $I_1 \restriction [0, \frac{\pi}{2}]$ is bounded.
$I_2 \restriction [0, \frac{\pi}{2}]$ is continuous and I_2 is integrable on $[0, \frac{\pi}{2}]$ and $I_2 \restriction [0, \frac{\pi}{2}]$ is bounded.
For every real number x such that $x \in [0, \frac{\pi}{2}]$ holds $I_2(x) \leqslant I_1(x)$. For every real number x such that $x \in [0, \frac{\pi}{2}]$ holds $I_1(x) \leqslant I_0(x)$. □

(44) (i) $1 \leqslant \frac{\mathrm{WallisInt}(2 \cdot n)}{\mathrm{WallisInt}(2 \cdot n + 1)}$, and

(ii) $\frac{\mathrm{WallisInt}(2 \cdot n)}{\mathrm{WallisInt}(2 \cdot n + 1)} = \frac{(\prod \text{odd-seq}(n))^2}{(\prod \text{even-seq}(n))^2} \cdot n \cdot ((1 + \frac{1}{2 \cdot n}) \cdot \pi)$.

The theorem is a consequence of (41), (13), (40), and (43).

(45) $\frac{1}{(1 + \frac{1}{2 \cdot n}) \cdot \pi} \leqslant \frac{(\prod \text{odd-seq}(n))^2}{(\prod \text{even-seq}(n))^2} \cdot n$. The theorem is a consequence of (44).

(46) $\frac{\mathrm{WallisInt}(2 \cdot n)}{\mathrm{WallisInt}(2 \cdot n + 1)} \leqslant 1 + \frac{1}{2 \cdot n}$. The theorem is a consequence of (40), (39), and (43).

(47) $\frac{(\prod \text{odd-seq}(n))^2}{(\prod \text{even-seq}(n))^2} \cdot n \leqslant \frac{1}{\pi}$. The theorem is a consequence of (44) and (46).

The functor WallisSeq yielding a sequence of real numbers is defined by

(Def. 6) for every natural number k, $\mathit{it}(k) = \frac{(\prod \text{odd-seq}(k+1))^2}{(\prod \text{even-seq}(k+1))^2} \cdot (k + 1)$.

The functor WSeq yielding a sequence of real numbers is defined by

(Def. 7) for every natural number n, $\mathit{it}(n) = \frac{1}{(1 + \frac{1}{2 \cdot (n+1)}) \cdot \pi}$.

Now we state the propositions:

(48) (i) $\lim \mathrm{WallisSeq} = \frac{1}{\pi}$, and

(ii) WallisSeq is convergent.

PROOF: For every real number r such that $r > 0$ there exists a natural number n_0 such that for every natural number n such that $n_0 \leqslant n$ holds $|\,\mathrm{WallisSeq}(n) - \frac{1}{\pi}| < r$. Consider a being a real number such that $a = \frac{1}{\pi}$ and for every real number r such that $r > 0$ there exists a natural number n_0 such that for every natural number n such that $n_0 \leqslant n$ holds $|\,\mathrm{WallisSeq}(n) - a| < r$. □

(49) WallisSeq is positive yielding.

Let us observe that WallisSeq is positive yielding. Now we state the propositions:

(50) Let us consider a positive yielding sequence s_2 of real numbers. Suppose s_2 is convergent. Then

(i) $\sqrt{s_2}$ is convergent, and

(ii) $\lim \sqrt{s_2} = \sqrt{\lim s_2}$.

Proof: Consider a being a real number such that for every real number r such that $0 < r$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s_2(m) - a| < r$. For every natural number k, $\sqrt{s_2(k)} = \sqrt{s_2}(k)$. $\sqrt{s_2}$ is convergent and $\lim \sqrt{s_2} = \sqrt{\lim s_2}$. □

(51) $\lim \sqrt{\mathrm{WallisSeq}} = \frac{1}{\sqrt{\pi}}$. The theorem is a consequence of (48) and (50).

(52) Let us consider a non zero natural number n. Then $\sqrt{\mathrm{WallisSeq}}(n -' 1) = \frac{\sqrt{n} \cdot (2 \cdot n)!}{4^n \cdot n!^2}$. The theorem is a consequence of (15) and (14).

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