

A Case Study of Transporting Urysohn's Lemma from Topology via Open Sets into Topology via Neighborhoods

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Summary. Józef Białas and Yatsuka Nakamura has completely formalized a proof of Urysohn's lemma in the article [4], in the context of a topological space defined via open sets. In the Mizar Mathematical Library (MML), the topological space is defined in this way by Beata Padlewska and Agata Darmochwał in the article [18]. In [7] the topological space is defined via neighborhoods. It is well known that these definitions are equivalent [5, 6].

In the definitions, an abstract structure (i.e. the article [17, `STRUCT_0`] and its descendants, all of them directly or indirectly using Mizar structures [3]) have been used (see [10], [9]). The first topological definition is based on the Mizar structure `TopStruct` and the topological space defined via neighborhoods with the Mizar structure: `FMT_Space_Str`. To emphasize the notion of a neighborhood, we rename `FMT.TopSpace` (topology from neighbourhoods) to `NTopSpace` (a neighborhood topological space).

Using Mizar [2], we transport the Urysohn's lemma from `TopSpace` to `NTopSpace`.

In some cases, Mizar allows certain techniques for transporting proofs, definitions or theorems. Generally speaking, there is no such automatic translating.

In Coq, Isabelle/HOL or homotopy type theory transport is also studied, sometimes with a more systematic aim [14], [21], [11], [12], [8], [19]. In [1], two co-existing Isabelle libraries: Isabelle/HOL and Isabelle/Mizar, have been aligned in a single foundation in the Isabelle logical framework.

In the MML, they have been used since the beginning: `reconsider`, `registration`, `cluster`, others were later implemented [13]: `identify`.

In some proofs, it is possible to define particular functors between different structures, mainly useful when results are already obtained in a given structure. This technique is used, for example, in [15] to define two functors `MXR2MXF` and `MXF2MXF` between `Matrix of REAL` and `Matrix of F-Real` and to transport the

definition of the addition from one structure to the other: [...] $A + B \rightarrow \text{Matrix of REAL equals MXF2MXR} ((\text{MXR2MXF } A) + (\text{MXR2MXF } B))$ [...].

In this paper, first we align the necessary topological concepts. For the formalization, we were inspired by the works of Claude Wagschal [20]. It allows us to transport more naturally the Urysohn's lemma ([4, URYSOHN3:20]) to the topological space defined via neighborhoods.

Nakasho and Shidama have developed a solution to explore the notions introduced in various ways <https://mimosa-project.github.io/mmlreference/current/> [16].

The definitions can be directly linked in the HTML version of the Mizar library (example: Urysohn's lemma <http://mizar.org/version/current/html/urysohn3.html#T20>).

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1. SOME REDEFINITIONS: NEIGHBORHOOD TOPOLOGICAL SPACE

From now on T denotes a topological space and A, B denote subsets of T .

Now we state the proposition:

- (1) If A misses B , then $\text{Int } A$ misses $\text{Int } B$.

A neighborhood topological space is a topology from neighbourhoods. Let X be a non empty topological space. We introduce the notation $\text{Top2NTop}(X)$ as a synonym of $\text{TopSpace2FMT } X$.

Let X be a topology from neighbourhoods. We introduce the notation $\text{NTop2-Top}(X)$ as a synonym of $\text{FMT2TopSpace } X$.

2. ALIGNMENT OF TOPOLOGICAL SPACE CONCEPTS DEFINED VIA OPEN SETS AND DEFINED VIA NEIGHBOURHOODS

Let N_1 be a non empty neighborhood topological space. Observe that Ω_{N_1} is open and $\emptyset_{N_1}$ is open.

Let N_1 be a U-FMT filter, non empty, strict formal topological space and x be an element of N_1 . Note that the functor $U_F(x)$ yields a filter of the carrier of N_1 .

[20, DEFINITION 2.11.2, P. 89]:

Let N_1 be a U-FMT filter, non empty, strict formal topological space and F be a filter of the carrier of N_1 . The functor $\text{LimFilter}(F)$ yielding a subset of N_1 is defined by the term

(Def. 1) $\{x, \text{ where } x \text{ is a point of } N_1 : F \text{ is finer than } U_F(x)\}.$

[20, DEFINITION 2.11.3, P. 92 AND PROPOSITION 2.11.4, P. 90]:

Let N_1, N_2 be U-FMT filter, non empty, strict formal topological spaces, f be a function from N_1 into N_2 , and F be a filter of the carrier of N_1 . The functor $\lim_F f$ yielding a subset of N_2 is defined by the term

(Def. 2) $\text{LimFilter}(\text{the image of filter } F \text{ under } f).$

[20, DEFINITION 2.10.1 (1), P. 83]:

Let N be a neighborhood topological space, A be a subset of N , and x be a point of N . We say that x is interior point of A if and only if

(Def. 3) A is a neighbourhood of x .

[20, DEFINITION 2.10.1 (2), P. 83]:

Let N be a neighborhood topological space, A be a subset of N , and x be a point of N . We say that x is adherent point of A if and only if

(Def. 4) for every element V of $U_F(x)$, V meets A .

The functor $\text{Int } A$ yielding a subset of N is defined by the term

(Def. 5) $\{x, \text{ where } x \text{ is a point of } N : x \text{ is interior point of } A\}.$

[20, DEFINITION 2.13.1, P. 97]:

Let N_1, N_2 be neighborhood topological spaces, f be a function from N_1 into N_2 , and x be a point of N_1 . We say that f is continuous at x if and only if

(Def. 6) for every filter F of the carrier of N_1 such that $x \in \text{LimFilter}(F)$ holds $f(x) \in \lim_F f$.

We say that f is continuous if and only if

(Def. 7) for every point x of N_1 , f is continuous at x .

Note that there exists a function from N_1 into N_2 which is continuous.

Let N be a neighborhood topological space and A be a subset of N .

[20, DEFINITION 2.10.1 (1), P. 83]: $\text{Int } A$ is open.

[20, DEFINITION 2.10.1 (2), P. 83]:

Let N be a neighborhood topological space and A be a subset of N . The functor $\overline{A}$ yielding a subset of N is defined by the term

(Def. 8) $\{x, \text{ where } x \text{ is a point of } N : x \text{ is adherent point of } A\}.$

[20, DEFINITION 2.9.3, P. 81]:

Let N_1 be a neighborhood topological space and A be a subset of N_1 . We say that A is closed if and only if

(Def. 9) $\Omega_{N_1} \setminus A$ is an open subset of N_1 .

One can check that there exists a subset of N_1 which is closed and Ω_{N_1} is closed as a subset of N_1 and $\emptyset_{N_1}$ is closed as a subset of N_1 and there exists a subset of N_1 which is non empty and closed.

Let S, T be non empty topological spaces and f be a function from S into T . The functor $\text{Top2NTop}(f)$ yielding a function from $\text{Top2NTop}(S)$ into $\text{Top2NTop}(T)$ is defined by the term

(Def. 10) f .

Let T_1 be a non empty topological space, T_2 be a non empty, strict topological space, and f be a continuous function from T_1 into T_2 . Observe that the functor $\text{Top2NTop}(f)$ yields a continuous function from $\text{Top2NTop}(T_1)$ into $\text{Top2NTop}(T_2)$ and is defined by the term

(Def. 11) f .

[20, DEFINITION 2.17.1, P. 111]:

Let N be a neighborhood topological space. We say that N is T_2 if and only if

(Def. 12) for every filter F of the carrier of N , $\text{LimFilter}(F)$ is trivial.

One can check that there exists a neighborhood topological space which is T_2 .

Let N be a neighborhood topological space. We say that N is normal if and only if

(Def. 13) for every closed subsets A, B of N such that A misses B there exists a neighbourhood V of A and there exists a neighbourhood W of B such that V misses W .

Let x be a point of N . The functor $\text{NTop2Top}(x)$ yielding a point of $\text{NTop2Top}(N)$ is defined by the term

(Def. 14) x .

Let T be a non empty topological space and x be a point of T . The functor $\text{Top2NTop}(x)$ yielding a point of $\text{Top2NTop}(T)$ is defined by the term

(Def. 15) x .

Let N be a neighborhood topological space and S be a subset of N . The functor $\text{NTop2Top}(S)$ yielding a subset of $\text{NTop2Top}(N)$ is defined by the term

(Def. 16) S .

Let T be a non empty topological space and S be a subset of T . The functor $\text{Top2NTop}(S)$ yielding a subset of $\text{Top2NTop}(T)$ is defined by the term

(Def. 17) S .

One can verify that there exists a neighborhood topological space which is non empty and normal.

Let T_1, T_2 be neighborhood topological spaces and f be a function from T_1 into T_2 . The functor $\text{NTop2Top}(f)$ yielding a function from $\text{NTop2Top}(T_1)$ into $\text{NTop2Top}(T_2)$ is defined by the term

(Def. 18) f .

The functor $\text{FMT-}\mathbb{R}^1$ yielding a neighborhood topological space is defined by the term

(Def. 19) $\text{Top2NTop}(\mathbb{R}^1)$.

Now we state the proposition:

(2) The carrier of $\text{FMT-}\mathbb{R}^1 = \mathbb{R}$.

One can verify that $\text{FMT-}\mathbb{R}^1$ is real-membered.

3. SOME PROPERTIES OF A NEIGHBORHOOD TOPOLOGY

From now on N , N_1 , N_2 denote neighborhood topological spaces, A , B denote subsets of N , O denotes an open subset of N , a denotes a point of N , X denotes a subset of N_1 , Y denotes a subset of N_2 , x denotes a point of N_1 , y denotes a point of N_2 , f denotes a function from N_1 into N_2 , and f_1 denotes a continuous function from N_1 into N_2 .

Now we state the propositions:

- (3) O is an open subset of $\text{NTop2Top}(N)$.
- (4) A is a subset of $\text{NTop2Top}(N)$.
- (5) (i) Ω_N is open, and
(ii) $\emptyset_N$ is open.
- (6) $N \mapsto y$ is continuous.
- (7) a is interior point of A if and only if there exists an open subset O of N such that $a \in O$ and $O \subseteq A$.
- (8) If $a \in O$, then a is interior point of O .
- (9) $\text{Int } A = \bigcup \{O, \text{ where } O \text{ is an open subset of } N : O \subseteq A\}$.
- (10) $\text{Int } A \subseteq A$.
- (11) [20, DEFINITION 2.10.1, P. 83]:
If $A \subseteq B$, then $\text{Int } A \subseteq \text{Int } B$.
- (12) [20, DEFINITION 2.10.2, P. 83]:
 A is open if and only if $\text{Int } A = A$.
- (13) $\text{Int } A = \text{Int Int } A$.
- (14) Let us consider a non empty, strict neighborhood topological space N , a subset A of N , and a point x of N . Suppose A is a neighbourhood of x . Then $\text{Int } A$ is an open neighbourhood of x . The theorem is a consequence of (12).
- (15) The image of filter $U_F(x)$ under $f = \{M, \text{ where } M \text{ is a subset of } N_2 : f^{-1}(M) \in U_F(x)\}$.

- (16) If f is continuous at x and $y = f(x)$, then for every element V of $U_F(y)$, there exists an element W of $U_F(x)$ such that $f^\circ W \subseteq V$.
- (17) If $y = f(x)$ and for every element V of $U_F(y)$, there exists an element W of $U_F(x)$ such that $f^\circ W \subseteq V$, then f is continuous at x .
- (18) [20, DEFINITION 2.13.1, P. 97]:
If $y = f(x)$, then f is continuous at x iff for every element V of $U_F(y)$, there exists an element W of $U_F(x)$ such that $f^\circ W \subseteq V$.
- (19) [20, PROPOSITION 2.13.3, P. 99]:
If f is continuous at x and x is adherent point of X and $y = f(x)$ and $Y = f^\circ X$, then y is adherent point of Y .
- (20) [20, THEOREM 2.13.4, P. 99, (1) $\Rightarrow$ (2)]:
 $f_1^\circ \overline{X} \subseteq \overline{f_1^\circ X}$.
- (21) Every closed subset of N is a closed subset of $\text{NTop2Top}(N)$.
- (22) [20, PROPOSITION 2.10.2, P. 84]:
If $B = \Omega_N \setminus A$, then $\Omega_N \setminus \overline{A} = \text{Int } B$.
- (23) [20, PROPOSITION 2.10.2, P. 84]:
If $B = \Omega_N \setminus A$, then $\Omega_N \setminus (\text{Int } A) = \overline{B}$.
- (24) $A \subseteq \overline{A}$.
- (25) [20, 2.10.6, P. 84]:
 A is closed if and only if $\overline{A} = A$.
- (26) [20, 2.10.5, P. 84]:
If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.
- (27) [20, THEOREM 2.13.4, P. 99, (2) $\Rightarrow$ (3)]:
If for every subset X of N_1 , $f^\circ \overline{X} \subseteq \overline{f^\circ X}$, then for every closed subset S of N_2 , $f^{-1}(S)$ is a closed subset of N_1 .
- (28) [20, DEFINITION 2.9.3, P. 81]:
If $B = \Omega_N \setminus A$, then A is open iff B is closed.
- (29) If $A = \Omega_N \setminus B$, then A is open iff B is closed.
- (30) [20, THEOREM 2.13.4, P. 99, (3) $\Rightarrow$ (4)]:
If for every closed subset S of N_2 , $f^{-1}(S)$ is a closed subset of N_1 , then for every open subset S of N_2 , $f^{-1}(S)$ is an open subset of N_1 .
- (31) [20, THEOREM 2.13.4, P. 99, (4) $\Rightarrow$ (1)]:
If for every open subset S of N_2 , $f^{-1}(S)$ is an open subset of N_1 , then f is continuous.
- (32) [20, THEOREM 2.13.4, P. 99, (1) $\Leftrightarrow$ (4)]:
 f is continuous if and only if for every open subset O of N_2 , $f^{-1}(O)$ is an open subset of N_1 .

- (33) [20, THEOREM 2.13.4, P. 99, (1) $\Leftrightarrow$ (3)]:
 f is continuous if and only if for every closed subset O of N_2 , $f^{-1}(O)$ is a closed subset of N_1 .
- (34) $\text{Int } A = \text{Int } \text{NTop2Top}(A)$.
- (35) If A is a neighbourhood of a , then $\text{NTop2Top}(A)$ is a neighbourhood of $\text{NTop2Top}(a)$. The theorem is a consequence of (34).
- (36) If A is a neighbourhood of B , then $\text{NTop2Top}(A)$ is a neighbourhood of $\text{NTop2Top}(B)$.
- (37) If A misses B , then $\text{NTop2Top}(A)$ misses $\text{NTop2Top}(B)$.
- (38) If A misses B , then $\text{Int } A$ misses $\text{Int } B$.

From now on N denotes a T_2 neighborhood topological space.

Now we state the propositions:

- (39) Let us consider points x, y of N . Suppose $x \neq y$. Then there exists an element V_1 of $U_F(x)$ and there exists an element V_2 of $U_F(y)$ such that V_1 misses V_2 .
- (40) $\text{NTop2Top}(N)$ is a T_2 , non empty, strict topological space. The theorem is a consequence of (39).
- (41) Let us consider a non empty, normal neighborhood topological space N . Then $\text{NTop2Top}(N)$ is normal. The theorem is a consequence of (36) and (1).

Let N be a non empty, normal neighborhood topological space. One can verify that $\text{NTop2Top}(N)$ is normal.

4. SOME CONNECTIONS BETWEEN NEIGHBORHOOD TOPOLOGY AND OPEN-SET TOPOLOGY

In the sequel T denotes a non empty topological space, A, B denote subsets of T , F denotes a closed subset of T , and O denotes an open subset of T .

Now we state the propositions:

- (42) A is a subset of $\text{Top2NTop}(T)$.
- (43) F is a closed subset of $\text{Top2NTop}(T)$.
- (44) O is an open subset of $\text{Top2NTop}(T)$.
- (45) If A misses B , then $\text{Top2NTop}(A)$ misses $\text{Top2NTop}(B)$.
- (46) Let us consider a T_2 , non empty topological space T . Then $\text{Top2NTop}(T)$ is a T_2 neighborhood topological space.

In the sequel T denotes a non empty, strict topological space, A, B denote subsets of T , and x denotes a point of T .

Now we state the propositions:

(47) $\text{Int } A = \text{Int Top2NTop}(A)$.

(48) If A is a neighbourhood of B , then $\text{Top2NTop}(A)$ is a neighbourhood of $\text{Top2NTop}(B)$.

(49) If A is a neighbourhood of x , then $\text{Top2NTop}(A)$ is a neighbourhood of $\text{Top2NTop}(x)$.

(50) Let us consider a non empty, normal, strict topological space T . Then $\text{Top2NTop}(T)$ is normal.

Let T be a non empty, normal, strict topological space. Note that $\text{Top2NTop}(T)$ is normal.

5. TRANSPORT FROM $\mathbb{R}^1$ TO $\text{FMT-}\mathbb{R}^1$

From now on A denotes a subset of $\text{FMT-}\mathbb{R}^1$, x denotes a point of $\text{FMT-}\mathbb{R}^1$, y denotes a point of the metric space of real numbers, z denotes a point of (the metric space of real numbers)_{top}, and r denotes a real number.

Now we state the propositions:

(51) $\text{NTop2Top}(\text{FMT-}\mathbb{R}^1) = \mathbb{R}^1$.

(52) The carrier of $\text{FMT-}\mathbb{R}^1 = \mathbb{R}$.

(53) Let us consider a neighborhood topological space N , and a function f from N into $\text{FMT-}\mathbb{R}^1$. Then $\text{NTop2Top}(f)$ is a function from $\text{NTop2Top}(N)$ into $\mathbb{R}^1$.

(54) Let us consider a non empty topological space T , and a function f from T into $\mathbb{R}^1$. Then $\text{Top2NTop}(f)$ is a function from $\text{Top2NTop}(T)$ into $\text{Top2NTop}(\mathbb{R}^1)$.

(55) A is open if and only if for every real number x such that $x \in A$ there exists r such that $r > 0$ and $]x - r, x + r[\subseteq A$.

(56) $\{]a, b[, \text{ where } a, b \text{ are real numbers : } a < b\}$ is a basis of $\mathbb{R}^1$.

(57) $\{]a, b[, \text{ where } a, b \text{ are real numbers : } a < b\}$ is a basis of $\text{FMT-}\mathbb{R}^1$.

PROOF: Set $B = \{]a, b[, \text{ where } a, b \text{ are real numbers : } a < b\}$. $B \subseteq 2^\alpha$, where α is the carrier of $\text{FMT-}\mathbb{R}^1$. $B \subseteq$ the open set family of $\text{FMT-}\mathbb{R}^1$.

□

(58) If $r > 0$, then $]x - r, x + r[$ is a neighbourhood of x . The theorem is a consequence of (57).

(59) Let us consider an object x . Then x is a point of $\text{FMT-}\mathbb{R}^1$ if and only if x is a point of the metric space of real numbers.

(60) If $x = y$, then $\text{Ball}(y, r) =]x - r, x + r[$.

(61) If $x = y$ and $r > 0$, then $\text{Ball}(y, r)$ is a neighbourhood of x . The theorem is a consequence of (58).

(62) If $x = z$, then $\text{Balls } z$ is a family of subsets of $\text{FMT-}\mathbb{R}^1$.

(63) Let us consider a family S of subsets of $\text{FMT-}\mathbb{R}^1$. If $x = z$ and $S = \text{Balls } z$, then $[S] = U_F(x)$. The theorem is a consequence of (61), (14), and (55).

The functor $\text{gen-NS-}\mathbb{R}^1$ yielding a function from the carrier of $\text{FMT-}\mathbb{R}^1$ into $2^{2^{(\text{the carrier of FMT-}\mathbb{R}^1)}}$ is defined by

(Def. 20) for every real number r , there exists a point x of (the metric space of real numbers) $_{\text{top}}$ such that $x = r$ and $it(r) = \text{Balls } x$.

The functor $\text{gen-}\mathbb{R}^1$ yielding a non empty, strict formal topological space is defined by the term

(Def. 21) $\langle \text{the carrier of FMT-}\mathbb{R}^1, \text{gen-NS-}\mathbb{R}^1 \rangle$.

Now we state the propositions:

(64) The carrier of $\text{gen-}\mathbb{R}^1 = \mathbb{R}$.

(65) Let us consider an element x of $\text{gen-}\mathbb{R}^1$. Then there exists a point y of (the metric space of real numbers) $_{\text{top}}$ such that

(i) $x = y$, and

(ii) $U_F(x) = \text{Balls } y$.

(66) $\text{dom}[\text{gen-}\mathbb{R}^1] = \mathbb{R}$.

(67) $\text{gen-filter gen-}\mathbb{R}^1 = \text{FMT-}\mathbb{R}^1$. The theorem is a consequence of (64), (65), and (58).

6. TRANSPORTING URYSOHN'S LEMMA ([4, URYSOHN3:20]) FROM AN OPEN-SET TOPOLOGICAL SPACE TO THE ASSOCIATED NEIGHBORHOOD TOPOLOGICAL SPACE

Now we state the proposition:

(68) **Main result** URYSOHN'S LEMMA IN A NEIGHBORHOOD TOPOLOGICAL SPACE:

Let us consider a non empty, normal neighborhood topological space N , and closed subsets A, B of N . Suppose A misses B . Then there exists a function F from N into $\text{FMT-}\mathbb{R}^1$ such that

(i) F is continuous, and

(ii) for every point x of N , $0 \leq F(x) \leq 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

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