

## An Enhanced Intuitionistic Fuzzy TOPSIS approach Based on Yager's Aggregation Operators and the Application in Selecting Green Suppliers

Lining Lai<sup>1</sup>, Zia ULLAH<sup>2</sup>, Muhammad Tariq Rahim<sup>2</sup>, Qaisar Khan<sup>3</sup>, Fawad Hussain<sup>2</sup>,  
Peide Liu<sup>4,\*</sup>

**Abstract.** The theory of intuitionistic fuzzy sets (IFS) is ideally suited to handle uncertainty and haziness. In this study, an enhanced fuzzy TOPSIS-based method for dealing with multi-attribute group decision making (MAGDM) problems under intuitionistic fuzzy information, where the weights of the decision-makers (DMs) and criteria are completely unknown. Firstly, the Yager operational rules are initiated for intuitionistic fuzzy numbers (IFNs) constructed on Yager T-norm (TN) and T-conorm and various core properties of these operational rules are investigated. Secondly, utilizing these operational various weighted aggregation operators such as, intuitionistic fuzzy Yager weighted averaging (IFYWA) operator, intuitionistic fuzzy Yager weighted ordered weighted averaging (IFYOWA) operator, intuitionistic fuzzy Yager weighted hybrid averaging (IFYWHA) operator, intuitionistic fuzzy Yager weighted geometric (IFYWG) operator, intuitionistic fuzzy Yager ordered weighted geometric (IFOWG) operator, intuitionistic fuzzy Yager hybrid weighted geometric (IFYHWG) operator are initiated. Thirdly, a few characteristics of the intended aggregation operators are investigated. Fourthly, a novel MAGDM model is constructed based on the intended aggregation operators to handle IF information. Finally, a numerical example related to the selection of suitable green suppliers is provided to show the efficacy and practicality of the initiated MAGDM approach, along with the comparison to some existing MAGDM approaches.

**Keywords:** Intuitionistic fuzzy sets, Yager's t-norm and t-conorm, TOPSIS and MAGDM.

---

<sup>\*1</sup> Shandong Special Equipment Inspection Institute Group CO., Ltd, Jinan 250101, Shandong China, E-mail: 10082611@sdic.com.cn

<sup>2</sup> Department of Mathematics Abbottabad University of Science and Technology, Abbottabad Pakistan, [ziaullah4678@gmail.com](mailto:ziaullah4678@gmail.com) (ULLAH), [Tariq@aust.edu.pk](mailto:Tariq@aust.edu.pk) (Rahim), [fawadhussain998@hotmail.com](mailto:fawadhussain998@hotmail.com) (Hussain)

<sup>3</sup> Department of Mathematics and Statistics University of Haripur, Haripur KPK Pakistan email: [qaisar.khan@uoh.edu.pk](mailto:qaisar.khan@uoh.edu.pk)

<sup>4</sup> School of Management Science and Engineering, Shandong University of Finance and Economics, Jinan, 250014, China Email: [peide.liu@gmail.com](mailto:peide.liu@gmail.com)

Corresponding author: Qaisar Khan and Peide Liu

## **1. Introduction**

The efficacy of supply chain management has grown in significance in the modern era to keep up with the escalating rivalry among businesses. Shortening the manufacturing cycle and lowering production costs are crucial for successfully impacting the market. For this reason, organizations use raw materials and parts that come from outside sources to concentrate on their primary activities. Getting the appropriate goods at the right time, from the right suppliers may save time and money when purchasing using several attributes but choosing an unsuitable source might cause the organization several issues in [6][15]. The performance of the supply chain is directly impacted by supplier selection, which also contributes to delivering high-quality goods that satisfy customers. Reduced purchasing risk, selection of the best suppliers, and development of long-lasting, intimate relationships between the supplier and the customer are the primary goals of the supplier selection process. Supplier selection is essentially the process of choosing the best supplier out of a group of potential ones in [7]. "Supplier selection" has been acknowledged as a crucial problem owing to the direct influence on the organizations, finances and performance to be in a strategically competitive position. The complexity of both qualitative and quantitative criteria for selection, as well as the number and variety of suppliers in [2], contribute to the complexity of the supplier evaluation procedure in [3]. However, the relevance and complexity of supplier selection decisions are increased by rising outsourcing tendencies, stricter regional and management rules, and competing organizational and operational aims. In this respect, the challenges include not only recognizing the importance of supplier management and practices, but also creating ways and plans for challenging supplier selection that both suppliers and supplier management experts must deal with. Resource conservation and environmental awareness have become more crucial considering the diminishing resources and rising environmental pollution in [11]. Global warming, rising public awareness of the issue, and significant environmental challenges including pressure from institutions, institutions of higher learning, and consumers are forcing organizations to concentrate on environmental competency in [21]. Green supply chain management is being examined in recent research to thrive in the face of escalating global market rivalry and environmental concerns in [17] [37].

Modern decision-making research focuses a lot of emphasis on multi-attribute decision making/group decision making (MADM/MAGDM), which is now applicable in many different domains. To rank the alternatives or compute a viable alternative, MADM/MAGDM techniques specify how attribute information is to be handled. The primary issues with any MADM/MAGDM approaches are how to completely aggregate the attribute value. In MADM/MAGDM approaches, the adaptability of the aggregate operators (AOs), which may rank the alternatives and aggregate the attribute values mutually, is commendable. The term fuzzy set (FS), first used by Zadeh in 1965, has gained traction from researchers and has been used in various fields [9][20]. In [4] established the notion of intuitionistic fuzzy set (IFS) as a generalization of FS to deal with decision-making issues that contains both membership degree (MD) and non-membership degree (N-MD). After their initiation, IFSs have gained much more attention from the researchers and have been applied to several fields in [5] [10][29]. When there is a list of qualities in an individual item of information, the notion of operators was developed to help choose the best option. In [33], various AOs, the weighted, ordered weighted, and hybrid averaging operators are introduced

to deal with IF data. Geometric weighted, geometric ordered weighted, and geometric hybrid operators were presented in [34] for use in an IF background information. In [19], the generalized ordered weighted averaging operators for IFS were initiated. In [32], the concept of induced geometric AOs under IF information was presented. In [39], examined the idea of IF Einstein hybrid AOs with the MADM issue. In [30][31] developed some IF Einstein aggregation operators and applied them to solve MADM problems. In [23] initiated some generalized aggregation operators and applied them to solve MAGDM problems under IF environment. In [12][13] initiated some novel IF operational laws and interactive IF Einstein aggregation operators and applied them to solve MADM problems. Various IF Dombi aggregation operators were initiated in [27] and applied them to solve MADM problems under IF context. In [14] introduced some IF Hamacher aggregation operators and applied them to solve MADM/MAGDM problems under IF context. Several entropies measures for IFSs have been initiated in [28] [40], by several authors and applied them to solve MADM/MAGDM problems under IF context in different fields. From the above discussion, one can notice that there are no such aggregation operators for IF numbers (IFNs), which are based on Yager's t-norm (TN) and t-conorm (T-CN) discussed in [35]. The significance of variation in parameter activities is emphasized by the Yager t-NM and t-CM. They also have the desirable property of preserving the intersection and union operational rules under multiplication and power by a scalar, which suggests that aggregation operators constructed using Yager possess the capacity to meet the requirements for aggregating general IF data and, through parameter modifications, to reflect the psychological preferences of decision-makers regarding profitability or risk avoidance. They also possess several nice algebraic characteristics and a low computation cost. Because of these benefits, researchers are introducing the Yager TNs and T-CN into the IF aggregation operator and creating new aggregation operators in response to address MCGDM issues in IF situations.

Numerous approaches for making decisions have been developed in the literature throughout the years, with Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) being one of the most well-known and widely employed. TOPSIS was introduced in [16] to deal with MADM problems. The optimal alternative in decision making problems is the one where the distance between it and the positive ideal solution (PIS) is the shortest and the distance between it and the negative ideal solution (NIS) is the greatest. The TOPSIS employing the fuzzy environment to solve the MAGDM problems was presented in [8]. Many researchers have recently developed an interest in TOPSIS and used it to actual MADM/MAGDM problems under various extended FS structures. In [22] developed IF TOPSIS and applied it to the performance of alternative vehicle technologies across their life cycles. In [26] initiated a novel distance based extended intuitionistic fuzzy TOPSIS approach with a potential application of credit risk assessment. In [18] proposed distance and entropy based TOPSIS approach for IFS and applied them to solve MADM problems under IF environment. In [24] developed IF TOPSIS and applied them to solve MADM problem about green supplier selection. A belief-plausibility TOPSIS is created in [36] to address multi-criteria decision-making issues. In [38] initiated TOPSIS approach for obtaining weights of decision makers with interval numbers. The modified slack-based measure TOPSIS approach is introduced in [1]. Laser Beam machining and TOPSIS are combined in [25] and apply them to solve MADM problems.

It should also be noted that the current TOPSIS algorithms have the limitation that to solve MADM/MAGDM problems either decision makers (DM) weights or criteria weights are known, or both and can obtained the aggregated information utilizing flexible aggregation

operators which can consist of generalized parameter under IF setting. Some researchers assigned unknown weight information to known criterion weights. Instead, other researchers solved MCGDM difficulties using DMs with known weights to manage uncertain criterion weights. Despite the worries of the authors, there can be no such instrument to solve MAGDM issues, when the weight data of DMs and criteria are fully unknown and flexible aggregation operators under IF information.

Motivated by the above discussion the following are listed as this article's contributions:

- To draught numerous IFN Yager operational laws while looking at the characteristics of these operational rules.
- To draught numerous IF Yager IF aggregation operators and investigate the characteristics of these aggregation operators.
- To develop an enhanced TOPSIS approach based on these Yager aggregation operators in which both DMs and criterion weights are fully unknown.
- To develop MAGDM approach based on enhanced TOPSIS approach.
- To apply the developed approach a numerical example related to green supplier selection is solved.
- To show the efficacy and practicality of the developed approach by comparing it with some existing approaches.

To do so, the rest of the article is organized as follows:

In section 2, some basic concepts about IFS, operational laws IFNs, distance among IFNs and entropy measure for IFN are given. In section 3, Yager operational laws for IFNs are introduced and their core properties are investigated. In section 4, based on these Yager operational laws various IF Yager weighted aggregation operators are introduced. Some core characteristics of these are also investigated. In section 5, a MAGDM technique, based on enhanced TOPSIS approach is initiated. In section 6, a numerical example about green supplier selection is put forward to show the effectiveness of the developed MAGDM approach. Finally, comparisons with some existing approaches are also given to show the practicality and advantages of the proposed approach. At the end conclusion and references are given.

## 2. Preliminaries

In this section, some basic concepts about IFS, operational laws IFNs, distance among IFNs and entropy measure for IFN are given.

**Definition 1 [4].** Let  $X$  be the universe set with conventional element in  $X$  exemplified by  $a$ . An IFS  $B$  in the reference set  $X$  is conveyed mathematically as follow:

$$B = \{ \langle a, \mathfrak{x}_B(a), \mathfrak{a}_B(a) \rangle \mid a \in X \}. \quad (1)$$

Where,  $\mathfrak{x}_B(a)$  and  $\mathfrak{a}_B(a)$  are the MD and the N-MD of an element  $a \in X$  to the IFS  $B$ , respectively,  $\mathfrak{x}_B(a), \mathfrak{a}_B(a) \in [0,1]$  and  $0 \leq \mathfrak{x}_B(a) + \mathfrak{a}_B(a) \leq 1$ . The hesitancy degree  $\mathfrak{s}_B(a)$  of number  $a$  belonging to IFS  $B$  is described by  $\mathfrak{s}_B(a) = 1 - (\mathfrak{x}_B(a) + \mathfrak{a}_B(a))$ , for all  $a \in X$ .

The pair  $(\mathfrak{x}_B, \mathfrak{a}_B)$  in the IFS  $B = \{ \langle a, \mathfrak{x}_B(a), \mathfrak{a}_B(a) \rangle | a \in X \}$  is said to be an IFN. To be easily understood, we shall represent IFN by  $\mathcal{K} = (\mathfrak{x}_{\mathcal{K}}, \mathfrak{a}_{\mathcal{K}})$ , where  $\mathfrak{x}_{\mathcal{K}}, \mathfrak{a}_{\mathcal{K}} \in [0, 1]$  and  $0 \leq \mathfrak{x}_{\mathcal{K}} + \mathfrak{a}_{\mathcal{K}} \leq 1$ .

**Definition 2 [10].** Let  $\mathcal{K}_1 = \langle \mathfrak{x}_1, \mathfrak{a}_1 \rangle$  and  $\mathcal{K}_2 = \langle \mathfrak{x}_2, \mathfrak{a}_2 \rangle$  be any two arbitrary IFNs,  $\wp > 0$ . Subsequently, the operational rules for IFNs are explained as follow:

$$1. \quad \mathcal{K}_1 \oplus \mathcal{K}_2 = \langle \mathfrak{x}_1 + \mathfrak{x}_2 - \mathfrak{x}_1 \mathfrak{x}_2, \mathfrak{a}_1 \mathfrak{a}_2 \rangle; \quad (2)$$

$$2. \quad \mathcal{K}_1 \otimes \mathcal{K}_2 = \langle \mathfrak{x}_1 \mathfrak{x}_2, \mathfrak{a}_1 + \mathfrak{a}_2 - \mathfrak{a}_1 \mathfrak{a}_2 \rangle; \quad (3)$$

$$3. \quad \wp \mathcal{K}_1 = \langle 1 - (1 - \mathfrak{x}_1)^\wp, \mathfrak{a}_1^\wp \rangle; \quad (4)$$

$$4. \quad \mathcal{K}_1^\wp = \langle (\mathfrak{x}_1)^\wp, 1 - (1 - \mathfrak{a}_1)^\wp \rangle. \quad (5)$$

**Definition 3 [33].** Let  $\mathcal{K} = \langle \mathfrak{x}, \mathfrak{a} \rangle$  be an IFN. The score function  $\overline{SRE}$  of the IFN  $\mathcal{K}$  is explained as follows:

$$\overline{SRE}(\mathcal{K}) = \mathfrak{x} - \mathfrak{a}; \quad (6)$$

Where, its score values located in  $[-1, 1]$ . The greater the score value  $\overline{SRE}(\mathcal{K})$ , the superior the IFN  $\mathcal{K}$ .

The accuracy function  $\overline{ACY}$  of the IFN  $\mathcal{K}$  is explained as follows:

$$\overline{ACY}(\mathcal{K}) = \mathfrak{x} + \mathfrak{a}; \quad (7)$$

Where, its accuracy values  $[0, 1]$ . The finer the accuracy value  $\overline{ACY}(\mathcal{K})$ , the superior the IFN  $\mathcal{K}$ .

**Definition 4 [33].** Let  $\mathcal{K}_1 = \langle \mathfrak{x}_1, \mathfrak{a}_1 \rangle$  and  $\mathcal{K}_2 = \langle \mathfrak{x}_2, \mathfrak{a}_2 \rangle$  be any two arbitrary IFNs. Then for comparing two IFNs, Xu [33] categorized the decision rules, which are enlightened as follow:

(1) If  $\overline{SRE}(\mathcal{K}_1) < \overline{SRE}(\mathcal{K}_2)$ , then  $\mathcal{K}_1 < \mathcal{K}_2$ ,

(2) If  $\overline{SRE}(\mathcal{K}_1) = \overline{SRE}(\mathcal{K}_2)$ , then

a. If  $\overline{ACY}(\mathcal{K}_1) < \overline{ACY}(\mathcal{K}_2)$ , afterwards  $\mathcal{K}_1 < \mathcal{K}_2$ ;

b. If  $\overline{ACY}(\mathcal{K}_1) = \overline{ACY}(\mathcal{K}_2)$ , afterwards  $\mathcal{K}_1 < \mathcal{K}_2$ .

**Definition 5 [5].** Let  $\mathcal{K}_1 = \langle \mathfrak{x}_1, \mathfrak{a}_1 \rangle$  and  $\mathcal{K}_2 = \langle \mathfrak{x}_2, \mathfrak{a}_2 \rangle$  be any two IFNs. Then the Hamming distance measure between  $\mathcal{K}_1$  and  $\mathcal{K}_2$  is defined as follows.

$$DIS(\mathcal{K}_1, \mathcal{K}_2) = \frac{1}{2} (|\mathfrak{x}_1 - \mathfrak{x}_2| + |\mathfrak{a}_1 - \mathfrak{a}_2|) \quad (8)$$

**Definition 6 [40].** The entropy measure of IFS  $B = \langle \mathfrak{x}_B, \mathfrak{a}_B \rangle$  over the universe of discourse set  $X$  is demarcated as follows:

$$\overline{E}(B) = \frac{1}{4n} \sum_{i=1}^n (1 + \mathfrak{s}_B(x_i)) (1 - |\mathfrak{x}_B(x_i) - \mathfrak{a}_B(x_i)| + \mathfrak{s}_B(x_i)), \quad x \in X \quad (9)$$

Where,  $\mathfrak{x}_B(x)$  and  $\mathfrak{a}_B(x)$  are the MD and the N-MD of an element  $x \in X$  to the IFS  $B$ , respectively,  $\mathfrak{x}_B(x), \mathfrak{a}_B(x) \in [0, 1]$  and  $0 \leq \mathfrak{x}_B(x) + \mathfrak{a}_B(x) \leq 1$ . The hesitancy degree  $\mathfrak{s}_B(x)$  of number  $x$  belonging to IFS  $B$  is described by  $\mathfrak{s}_B(x) = 1 - (\mathfrak{x}_B(x) + \mathfrak{a}_B(x))$ , for all  $x \in X$ .

### 3. Yager's operational laws for IFN

In this part, Yager's operational laws for IFNs are initiated and some core features of these operational laws are investigated.

**Definition 7[35].** Let  $\mathcal{E}$  and  $\mathcal{A}$  be any two real numbers. Then, the Yager's t-norm  $R$  and t-conorm  $N$  are defined as follows:

$$\begin{aligned} 1 \quad R(\mathcal{E}, \mathcal{A}) &= 1 - \min \left( 1, \left( (1 - \mathcal{E})^{\mathcal{I}\mathcal{O}} + (1 - \mathcal{A})^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right); \\ 2 \quad N(\mathcal{E}, \mathcal{A}) &= \min \left( 1, \left( \mathcal{E}^{\mathcal{I}\mathcal{O}} + \mathcal{A}^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \quad \mathcal{I}\mathcal{O} \in [1, +\infty]. \end{aligned}$$

**Definition 8.** Let  $\mathcal{K}_1 = \langle \mathcal{a}_1, \mathcal{a}_1 \rangle$  and  $\mathcal{K}_2 = \langle \mathcal{a}_2, \mathcal{a}_2 \rangle$  be any two IFNs with  $\overline{\mathcal{A}}, \mathcal{I}\mathcal{O} > 0$ . Then, the Yager's operational laws (YOLs) for IFNs are explained as follows.

$$1) \quad \mathcal{K}_1 \oplus \mathcal{K}_2 = \left\langle \min \left( 1, \left( \mathcal{a}_1^{\mathcal{I}\mathcal{O}} + \mathcal{a}_2^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( (1 - \mathcal{a}_1)^{\mathcal{I}\mathcal{O}} + (1 - \mathcal{a}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle, \quad (10)$$

$$2) \quad \mathcal{K}_1 \otimes \mathcal{K}_2 = \left\langle 1 - \min \left( 1, \left( (1 - \mathcal{a}_1)^{\mathcal{I}\mathcal{O}} + (1 - \mathcal{a}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \mathcal{a}_1^{\mathcal{I}\mathcal{O}} + \mathcal{a}_2^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle, \quad (11)$$

$$3) \quad \overline{\mathcal{A}}\mathcal{K} = \left\langle \min \left( 1, \left( \overline{\mathcal{A}}\mathcal{a}_1^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \overline{\mathcal{A}}(1 - \mathcal{a}_1)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle; \quad (12)$$

$$4) \quad \mathcal{K}^{\overline{\mathcal{A}}} = \left\langle 1 - \min \left( 1, \left( \overline{\mathcal{A}}(1 - \mathcal{a}_1)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \overline{\mathcal{A}}\mathcal{a}_1^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle. \quad (13)$$

**Example 1.** Let  $\mathcal{K}_1 = \langle 0.6, 0.3 \rangle$  and  $\mathcal{K}_2 = \langle 0.8, 0.2 \rangle$  be two IFNs. Then by Definition 8 for  $\mathcal{I}\mathcal{O} = 3, \overline{\mathcal{A}} = 5$ , we have

$$\begin{aligned} 1. \quad \mathcal{K}_1 \oplus \mathcal{K}_2 &= \left\langle \min \left( 1, \left( 0.6^3 + 0.8^3 \right)^{\frac{1}{3}} \right), 1 - \min \left( 1, \left( (1 - 0.3)^3 + (1 - 0.2)^3 \right)^{\frac{1}{3}} \right) \right\rangle = \langle 0.8996, 0.0509 \rangle; \\ 2. \quad \mathcal{K}_1 \otimes \mathcal{K}_2 &= \left\langle 1 - \min \left( 1, \left( (1 - 0.6)^3 + (1 - 0.8)^3 \right)^{\frac{1}{3}} \right), \min \left( 1, \left( 0.3^3 + 0.2^3 \right)^{\frac{1}{3}} \right) \right\rangle = \langle 0.5840, 0.3271 \rangle; \\ 3. \quad 5 * \mathcal{K}_1 &= \left\langle \min \left( 1, \left( 5 * (0.6^3) \right)^{\frac{1}{3}} \right), 1 - \min \left( 1, \left( 5 * (1 - 0.3)^3 \right)^{\frac{1}{3}} \right) \right\rangle = \langle 1, 0 \rangle; \\ 4. \quad \mathcal{K}_1^5 &= \left\langle 1 - \min \left( 1, \left( 5 * (1 - 0.6)^3 \right)^{\frac{1}{3}} \right), \min \left( 1, \left( 5 * (0.3^3) \right)^{\frac{1}{3}} \right) \right\rangle = \langle 0.3160, 0.5130 \rangle. \end{aligned}$$

**Theorem 1.** Let  $\mathcal{K} = \langle \mathcal{a}, \mathcal{a} \rangle, \mathcal{K}_1 = \langle \mathcal{a}_1, \mathcal{a}_1 \rangle$  and  $\mathcal{K}_2 = \langle \mathcal{a}_2, \mathcal{a}_2 \rangle$  be any three IFNs. Then

- 1)  $\mathcal{K}_1 \oplus \mathcal{K}_2 = \mathcal{K}_2 \oplus \mathcal{K}_1$
- 2)  $\mathcal{K}_1 \otimes \mathcal{K}_2 = \mathcal{K}_2 \otimes \mathcal{K}_1$ ;

- 3)  $\overline{\overline{\mathcal{K}}}(\mathcal{K}_1 \oplus \mathcal{K}_2) = \overline{\overline{\mathcal{K}}}\mathcal{K}_1 \oplus \overline{\overline{\mathcal{K}}}\mathcal{K}_2; \overline{\overline{\mathcal{K}}} > 0;$
- 4)  $\overline{\overline{\mathcal{K}}}\mathcal{K} \oplus \overline{\overline{\mathcal{K}}}\mathcal{K} = \left(\overline{\overline{\mathcal{K}}} + \overline{\overline{\mathcal{K}}}\right)\mathcal{K}; \overline{\overline{\mathcal{K}}}, \overline{\overline{\mathcal{K}}} > 0;$
- 5)  $(\mathcal{K}_1 \otimes \mathcal{K}_2)^{\overline{\overline{\mathcal{K}}}} = \mathcal{K}_1^{\overline{\overline{\mathcal{K}}}} \otimes \mathcal{K}_2^{\overline{\overline{\mathcal{K}}}}, \overline{\overline{\mathcal{K}}} > 0;$
- 6)  $\mathcal{K}^{\overline{\overline{\mathcal{K}}}_{\mathcal{K}_1}} \otimes \mathcal{K}^{\overline{\overline{\mathcal{K}}}_{\mathcal{K}_2}} = \mathcal{K}^{\left(\overline{\overline{\mathcal{K}}}_{\mathcal{K}_1} + \overline{\overline{\mathcal{K}}}_{\mathcal{K}_2}\right)}.$

**Proof.** For three IFNs  $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ , and  $\overline{\overline{\mathcal{K}}}, \overline{\overline{\mathcal{K}}}_{\mathcal{K}_1}, \overline{\overline{\mathcal{K}}}_{\mathcal{K}_2} > 0$ , by Definition (8), we can have

$$\begin{aligned}
 (1). \mathcal{K}_1 \oplus \mathcal{K}_2 &= \left\langle \min \left( 1, \left( \mathcal{K}_1^{\mathcal{K}} + \mathcal{K}_2^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), 1 - \min \left( 1, \left( (1 - \mathcal{K}_1)^{\mathcal{K}} + (1 - \mathcal{K}_2)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle \\
 &= \left\langle \min \left( 1, \left( \mathcal{K}_2^{\mathcal{K}} + \mathcal{K}_1^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), 1 - \min \left( 1, \left( (1 - \mathcal{K}_2)^{\mathcal{K}} + (1 - \mathcal{K}_1)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle \\
 &= \mathcal{K}_2 \oplus \mathcal{K}_1 \\
 (2). \mathcal{K}_1 \otimes \mathcal{K}_2 &= \left\langle 1 - \min \left( \left( (1 - \mathcal{K}_1)^{\mathcal{K}} + (1 - \mathcal{K}_2)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), \min \left( 1, \left( \mathcal{K}_1^{\mathcal{K}} + \mathcal{K}_2^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle, \\
 &= \left\langle 1 - \min \left( \left( (1 - \mathcal{K}_2)^{\mathcal{K}} + (1 - \mathcal{K}_1)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), \min \left( 1, \left( \mathcal{K}_2^{\mathcal{K}} + \mathcal{K}_1^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle, \\
 &= \mathcal{K}_2 \otimes \mathcal{K}_1. \\
 (3) \overline{\overline{\mathcal{K}}}(\mathcal{K}_1 \oplus \mathcal{K}_2) &= \overline{\overline{\mathcal{K}}}\mathcal{K}_1 \oplus \overline{\overline{\mathcal{K}}}\mathcal{K}_2 \\
 \overline{\overline{\mathcal{K}}}(\mathcal{K}_1 \oplus \mathcal{K}_2) &= \overline{\overline{\mathcal{K}}} \left\langle \min \left( 1, \left( \mathcal{K}_1^{\mathcal{K}} + \mathcal{K}_2^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), 1 - \min \left( 1, \left( (1 - \mathcal{K}_1)^{\mathcal{K}} + (1 - \mathcal{K}_2)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle, \\
 &= \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{K}}}\mathcal{K}_1^{\mathcal{K}} + \overline{\overline{\mathcal{K}}}\mathcal{K}_2^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), 1 - \min \left( 1, \left( \overline{\overline{\mathcal{K}}}(1 - \mathcal{K}_1)^{\mathcal{K}} + \overline{\overline{\mathcal{K}}}(1 - \mathcal{K}_2)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle; \\
 \overline{\overline{\mathcal{K}}}(\mathcal{K}_1 \oplus \mathcal{K}_2) &= \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{K}}}\mathcal{K}_1^{\mathcal{K}} + \overline{\overline{\mathcal{K}}}\mathcal{K}_2^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), 1 - \min \left( 1, \left( \overline{\overline{\mathcal{K}}}(1 - \mathcal{K}_1)^{\mathcal{K}} + \overline{\overline{\mathcal{K}}}(1 - \mathcal{K}_2)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle; \quad (14)
 \end{aligned}$$

Now  $\overline{\overline{\mathcal{K}}}\mathcal{K}_1 \oplus \overline{\overline{\mathcal{K}}}\mathcal{K}_2$

Since

$$\overline{\overline{\mathcal{K}}}\mathcal{K}_1 = \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{K}}}\mathcal{K}_1^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), 1 - \min \left( 1, \left( \overline{\overline{\mathcal{K}}}(1 - \mathcal{K}_1)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle;$$

and

$$\overline{\overline{\mathcal{K}}}\mathcal{K}_2 = \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{K}}}\mathcal{K}_2^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right), 1 - \min \left( 1, \left( \overline{\overline{\mathcal{K}}}(1 - \mathcal{K}_2)^{\mathcal{K}} \right)^{\frac{1}{\mathcal{K}}} \right) \right\rangle,$$

then

$$\begin{aligned}\overline{\overline{\mathcal{J}}}\mathcal{K}_1 \oplus \overline{\overline{\mathcal{J}}}\mathcal{K}_2 &= \left\langle \left( \min \left( 1, \left( \overline{\overline{\mathcal{J}}}\mathcal{K}_1^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right), 1 - \min \left( 1, \left( \overline{\overline{\mathcal{J}}}(1 - \mathcal{K}_1)^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right) \right) \right. \\ &\quad \left. \oplus \left( \min \left( 1, \left( \overline{\overline{\mathcal{J}}}\mathcal{K}_2^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right), 1 - \min \left( 1, \left( \overline{\overline{\mathcal{J}}}(1 - \mathcal{K}_2)^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right) \right) \right\rangle; \\ \overline{\overline{\mathcal{J}}}\mathcal{K}_1 \oplus \overline{\overline{\mathcal{J}}}\mathcal{K}_2 &= \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{J}}}\mathcal{K}_1^{\mathcal{I}0} + \overline{\overline{\mathcal{J}}}\mathcal{K}_2^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right), 1 - \min \left( 1, \left( \overline{\overline{\mathcal{J}}}(1 - \mathcal{K}_1)^{\mathcal{I}0} + \overline{\overline{\mathcal{J}}}(1 - \mathcal{K}_2)^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right) \right\rangle. \quad (15)\end{aligned}$$

Hence,

$$\overline{\overline{\mathcal{J}}}(\mathcal{K}_1 \oplus \mathcal{K}_2) = \overline{\overline{\mathcal{J}}}\mathcal{K}_1 \oplus \overline{\overline{\mathcal{J}}}\mathcal{K}_2$$

(4) Since,

$$\overline{\overline{\mathcal{J}}}_1\mathcal{K}_1 = \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{J}}}_1(\mathcal{K}_1^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right), 1 - \left( \min(1, (\overline{\overline{\mathcal{J}}}_1((1 - \mathcal{K}_1))^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right),$$

and

$$\overline{\overline{\mathcal{J}}}_2\mathcal{K}_1 = \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{J}}}_2(\mathcal{K}_1^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right), 1 - \left( \min(1, (\overline{\overline{\mathcal{J}}}_2((1 - \mathcal{K}_1))^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right),$$

So,

$$\begin{aligned}\overline{\overline{\mathcal{J}}}_1\mathcal{K}_1 \oplus \overline{\overline{\mathcal{J}}}_2\mathcal{K}_1 &= \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{J}}}_1(\mathcal{K}_1^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right), 1 - \left( \min(1, (\overline{\overline{\mathcal{J}}}_1((1 - \mathcal{K}_1))^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right) \right. \\ &\quad \left. \oplus \overline{\overline{\mathcal{J}}}_2\mathcal{K}_1 = \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{J}}}_2(\mathcal{K}_1^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right), 1 - \left( \min(1, (\overline{\overline{\mathcal{J}}}_2((1 - \mathcal{K}_1))^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right) \right\rangle, \\ &= \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{J}}}_1(\mathcal{K}_1^{\mathcal{I}0}) + \overline{\overline{\mathcal{J}}}_2(\mathcal{K}_1^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right), 1 - \min \left( 1, \left( \overline{\overline{\mathcal{J}}}_1(1 - \mathcal{K}_1)^{\mathcal{I}0} + \overline{\overline{\mathcal{J}}}_2(1 - \mathcal{K}_1)^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right) \right\rangle, \\ &= \left\langle \min \left( 1, \left( \overline{\overline{\mathcal{J}}}_1 + \overline{\overline{\mathcal{J}}}_2 \right) (\mathcal{K}_1^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}}, 1 - \min \left( 1, \left( \left( \overline{\overline{\mathcal{J}}}_1 + \overline{\overline{\mathcal{J}}}_2 \right) (1 - \mathcal{K}_1)^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right) \right\rangle = \left( \overline{\overline{\mathcal{J}}}_1 + \overline{\overline{\mathcal{J}}}_2 \right) \mathcal{K}_1 \\ &\quad \overline{\overline{\mathcal{J}}}_1\mathcal{K}_1 + \overline{\overline{\mathcal{J}}}_2\mathcal{K}_1 = \left( \overline{\overline{\mathcal{J}}}_1 + \overline{\overline{\mathcal{J}}}_2 \right) \mathcal{K}_1\end{aligned}$$

(5) Since,

$$\mathcal{K}_1^{\overline{\overline{\mathcal{J}}}} = \left\langle 1 - \min \left( 1, \left( \overline{\overline{\mathcal{J}}}(1 - \mathcal{K}_1)^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right), \min \left( 1, \left( \overline{\overline{\mathcal{J}}}(\mathcal{K}_1^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right) \right\rangle;$$

and

$$\mathcal{K}_2^{\overline{\overline{\mathcal{J}}}} = \left\langle 1 - \min \left( 1, \left( \overline{\overline{\mathcal{J}}}(1 - \mathcal{K}_2)^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right), \min \left( 1, \left( \overline{\overline{\mathcal{J}}}(\mathcal{K}_2^{\mathcal{I}0}) \right)^{\frac{1}{\mathcal{I}0}} \right) \right\rangle,$$

$$(\mathcal{K}_1 \otimes \mathcal{K}_2)^{\overline{\overline{\mathcal{J}}}} = \left\langle \left( 1 - \min \left( 1, \left( (1 - \mathcal{K}_1)^{\mathcal{I}0} + (1 - \mathcal{K}_2)^{\mathcal{I}0} \right)^{\frac{1}{\mathcal{I}0}} \right), \min(1, (\mathcal{K}_1^{\mathcal{I}0} + \mathcal{K}_2^{\mathcal{I}0})^{\frac{1}{\mathcal{I}0}} \right) \right)^{\overline{\overline{\mathcal{J}}}},$$

$$(\mathcal{K}_1 \otimes \mathcal{K}_2)^{\bar{\mathcal{J}}} = \left\langle 1 - \min \left( 1, \left( \bar{\mathcal{J}}(1 - \mathfrak{x}_1)^{\mathcal{I}\mathcal{O}} + \bar{\mathcal{J}}(1 - \mathfrak{x}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \bar{\mathcal{J}}\mathfrak{a}_1^{\mathcal{I}\mathcal{O}} + \bar{\mathcal{J}}\mathfrak{a}_2^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle; \quad (16)$$

Now  $\mathcal{K}_1^{\bar{\mathcal{J}}} \otimes \mathcal{K}_2^{\bar{\mathcal{J}}}$

Since

$$\mathcal{K}_1^{\bar{\mathcal{J}}} = \left\langle 1 - \min \left( 1, \left( \bar{\mathcal{J}}(1 - \mathfrak{x}_1)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \left( \min \left( 1, \left( \bar{\mathcal{J}}\mathfrak{a}_1^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right) \right\rangle;$$

and

$$\mathcal{K}_2^{\bar{\mathcal{J}}} = \left\langle 1 - \min \left( 1, \left( \bar{\mathcal{J}}(1 - \mathfrak{x}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \left( \min \left( 1, \left( \bar{\mathcal{J}}\mathfrak{a}_2^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right) \right\rangle,$$

then

$$\begin{aligned} \mathcal{K}_1^{\bar{\mathcal{J}}} \otimes \mathcal{K}_2^{\bar{\mathcal{J}}} &= \left\langle 1 - \min \left( 1, \left( \bar{\mathcal{J}}(1 - \mathfrak{x}_1)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \left( \min \left( 1, \left( \bar{\mathcal{J}}\mathfrak{a}_1^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right) \right\rangle \\ &\quad \otimes \left\langle 1 - \min \left( 1, \left( \bar{\mathcal{J}}(1 - \mathfrak{x}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \left( \min \left( 1, \left( \bar{\mathcal{J}}\mathfrak{a}_2^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right) \right\rangle; \\ \mathcal{K}_1^{\bar{\mathcal{J}}} \oplus \mathcal{K}_2^{\bar{\mathcal{J}}} &= \left\langle 1 - \min \left( 1, \left( \bar{\mathcal{J}}(1 - \mathfrak{x}_1)^{\mathcal{I}\mathcal{O}} + \bar{\mathcal{J}}(1 - \mathfrak{x}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \bar{\mathcal{J}}\mathfrak{a}_1^{\mathcal{I}\mathcal{O}} + \bar{\mathcal{J}}\mathfrak{a}_2^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle. \end{aligned} \quad (17)$$

Hence,

$$(\mathcal{K}_1 \otimes \mathcal{K}_2)^{\bar{\mathcal{J}}} = \mathcal{K}_1^{\bar{\mathcal{J}}} \otimes \mathcal{K}_2^{\bar{\mathcal{J}}}$$

(6) Since,

$$\mathcal{K}_1^{\bar{\mathcal{J}}_1} = \left\langle 1 - \left( \min(1, (\bar{\mathcal{J}}_1((1 - \mathfrak{x}_1))^{\mathcal{I}\mathcal{O}}))^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \bar{\mathcal{J}}_1(\mathfrak{a}_1^{\mathcal{I}\mathcal{O}})^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right) \right\rangle,$$

and

$$\mathcal{K}_1^{\bar{\mathcal{J}}_2} = \left\langle 1 - \left( \min(1, (\bar{\mathcal{J}}_2((1 - \mathfrak{x}_1))^{\mathcal{I}\mathcal{O}}))^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \bar{\mathcal{J}}_2(\mathfrak{a}_1^{\mathcal{I}\mathcal{O}})^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right) \right\rangle,$$

So,

$$\begin{aligned} \mathcal{K}_1^{\bar{\mathcal{J}}_1} \oplus \mathcal{K}_1^{\bar{\mathcal{J}}_2} &= \left\langle 1 - \left( \min(1, (\bar{\mathcal{J}}_1((1 - \mathfrak{x}_1))^{\mathcal{I}\mathcal{O}}))^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \bar{\mathcal{J}}_1(\mathfrak{a}_1^{\mathcal{I}\mathcal{O}})^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right) \right\rangle \\ &\quad \otimes \left\langle 1 - \left( \min(1, (\bar{\mathcal{J}}_2((1 - \mathfrak{x}_1))^{\mathcal{I}\mathcal{O}}))^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \bar{\mathcal{J}}_2(\mathfrak{a}_1^{\mathcal{I}\mathcal{O}})^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right) \right\rangle, \\ &= \left\langle 1 - \min \left( 1, \left( \bar{\mathcal{J}}_1(1 - \mathfrak{x}_1)^{\mathcal{I}\mathcal{O}} + \bar{\mathcal{J}}_2(1 - \mathfrak{x}_1)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), \min \left( 1, \left( \bar{\mathcal{J}}_1(\mathfrak{a}_1^{\mathcal{I}\mathcal{O}}) + \bar{\mathcal{J}}_2(\mathfrak{a}_1^{\mathcal{I}\mathcal{O}}) \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle, \end{aligned}$$

$$= \left\langle 1 - \min \left( 1, \left( \left( \overline{\mathcal{J}}_1 + \overline{\mathcal{J}}_2 \right) (1 - \alpha_1)^{j0} \right)^{\frac{1}{j0}} \right), \min \left( 1, \left( \overline{\mathcal{J}}_1 + \overline{\mathcal{J}}_2 \right) (\alpha_1^{j0})^{\frac{1}{j0}} \right) \right\rangle = \mathcal{K}_1^{\left( \overline{\mathcal{J}}_1 + \overline{\mathcal{J}}_2 \right)}.$$

#### 4. Intuitionistic Fuzzy Yager's Weighted Averaging Operators

In this section, based on Yager's operational law, some IF Yager's weighted averaging and IF Yager's weighted geometric operators are initiated. Various core properties of the initiated aggregation operators are investigated.

##### 4.1 Intuitionistic Fuzzy Yager's Weighted Averaging operator

In this subsection, IFWA, IFYOWA and IFYAHWA operators are anticipated, and various core properties are investigated.

**Definition 9.** Let  $\mathcal{H}_\beta = \langle \alpha_\beta, \alpha_\beta \rangle$  ( $\beta=1, \dots, z$ ) be a group of IFNs. Then, The IFYWA operator is explained and symbolized by:

$$IFYWA(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \dots, \mathcal{H}_z) = \sum_{\beta=1}^z \mathcal{J}_\beta \mathcal{H}_\beta; \quad (18)$$

where  $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_z)^T$  is the importance degree such that  $\mathcal{J}_i \in [0, 1]$  with the condition

$$\sum_{i=1}^z \mathcal{J}_i = 1.$$

**Theorem 2.** Let  $\mathcal{H}_\beta = \langle \alpha_\beta, \alpha_\beta \rangle$  ( $\beta=1, \dots, z$ ) be a group of IFNs and the corresponding weight vectors is  $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_z)^T$  such that  $\mathcal{J}_\beta \in [0, 1]$  with the condition  $\sum_{\beta=1}^z \mathcal{J}_\beta = 1$ .

Then, the aggregated value obtained from Equation (18) is still an IFN, and

$$\begin{aligned} IFYWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) &= \sum_{\beta=1}^z \mathcal{J}_\beta \mathcal{H}_\beta \\ &= \left\langle \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{J}_\beta \alpha_\beta^{j0} \right)^{\frac{1}{j0}} \right), 1 - \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{J}_\beta (1 - \alpha_\beta)^{j0} \right)^{\frac{1}{j0}} \right) \right\rangle. \end{aligned} \quad (19)$$

**Proof.** By using mathematical induction  $z$ , we shall demonstrate Equation (19).

(1) For  $z = 2$ , we have

$$IFYWA(\mathcal{H}_1, \mathcal{H}_2) = \sum_{\beta=1}^2 \mathcal{J}_\beta \mathcal{H}_\beta = \mathcal{J}_1 \mathcal{H}_1 \oplus \mathcal{J}_2 \mathcal{H}_2$$

Where

$$\mathcal{J}_1 \mathcal{H}_1 = \left\langle \min \left( 1, \left( \mathcal{J}_1 (\alpha_1)^{j0} \right)^{\frac{1}{j0}} \right), 1 - \min \left( 1, \left( \mathcal{J}_1 (1 - \alpha_1)^{j0} \right)^{\frac{1}{j0}} \right) \right\rangle$$

and

$$\mathcal{L}_2 \mathcal{H}_2 = \left\langle \min \left( 1, \left( \mathcal{L}_2 (\mathcal{A}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \mathcal{L}_2 (1 - \mathcal{A}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle.$$

Now,

$$\begin{aligned} \mathcal{L}_1 \mathcal{H}_1 + \mathcal{L}_2 \mathcal{H}_2 &= \left\langle \min \left( 1, \left( \mathcal{L}_1 (\mathcal{A}_1)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \mathcal{L}_1 (1 - \mathcal{A}_1)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle \oplus \left\langle \min \left( 1, \left( \mathcal{L}_2 (\mathcal{A}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \mathcal{L}_2 (1 - \mathcal{A}_2)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle \\ &= \left\langle \min \left( 1, \left( \sum_{i=1}^2 \mathcal{L}_i \mathcal{A}_i^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \sum_{i=1}^2 \mathcal{L}_i (1 - \mathcal{A}_i)^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle. \end{aligned}$$

(2) Assuming that Equation (19) is valid for  $z = e$  we obtain.

$$\begin{aligned} IFYWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_e) &= \sum_{\mathcal{B}=1}^e \mathcal{L}_{\mathcal{B}} \mathcal{H}_{\mathcal{B}} \\ &= \left\langle \min \left( 1, \left( \sum_{\mathcal{B}=1}^e \mathcal{L}_{\mathcal{B}} \mathcal{A}_{\mathcal{B}}^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \sum_{\mathcal{B}=1}^e \mathcal{L}_{\mathcal{B}} (1 - \mathcal{A}_{\mathcal{B}})^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle. \end{aligned}$$

(3) We are now required to show that equation (19), valid for  $z = e + 1$ ,

$$\begin{aligned} IFYWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_e) &= \sum_{\mathcal{B}=1}^e \mathcal{L}_{\mathcal{B}} \mathcal{H}_{\mathcal{B}} \oplus \mathcal{L}_{e+1} \mathcal{H}_{e+1} \\ &= \left\langle \min \left( 1, \left( \sum_{\mathcal{B}=1}^e \mathcal{L}_{\mathcal{B}} \mathcal{A}_{\mathcal{B}}^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \sum_{\mathcal{B}=1}^e \mathcal{L}_{\mathcal{B}} (1 - \mathcal{A}_{\mathcal{B}})^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle \oplus \left\langle \min \left( 1, \left( \mathcal{L}_{e+1} \mathcal{A}_{e+1}^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \mathcal{L}_{e+1} (1 - \mathcal{A}_{e+1})^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle \\ &= \left\langle \min \left( 1, \left( \sum_{\mathcal{B}=1}^{e+1} \mathcal{L}_{\mathcal{B}} \mathcal{A}_{\mathcal{B}}^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \sum_{\mathcal{B}=1}^{e+1} \mathcal{L}_{\mathcal{B}} (1 - \mathcal{A}_{\mathcal{B}})^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle \end{aligned}$$

Hence Equation (19) is therefore true for  $z = e + 1$ . Therefore Equation (19) is true for any  $z$ .

**Example 2.** Let  $\mathcal{H}_1 = \langle 0.5, 0.4 \rangle$ ,  $\mathcal{H}_2 = \langle 0.6, 0.4 \rangle$ ,  $\mathcal{H}_3 = \langle 0.3, 0.6 \rangle$  and  $\mathcal{H}_4 = \langle 0.8, 0.2 \rangle$  be four IFNs, with importance degree  $\mathcal{L} = (0.2, 0.3, 0.25, 0.25)^T$ . Then by applying Theorem (2), the aggregated value of IFNs for  $\mathcal{I}\mathcal{O} = 3$ , is

$$\begin{aligned} IFYWA(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) &= \left\langle \min \left( 1, \left( \sum_{\mathcal{B}=1}^4 \mathcal{L}_{\mathcal{B}} \mathcal{A}_{\mathcal{B}}^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right), 1 - \min \left( 1, \left( \sum_{\mathcal{B}=1}^4 \mathcal{L}_{\mathcal{B}} (1 - \mathcal{A}_{\mathcal{B}})^{\mathcal{I}\mathcal{O}} \right)^{\frac{1}{\mathcal{I}\mathcal{O}}} \right) \right\rangle, \\ &= \left\langle \min \left( 1, ((0.2 * (0.5^3)) + (0.3 * (0.6^3)) + (0.25 * (0.3^3)) + (0.25 * (0.8^3)))^{\frac{1}{3}} \right), \right. \\ &\quad \left. 1 - \min(1, ((0.2 * (1 - 0.4)^3) + (0.3 * (1 - 0.4)^3) + (0.25 * (1 - 0.6)^3) + (0.25 * (1 - 0.2)^3)) \right\rangle \\ &= \langle 0.6078, 0.3684 \rangle \end{aligned}$$

Next, some core properties of the initiated AOs are investigated.

**Theorem 3.** (Idempotency) Let  $\mathcal{H}_{\mathcal{B}} = \langle \mathcal{A}_{\mathcal{B}}, \mathcal{A}_{\mathcal{B}} \rangle$  ( $\mathcal{B} = 1, \dots, z$ ) be a group of IFNs. If  $\mathcal{H}_{\mathcal{B}} = \langle \mathcal{A}_{\mathcal{B}}, \mathcal{A}_{\mathcal{B}} \rangle = \langle \mathcal{A}, \mathcal{A} \rangle = \mathcal{H}$ . Then

$$IFYWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) = \mathcal{H} \quad (20)$$

**Proof.** Since  $\mathcal{H}_\beta = \langle \mathcal{A}_\beta, \mathcal{O}_\beta \rangle = \langle \mathcal{A}, \mathcal{O} \rangle = \mathcal{H}$ . Then, by Theorem 2, we can have.

$$\begin{aligned} IFYWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) &= \left\langle \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{L}_\beta \mathcal{A}_\beta^{1/\theta} \right)^{1/\theta} \right), 1 - \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{L}_\beta (1 - \mathcal{O}_\beta)^{1/\theta} \right)^{1/\theta} \right) \right\rangle \\ &= \left\langle \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{L}_\beta \mathcal{A}^{1/\theta} \right)^{1/\theta} \right), 1 - \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{L}_\beta (1 - \mathcal{O})^{1/\theta} \right)^{1/\theta} \right) \right\rangle \\ &= \left\langle \min \left( 1, (\mathcal{A}^{1/\theta})^{1/\theta} \right), 1 - \min \left( 1, ((1 - \mathcal{O})^{1/\theta})^{1/\theta} \right) \right\rangle \\ &= \langle \mathcal{A}, \mathcal{O} \rangle = \mathcal{H} \end{aligned}$$

**Theorem 4.** (Commutativity): Assume that the  $(\mathcal{H}_1'', \mathcal{H}_2'', \dots, \mathcal{H}_z'')$  is any permutation of  $(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z)$ . If  $\mathcal{A}_\beta \leq \mathcal{A}_\beta'', \mathcal{O}_\beta \leq \mathcal{O}_\beta''$ . Then

$$IFYWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) \leq IFYWA(\mathcal{H}_1'', \mathcal{H}_2'', \dots, \mathcal{H}_z''). \quad (21)$$

**Proof.** The proof is like the above theorem, so here we eliminate.

**Theorem 5.** (Boundedness): Let  $\mathcal{H}_\beta = \langle \mathcal{A}_\beta, \mathcal{O}_\beta \rangle$  ( $\beta=1, \dots, z$ ) be a group of IFNs. If

$\mathcal{H}_\beta^- = \langle \min \mathcal{A}_\beta, \max \mathcal{O}_\beta \rangle$  and  $\mathcal{H}_\beta^+ = \langle \max \mathcal{A}_\beta, \min \mathcal{O}_\beta \rangle$ . Then

$$\mathcal{H}_\beta^- \leq IFYWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) \leq \mathcal{H}_\beta^+. \quad (22)$$

**Proof.** Since  $\min(\mathcal{A}_\beta) \leq \mathcal{A}_\beta \leq \max(\mathcal{A}_\beta), \min(\mathcal{O}_\beta) \leq \mathcal{O}_\beta \leq \max(\mathcal{O}_\beta)$  induced as,

$$\min \left( \sum_{\beta=1}^z \mathcal{L}_\beta (\mathcal{A}_\beta)^{1/\theta} \right)^{1/\theta} \leq \left( \sum_{\beta=1}^z \mathcal{L}_\beta (\mathcal{A}_\beta)^{1/\theta} \right)^{1/\theta} \leq \max \left( \sum_{\beta=1}^z \mathcal{L}_\beta (\mathcal{A}_\beta)^{1/\theta} \right)^{1/\theta}.$$

And

$$\min \left( 1 - \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{L}_\beta (1 - \mathcal{O}_\beta)^{1/\theta} \right)^{1/\theta} \right) \right) \leq 1 - \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{L}_\beta (1 - \mathcal{O}_\beta)^{1/\theta} \right)^{1/\theta} \right) \leq \max \left( 1 - \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{L}_\beta (1 - \mathcal{O}_\beta)^{1/\theta} \right)^{1/\theta} \right) \right).$$

Hence,  $\mathcal{H}_\beta^- \leq IFYWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) \leq \mathcal{H}_\beta^+$  holds.

**Definition 9.** Let  $\mathcal{H}_\beta = \langle \mathcal{A}_\beta, \mathcal{O}_\beta \rangle$  ( $\beta=1, \dots, z$ ) be a group of IFNs. Then, the IF Yager ordered weighted averaging (IFYOWA) operator is explained as follows:

$$IFYOWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) = \sum_{\beta=1}^z \mathcal{L}_\beta \mathcal{H}_{\beta^{(z)}}. \quad (23)$$

Where,  $\Psi(\beta)$  expresses the ordered and  $(\Psi(1), \Psi(2), \dots, \Psi(z))$  is a permutation of  $(1, 2, \dots, z)$  subject to  $\aleph_{\Psi(\beta-1)} \geq \aleph_{\Psi(\beta)}$  for all  $\beta$ . Also, the weights  $(\aleph_1, \aleph_2, \aleph_3, \dots, \aleph_z)$  of  $\mathcal{H}_\beta$  having  $\aleph_\beta \geq 0$  and  $\sum_{\beta=1}^z \aleph_\beta = 1$ .

**Theorem 6.** Let  $\mathcal{H}_\beta = \langle \mathfrak{a}_\beta, \mathfrak{a}_\beta \rangle$  ( $\beta = 1, \dots, z$ ) be a group of IFNs and the corresponding weight vectors is  $\aleph = (\aleph_1, \aleph_2, \dots, \aleph_z)^T$  such that  $\aleph_i \in [0, 1]$  with the condition  $\sum_{i=1}^n \aleph_i = 1$ . Then, the aggregated value obtained from Equation (23) is still an IFN, and

$$\begin{aligned} IFYOWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) &= \sum_{\beta=1}^z \aleph_\beta \mathcal{H}_{\Psi(\beta)} \\ &= \left\langle \min \left( 1, \left( \sum_{\beta=1}^z \aleph_\beta \mathfrak{a}_{\Psi(\beta)}^{\aleph_\beta} \right)^{\frac{1}{\aleph_\beta}} \right), 1 - \min \left( 1, \left( \sum_{\beta=1}^z \aleph_\beta (1 - \mathfrak{a}_{\Psi(\beta)})^{\aleph_\beta} \right)^{\frac{1}{\aleph_\beta}} \right) \right\rangle. \end{aligned} \quad (24)$$

**Remark 1.** IFYOWA operators satisfy the properties 3, 4 and 5.

**Definition 11.** Let  $\mathcal{H}_i = \langle \mathfrak{a}_i, \mathfrak{a}_i \rangle$  ( $i = 1, \dots, z$ ) be a group of IFNs. Then, the IF Yager hybrid weighted averaging (IFYHWA) operator is explained as follows:

$$IFYHWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) = \sum_{\beta=1}^z \Theta_\beta \mathcal{H}'_{\Psi(\beta)}. \quad (25)$$

Where, the weights  $(\aleph_1, \aleph_2, \aleph_3, \dots, \aleph_z)$  of  $\mathcal{H}_i$  having  $\aleph_\beta \geq 0$  and  $\sum_{\beta=1}^z \aleph_\beta = 1$  and

the  $j^{\text{th}}$  largest weighted value is  $\mathcal{H}'_{\Psi(\beta)}$  ( $\mathcal{H}'_{\Psi(\beta)} = n \aleph_\beta \mathcal{H}_{\Psi(\beta)} \mid \beta = 1, 2, \dots, z$ ), therefore, by the overall order  $(\Psi(1), \Psi(2), \dots, \Psi(z))$ . Also, associated weights  $(\Theta_1, \Theta_2, \dots, \Theta_z)$  of  $\mathcal{H}_\beta$

having  $\Theta_\beta \geq 0$  and  $\sum_{\beta=1}^z \Theta_\beta = 1$ .

**Theorem 7.** Let  $\mathcal{H}_i = \langle \mathfrak{a}_i, \mathfrak{a}_i \rangle$  ( $i = 1, \dots, z$ ) be a group of IFNs with the corresponding

weight vectors is  $\aleph = (\aleph_1, \aleph_2, \dots, \aleph_z)^T$  such that  $\aleph_i \in [0, 1]$  with the condition  $\sum_{\beta=1}^z \aleph_\beta = 1$ ,

and the associated  $(\Theta_1, \Theta_2, \dots, \Theta_\beta)$  of  $\mathcal{H}_\beta$  having  $\Theta_\beta \geq 0$  and  $\sum_{\beta=1}^z \Theta_\beta = 1$ . Then, the accumulated value attained from Equation (25) is still an IFN, and

$$\begin{aligned}
 IFYHWA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) &= \sum_{\beta=1}^z \Theta_{\beta} \mathcal{H}'_{\beta(\beta)} \\
 &= \left\langle \min \left( 1, \left( \sum_{\beta=1}^z \Theta_{\beta} \mathcal{H}'_{\beta(\beta)} \right)^{\frac{1}{\beta}} \right), 1 - \min \left( 1, \left( \sum_{\beta=1}^z \Theta_{\beta} (1 - \mathcal{H}'_{\beta(\beta)})^{\beta} \right)^{\frac{1}{\beta}} \right) \right\rangle. \quad (26)
 \end{aligned}$$

## 4.2 Intuitionistic fuzzy Yager's weighted geometric averaging operator

In this subsection, IF Yager AOs are anticipated and various core properties are investigated.

**Definition 12.** Let  $\mathcal{H}_{\beta} = \langle \mathcal{H}_{\beta}, \mathcal{H}'_{\beta} \rangle$  ( $\beta = 1, \dots, z$ ) be a group of IFNs. Then, The IFYWG operator is explained and symbolized by:

$$IFYWG(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \dots, \mathcal{H}_z) = \prod_{\beta=1}^z \mathcal{H}_{\beta}^{\mathcal{H}_{\beta}}; \quad (27)$$

where  $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z)^T$  is the importance degree such that  $\mathcal{H}_i \in [0, 1]$  with the condition

$$\sum_{\beta=1}^z \mathcal{H}_{\beta} = 1.$$

**Theorem 8.** Let  $\mathcal{H}_{\beta} = \langle \mathcal{H}_{\beta}, \mathcal{H}'_{\beta} \rangle$  ( $\beta = 1, \dots, z$ ) be a group of IFNs and the corresponding

weight vectors is  $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z)^T$  such that  $\mathcal{H}_{\beta} \in [0, 1]$  with the condition  $\sum_{\beta=1}^z \mathcal{H}_{\beta} = 1$ .

Then, the aggregated value obtained from Equation (27) is still an IFN, and

$$\begin{aligned}
 IFYWG(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) &= \prod_{\beta=1}^z \mathcal{H}_{\beta}^{\mathcal{H}_{\beta}} \\
 &= \left\langle 1 - \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{H}_{\beta} (1 - \mathcal{H}_{\beta})^{\beta} \right)^{\frac{1}{\beta}} \right), \min \left( 1, \left( \sum_{\beta=1}^z \mathcal{H}_{\beta} \mathcal{H}'_{\beta} \right)^{\frac{1}{\beta}} \right) \right\rangle. \quad (28)
 \end{aligned}$$

**Proof.** By using mathematical induction on  $z$ , we shall demonstrate Equation (28).

(1) For  $z = 2$ , we have

$$IFYWG(\mathcal{H}_1, \mathcal{H}_2) = \prod_{\beta=1}^2 \mathcal{H}_{\beta}^{\mathcal{H}_{\beta}} = \mathcal{H}_1^{\mathcal{H}_1} \otimes \mathcal{H}_2^{\mathcal{H}_2}.$$

Where,

$$\mathcal{H}_1^{\mathcal{H}_1} = \left\langle 1 - \min \left( 1, \left( \mathcal{H}_1 (1 - \mathcal{H}_1)^{\beta} \right)^{\frac{1}{\beta}} \right), \min \left( 1, \left( \mathcal{H}_1 \mathcal{H}'_1 \right)^{\frac{1}{\beta}} \right) \right\rangle;$$

and

$$\mathcal{H}_2^{\mathcal{H}_2} = \left\langle 1 - \min \left( 1, \left( \mathcal{H}_2 (1 - \mathcal{H}_2)^{\beta} \right)^{\frac{1}{\beta}} \right), \min \left( 1, \left( \mathcal{H}_2 \mathcal{H}'_2 \right)^{\frac{1}{\beta}} \right) \right\rangle.$$

Now,

$$\begin{aligned}
 & \mathcal{H}_1^{\mathcal{A}_1} \otimes \mathcal{H}_2^{\mathcal{A}_2} \\
 &= \left\langle 1 - \min \left( 1, \left( \mathcal{A}_1 (1 - \mathcal{A}_1)^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right), \min \left( 1, \left( \mathcal{A}_1 (\mathcal{A}_1)^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right) \right\rangle \otimes \left\langle 1 - \min \left( 1, \left( \mathcal{A}_2 (1 - \mathcal{A}_2)^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right), \min \left( 1, \left( \mathcal{A}_2 (\mathcal{A}_2)^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right) \right\rangle \\
 &= \left\langle 1 - \min \left( 1, \left( \sum_{\beta=1}^2 \mathcal{A}_{\beta} (1 - \mathcal{A}_{\beta})^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right), \min \left( 1, \left( \sum_{\beta=1}^2 \mathcal{A}_{\beta} \mathcal{A}_{\beta}^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right) \right\rangle.
 \end{aligned}$$

(2) Suppose Equation (28) is true for  $z = e$ , and, we have

$$\begin{aligned}
 IFYWGA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_e) &= \prod_{\beta=1}^e \mathcal{H}_{\beta}^{\mathcal{A}_{\beta}} \\
 &= \left\langle 1 - \min \left( 1, \left( \sum_{\beta=1}^e \mathcal{A}_{\beta} (1 - \mathcal{A}_{\beta})^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right), \min \left( 1, \left( \sum_{\beta=1}^e \mathcal{A}_{\beta} \mathcal{A}_{\beta}^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right) \right\rangle.
 \end{aligned}$$

(3) We must now demonstrate that equation (28), holds for  $z = e + 1$ ,

$$\begin{aligned}
 IFYWG(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{e+1}) &= \prod_{\beta=1}^e \mathcal{H}_{\beta}^{\mathcal{A}_{\beta}} \oplus \mathcal{H}_{e+1}^{\mathcal{A}_{e+1}} \\
 &= \left\langle 1 - \min \left( 1, \left( \sum_{\beta=1}^e \mathcal{A}_{\beta} (1 - \mathcal{A}_{\beta})^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right), \min \left( 1, \left( \sum_{\beta=1}^e \mathcal{A}_{\beta} \mathcal{A}_{\beta}^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right) \right\rangle \otimes \left\langle 1 - \min \left( 1, \left( \mathcal{A}_{e+1} (1 - \mathcal{A}_{e+1})^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right), \min \left( 1, \left( \mathcal{A}_{e+1} \mathcal{A}_{e+1}^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right) \right\rangle \\
 &= \left\langle 1 - \min \left( 1, \left( \sum_{\beta=1}^{e+1} \mathcal{A}_{\beta} (1 - \mathcal{A}_{\beta})^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right), \min \left( 1, \left( \sum_{\beta=1}^{e+1} \mathcal{A}_{\beta} \mathcal{A}_{\beta}^{\mathcal{A}} \right)^{\frac{1}{\mathcal{A}}} \right) \right\rangle.
 \end{aligned}$$

Hence, Equation (37) is true for  $z = e + 1$ . Therefore, Equation (28) is true for any  $z$ .

**Theorem 9.** (Idempotency) Let  $\mathcal{H}_{\beta} = \langle \mathcal{A}_{\beta}, \mathcal{A}_{\beta} \rangle$  ( $\beta = 1, \dots, z$ ) be a group of IFNs. If  $\mathcal{H}_{\beta} = \langle \mathcal{A}_{\beta}, \mathcal{A}_{\beta} \rangle = \langle \mathcal{A}, \mathcal{A} \rangle = \mathcal{H}$ . Then

$$IFYWG(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) = \mathcal{H} \quad (29)$$

**Theorem 10.** (Commutativity): Assume that the  $(\mathcal{H}_1'', \mathcal{H}_2'', \dots, \mathcal{H}_z'')$  is any permutation of  $(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z)$ . If  $\mathcal{A}_{\beta} \leq \mathcal{A}_{\beta}'', \mathcal{A}_{\beta} \leq \mathcal{A}_{\beta}''$ . Then

$$IFYWGA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) \leq IFYWGA(\mathcal{H}_1'', \mathcal{H}_2'', \dots, \mathcal{H}_z'') \quad (30)$$

**Theorem 11.** (Boundedness): Let  $\mathcal{H}_{\beta} = \langle \mathcal{A}_{\beta}, \mathcal{A}_{\beta} \rangle$  ( $\beta = 1, \dots, z$ ) be a group of IFNs. If  $\mathcal{H}_{\beta}^- = \langle \min \mathcal{A}_{\beta}, \max \mathcal{A}_{\beta} \rangle$  and  $\mathcal{H}_{\beta}^+ = \langle \max \mathcal{A}_{\beta}, \min \mathcal{A}_{\beta} \rangle$ . Then

$$\mathcal{H}_{\beta}^- \leq IFYWGA(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) \leq \mathcal{H}_{\beta}^+. \quad (31)$$

**Definition 13.** Let  $\mathcal{H}_{\beta} = \langle \mathcal{A}_{\beta}, \mathcal{A}_{\beta} \rangle$  ( $\beta = 1, \dots, z$ ) be a group of IFNs. Then, the IF Yager ordered weighted geometric (YIFOWG) operator is explained as follows:

$$IFYOWG(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) = \prod_{\beta=1}^z (\mathcal{H}_{\beta})^{\mathcal{A}_{\beta}}. \quad (32)$$

Where,  $\Psi(\beta)$  express the ordered and  $(\Psi(1), \Psi(2), \dots, \Psi(z))$  is a permutation of  $(1, 2, \dots, z)$  subject to  $\aleph_{\Psi(\beta-1)} \geq \aleph_{\Psi(\beta)}$  for all  $\beta$ . Also the weights  $(\aleph_1, \aleph_2, \aleph_3, \dots, \aleph_z)$  of  $\mathcal{H}_\beta$  having  $\aleph_\beta \geq 0$  and  $\sum_{\beta=1}^z \aleph_\beta = 1$ .

**Theorem 12.** Let  $\mathcal{H}_\beta = \langle \mathfrak{a}_\beta, \mathfrak{a}_\beta \rangle$  ( $\beta = 1, \dots, z$ ) be a group of IFNs and the corresponding weight vectors is  $\aleph = (\aleph_1, \aleph_2, \dots, \aleph_z)^T$  such that  $\aleph_\beta \in [0, 1]$  with the condition  $\sum_{\beta=1}^z \aleph_\beta = 1$ . Then, the aggregated value obtained from Equation (32) is still an IFN, and

$$\begin{aligned} IFYOWG(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) &= \prod_{\beta=1}^z (\mathcal{H}_{\Psi(\beta)})^{\aleph_\beta} \\ &= \left\langle 1 - \min \left( 1, \left( \sum_{\beta=1}^z \aleph_\beta (1 - \mathfrak{a}_{\Psi(\beta)})^{\aleph_\beta} \right)^{\frac{1}{\aleph_\beta}} \right), \min \left( 1, \left( \sum_{\beta=1}^z \aleph_\beta \mathfrak{a}_{\Psi(\beta)}^{\aleph_\beta} \right)^{\frac{1}{\aleph_\beta}} \right) \right\rangle. \end{aligned} \quad (33)$$

**Remark 1.** IFYOWG operators satisfy the properties 9, 10 and 11.

**Definition 14.** Let  $\mathcal{H}_i = \langle \mathfrak{a}_i, \mathfrak{a}_i \rangle$  ( $i = 1, \dots, z$ ) be a group of IFNs. Then, the IF Yager hybrid weighted geometric (IFYHWG) operator is explained as follows:

$$IFYHWG(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) = \prod_{\beta=1}^z (\mathcal{H}'_{\Psi(\beta)})^{\Theta_\beta}. \quad (34)$$

Where, the weights  $(\aleph_1, \aleph_2, \aleph_3, \dots, \aleph_z)$  of  $\mathcal{H}_\beta$  having  $\aleph_\beta \geq 0$  and  $\sum_{\beta=1}^z \aleph_\beta = 1$  and the  $j^{\text{th}}$  largest weighted value is  $\mathcal{H}'_{\Psi(\beta)} (\mathcal{H}'_{\Psi(\beta)} = z \aleph_\beta \mathcal{H}_{\Psi(\beta)} \mid \beta = 1, 2, \dots, z)$ , therefore, by the overall order  $(\Psi(1), \Psi(2), \dots, \Psi(z))$ . Also, associated weights  $(\Theta_1, \Theta_2, \dots, \Theta_z)$  of  $\mathcal{H}_\beta$  having  $\Theta_\beta \geq 0$  and  $\sum_{\beta=1}^z \Theta_\beta = 1$ .

**Theorem 13.** Let  $\mathcal{H}_\beta = \langle \mathfrak{a}_\beta, \mathfrak{a}_\beta \rangle$  ( $\beta = 1, \dots, z$ ) be a group of IFNs with the corresponding weight vectors is  $\aleph = (\aleph_1, \aleph_2, \dots, \aleph_z)^T$  such that  $\aleph_\beta \in [0, 1]$  with the condition  $\sum_{\beta=1}^z \aleph_\beta = 1$ , and the associated  $(\Theta_1, \Theta_2, \dots, \Theta_z)$  of  $\mathcal{H}_\beta$  having  $\Theta_\beta \geq 0$  and  $\sum_{\beta=1}^z \Theta_\beta = 1$ . Then, the accumulated value attained from Equation (34) is still an IFN, and

$$IFYHGW(\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_z) = \prod_{\beta=1}^z (\mathcal{H}'_{\beta(\beta)})^{\Theta_{\beta}} \\ = \left\langle 1 - \min \left( 1, \left( \sum_{\beta=1}^z \Theta_{\beta} (1 - \mathcal{H}'_{\beta(\beta)})^{\frac{1}{\beta}} \right)^{\frac{1}{\beta}} \right), \min \left( 1, \left( \sum_{\beta=1}^z \Theta_{\beta} \mathcal{H}'_{\beta(\beta)}^{\frac{1}{\beta}} \right)^{\frac{1}{\beta}} \right) \right\rangle. \quad (35)$$

## 5. Intuitionistic Fuzzy MAGDM Approach Based on Enhanced Fuzzy TOPSIS

This section enclosed the applications of these operators in the MAGDM.

### 5.1 Intuitionistic Fuzzy MAGDM problem

In this sub-section, a model is initiated to handle MAGDM problems under IF information. The MAGDM problems can be determined by means of a decision matrix, in which the rows signify alternatives, and the columns signify a group of attributes. Thus for the decision matrix  $\Xi_{n \times m}$  affect a group of  $n$  alternatives  $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots, \mathcal{A}_n\}$  and  $m$  attributes  $\{\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2, \bar{\mathcal{A}}_3, \dots, \bar{\mathcal{A}}_m\}$ . The importance degrees of decision makers and that of attributes are unknown and are indicated by  $q = \{q_1, q_2, \dots, q_z\}$  and  $u = \{u_1, u_2, u_3, \dots, u_m\}$  respectively, subject to the constraints  $q_g, u_z \in [0, 1], (g = 1, 2, \dots, n, z = 1, 2, \dots, m)$  with  $\sum_{g=1}^n q_g = 1, \sum_{z=1}^m u_z = 1$ . Assumed that the IF decision matrix is signified by  $\Xi^k = [\mathcal{H}_{ij}^k]_{n \times m} = \langle \mathcal{H}_{ij}^k, \mathcal{H}_{ij}^k \rangle$ , where  $\mathcal{H}_{ij}^k$  characterizes the degree of the alternative  $\mathcal{A}_i$  indulges the attribute  $\bar{\mathcal{A}}_j$  judged by the decision makers (DMs), and  $\mathcal{H}_{ij}^k$  epitomizes the degree of the alternative  $\mathcal{A}_i$  doesn't indulges the attribute/criteria  $\bar{\mathcal{A}}_j$  judged by the DMs.

$$\Xi_{n \times m}^k = \begin{matrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \vdots \\ \mathcal{A}_n \end{matrix} \begin{bmatrix} \bar{\mathcal{A}}_1 & \bar{\mathcal{A}}_2 & \cdots & \bar{\mathcal{A}}_m \\ \langle \mathcal{H}_{11}^k, \mathcal{H}_{11}^k \rangle & \langle \mathcal{H}_{12}^k, \mathcal{H}_{12}^k \rangle & \cdots & \langle \mathcal{H}_{1m}^k, \mathcal{H}_{1m}^k \rangle \\ \langle \mathcal{H}_{21}^k, \mathcal{H}_{21}^k \rangle & \langle \mathcal{H}_{22}^k, \mathcal{H}_{22}^k \rangle & \cdots & \langle \mathcal{H}_{2m}^k, \mathcal{H}_{2m}^k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathcal{H}_{n1}^k, \mathcal{H}_{n1}^k \rangle & \langle \mathcal{H}_{n2}^k, \mathcal{H}_{n2}^k \rangle & \cdots & \langle \mathcal{H}_{nm}^k, \mathcal{H}_{nm}^k \rangle \end{bmatrix}; \quad (36)$$

Note that none of the information on the weights of the DMs and criteria are known in the context of decision-making.

**Table 1. Assessments of green suppliers companies in view of linguistic terms**

| Linguistic term          | Abbreviation | IFN                         |
|--------------------------|--------------|-----------------------------|
| Extremely significance   | ES           | $\langle 0.9, 0.1 \rangle$  |
| Very significance        | VS           | $\langle 0.8, 0.15 \rangle$ |
| Significant              | S            | $\langle 0.7, 0.25 \rangle$ |
| Moderately               | M            | $\langle 0.5, 0.45 \rangle$ |
| Insignificance           | IS           | $\langle 0.4, 0.55 \rangle$ |
| Very insignificance      | VIS          | $\langle 0.2, 0.75 \rangle$ |
| Extremely insignificance | EIS          | $\langle 0.1, 0.9 \rangle$  |

## 5.2 IF- Enhanced TOPSIS Method Based on IFYWA Operator

The IF-Enhanced TOPSIS method contains three main portions. In the first portion, utilizing the suggested aggregation operators the importance degrees of the DMs are calculated. The second section describes how to calculate the weights of the criterion using the current IFN entropy measured and the proposed recommended aggregation operator. The third part is a classification method using positive ideal solutions (POIS) and negative ideal solutions (NEIS) based on how similar they are to ideal solutions. The following procedures are provided to solve the IF MAGDM Problems utilizing the IF-Enhanced TOPSIS-based method.

**Step 1.** The optimal group decision ideal solution (GDES) should be calculated by taking the average of all the opinions of each individual DMs, since it is closer to the GDES than any one DM's opinions. So, in this stage, instead of taking average of all opinions, we compute the GDES by using the IFYWA operator values of alternatives that meet the criteria provided by decision makers and considering the same weighting of the DM's values as follows.

$$GDES = \begin{pmatrix} ES_{11} & \dots & ES_{1n} \\ \vdots & \ddots & \vdots \\ ES_{m1} & \dots & ES_{mn} \end{pmatrix} \quad (37)$$

Where  $q_i = \frac{1}{k}$  and

$$ES = \sum_{i=1}^k \frac{1}{k} \mathcal{H}_S = \left\langle \min \left( 1, \left( \sum_{i=1}^k \frac{1}{k} \alpha_i^{\rho} \right)^{\frac{1}{\rho}} \right), 1 - \min \left( 1, \left( \sum_{i=1}^k \frac{1}{k} (1 - \alpha_i)^{\rho} \right)^{\frac{1}{\rho}} \right) \right\rangle. \quad (38)$$

**Step 2.** Following are the results of computing the group right decision (GRGID) and group left ideal decision matrices (GLTID).

$$GRGID = \begin{pmatrix} RGID_{11} & RGID_{12} & \dots & RGID_{1n} \\ RGID_{21} & RGID_{22} & \dots & RGID_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ RGID_{m1} & RGID_{m2} & \dots & RGID_{mn} \end{pmatrix} \quad (39)$$

Where ,

$$RGID_{ij} = \left\{ (\mathcal{H}_{ij}^{(k)}) : \max_i \overline{\overline{SRE}}(\mathcal{H}_{ij}) \right\},$$

And

$$GLTID = \begin{pmatrix} LTID_{11} & LTID_{12} & \dots & LTID_{1n} \\ LTID_{21} & LTID_{22} & \dots & LTID_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ LTID_{m1} & LTID_{m2} & \dots & LTID_{mn} \end{pmatrix} \quad (40)$$

$$\text{Where } LTID_{ij} = \left\{ (\mathcal{H}_{ij}^{(k)}) : \min_i \overline{\overline{SRE}}(\mathcal{H}_{ij}) \right\}$$

**Step 3.** This stage involves computing the decision matrices distance from GDES, GRGID, and GLTID using Definition 5. The distances are represented symbolically by the abbreviations DGDES, DGRGID, and DGLTID, respectively. Where

$$DGDES_i^k = \frac{1}{2n} \sum_{j=1}^n \left( |\mathcal{A}_{\mathcal{H}_{ij}^k} - \mathcal{A}_{ES_{ij}}| + |\mathcal{A}_{\mathcal{H}_{ij}^k} - \mathcal{A}_{ES_{ij}}| \right) \quad (41)$$

$$GRGID_i^k = \frac{1}{2n} \sum_{j=1}^n \left( |\mathcal{A}_{\mathcal{H}_{ij}^k} - \mathcal{A}_{RGID_{ij}}| + |\mathcal{A}_{\mathcal{H}_{ij}^k} - \mathcal{A}_{RGID_{ij}}| \right) \quad (42)$$

$$GLTID_i^k = \frac{1}{2n} \sum_{j=1}^n \left( |\mathcal{A}_{\mathcal{H}_{ij}^k} - \mathcal{A}_{LTID_{ij}}| + |\mathcal{A}_{\mathcal{H}_{ij}^k} - \mathcal{A}_{LTID_{ij}}| \right) \quad (43)$$

For  $i = 1, 2, 3, \dots, m$  and  $k = 1, 2, 3, \dots, e$ .

**Step 4.** In this stage, we compute the closeness indices (CINs) in accordance with the model put out by Yue [38].

$$CIN^{(k)} = \frac{\sum_{i=1}^m DRGID_i^{(k)} + \sum_{i=1}^m DLTID_i^{(k)}}{\sum_{i=1}^m DGDES_i^{(k)} + \sum_{i=1}^m DRGID_i^{(k)} + \sum_{i=1}^m DLTID_i^{(k)}} \quad (44)$$

For  $k = 1, 2, 3, \dots, e$ .

**Step 5.** In this step,  $DMs$ , weights are calculated as follows:

$$q^{(k)} = \frac{CIN^{(k)}}{\sum_{k=1}^e CIN^{(k)}} \quad (45)$$

**Step 6.** Using the suggested IF entropy measure in Definition 6 to generate the attribute weights, the revised group decision (RGDIS) has been determined as follows:

$$RES_{ij} = \sum_{k=1}^e q^{(k)} \mathcal{H}_{ij}^{(k)} = \left\langle \min \left( 1, \left( \sum_{i=1}^n q^{(k)} \mathcal{A}_{\mathcal{H}_{ij}}^\xi \right)^{\frac{1}{\xi}} \right), 1 - \min \left( 1, \left( \sum_{i=1}^n q^{(k)} (1 - \mathcal{A}_{\mathcal{H}_{ij}})^\xi \right)^{\frac{1}{\xi}} \right) \right\rangle. \quad (46)$$

**Step 7.** Equation (9) is used to calculate the IF entropy measure for each attribute.

$$\overline{E\mathcal{C}}_j = E(RES_{1j}, RES_{2j}, \dots, RES_{mj}), j = 1, 2, 3, \dots, n. \quad (47)$$

**Step 8.** In this stage, the weights of the attributes are calculated as follows:

$$\varphi \mathcal{E}_j = \frac{1 - \overline{\mathcal{E}}_j}{n - \prod_{j=1}^n \overline{\mathcal{E}}_j}, j = 1, 2, 3, \dots, n. \quad (48)$$

**Step 9.** The weighted standardized decision matrices are generated as follows using the attributes weight vector:

$$DM(\mathcal{H})_{ij}^{(k)} = \sum_{k=1}^e \varphi \mathcal{E}_j \mathcal{H}_{ij}^{(k)} = \left\langle \min \left( 1, \left( \varphi \mathcal{E}_j \mathcal{H}_{ij}^{(k)} \right)^{\frac{1}{\xi}} \right), 1 - \min \left( 1, \left( \varphi \mathcal{E}_j (1 - \mathcal{H}_{ij}^{(k)}) \right)^{\frac{1}{\xi}} \right) \right\rangle, \quad (49)$$

For each  $k = 1, 2, 3, \dots, e$ .

**Step 10.** Utilizing the weight normalized decision matrices  $\overline{\mathcal{E}}(\mathcal{H}_{ij}^{(k)})$ , computed the  $POIS^{(k)}$  and  $NEIS^{(k)}$  for each DMs as follows:

$$POIS^{(k)} = \left\{ \overline{\mathcal{E}}(\mathcal{H}_{ij}^{(k)}) : \max_i \overline{SRE}(\mathcal{H}_{ij}^{(k)}) \right\}, j = 1, 2, 3, \dots, n \quad (50)$$

$$NEIS^{(k)} = \left\{ \overline{\mathcal{E}}(\mathcal{H}_{ij}^{(k)}) : \min_i \overline{SRE}(\mathcal{H}_{ij}^{(k)}) \right\}, j = 1, 2, 3, \dots, n \quad (51)$$

**Step 11.** Calculate the WGDM by using Definition 5 from  $\overline{\mathcal{E}}(\mathcal{H}_{ij}^{(k)})$  to  $PIS^{(k)}$  and  $NIS^{(k)}$  are designated and demarcated as follows:

$$DIS_i^{+k} = \frac{1}{2n} \sum_{j=1}^n \varphi \mathcal{E}_j \left( \left| \mathcal{A}_{DM(\mathcal{H}_{ij}^{(k)})} - \mathcal{A}_{POIS_j^{(k)}} \right| + \left| \mathcal{A}_{\mathcal{H}_{ij}^{(k)}} - \mathcal{A}_{POIS_j^{(k)}} \right| \right) \quad (52)$$

$$NEIS_i^{+k} = \frac{1}{2n} \sum_{j=1}^n \varphi \mathcal{E}_j \left( \left| \mathcal{A}_{DM(\mathcal{H}_{ij}^{(k)})} - \mathcal{A}_{NEIS_j^{(k)}} \right| + \left| \mathcal{A}_{\mathcal{H}_{ij}^{(k)}} - \mathcal{A}_{NEIS_j^{(k)}} \right| \right) \quad (53)$$

For each  $i = 1, 2, 3, \dots, m$ .

**Step 12.** The following formula is used to calculate each DM's revised closeness indices (RCIs):

$$RCI_i^{(k)} = \frac{DIS_i^{-k}}{DIS_i^{+(k)} + DIS_i^{-k}} \quad (54)$$

**Step 13.** Utilizing the DMs weights, compute the final revised closeness indices (FRCINs) as follows:

$$FRCIN_i = \sum_{k=1}^e q^{(k)} RCI_i^{(k)} \quad (55)$$

Sort the computed FRCIN values in ascending order; the option with the highest value is the best option.

### 5.3. IF- Enhanced TOPSIS Method Based on IFYWG Operator

**Step 1.** The optimal group decision ideal solution (GDES) should be calculated by taking the average of all the opinions of each individual DMs, since it is closer to the GDES than any one DM's opinions. So, in this stage, instead of taking average of all opinions, we compute

the GDES by using the IFYWG operator values of alternatives that meet the criteria provided by decision makers and considering the same weighting of the DM's values as follows.

$$GDES = \begin{pmatrix} ES_{11} & \dots & ES_{1n} \\ \vdots & \ddots & \vdots \\ ES_{m1} & \dots & ES_{mn} \end{pmatrix} \quad (56)$$

Where  $q_i = \frac{1}{k}$  and

$$ES = \prod_{i=1}^k \mathcal{H}_i^{\frac{1}{k}} = \left\langle 1 - \min \left( 1, \left( \sum_{i=1}^k \frac{1}{k} (1 - \mathfrak{a}_i)^{\rho} \right)^{\frac{1}{\rho}} \right), \min \left( 1, \left( \sum_{i=1}^k \frac{1}{k} \mathfrak{a}_i^{\rho} \right)^{\frac{1}{\rho}} \right) \right\rangle. \quad (57)$$

**Step 2.** Figured out the group right decision (*GRGID*) and group left ideal decision (*GLTID*) as follows:

$$GRGID = \begin{pmatrix} RGID_{11} & RGID_{12} & \dots & RGID_{1n} \\ RGID_{21} & RGID_{22} & \dots & RGID_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ RGID_{m1} & RGID_{m2} & \dots & RGID_{mn} \end{pmatrix} \quad (58)$$

Where ,

$$RGID_{ij} = \left\{ \left( \mathcal{H}_{ij}^{(k)} \right) : \max_i \overline{SRE} \left( \mathcal{H}_{ij} \right) \right\},$$

And

$$GLTID = \begin{pmatrix} LTID_{11} & LTID_{12} & \dots & LTID_{1n} \\ LTID_{21} & LTID_{22} & \dots & LTID_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ LTID_{m1} & LTID_{m2} & \dots & LTID_{mn} \end{pmatrix} \quad (59)$$

$$\text{Where } LTID_{ij} = \left\{ \left( \mathcal{H}_{ij}^{(k)} \right) : \min_i \overline{SRE} \left( \mathcal{H}_{ij} \right) \right\}$$

**Step 3.** This stage involves computing the decision matrices distance from GDES, GRGID, and GLTID using Definition 5. The distances are represented symbolically by the abbreviations DGDES, DGRGID, and DGLTID, respectively. Where

$$DGDES_i^k = \frac{1}{2n} \sum_{j=1}^n \left( \left| \mathfrak{a}_{\mathcal{H}_{ij}^k} - \mathfrak{a}_{ES_{ij}} \right| + \left| \mathfrak{a}_{\mathcal{H}_{ij}^k} - \mathfrak{a}_{ES_{ij}} \right| \right) \quad (60)$$

$$GRGID_i^k = \frac{1}{2n} \sum_{j=1}^n \left( \left| \mathfrak{a}_{\mathcal{H}_{ij}^k} - \mathfrak{a}_{RGID_{ij}} \right| + \left| \mathfrak{a}_{\mathcal{H}_{ij}^k} - \mathfrak{a}_{RGID_{ij}} \right| \right) \quad (61)$$

$$GLTID_i^k = \frac{1}{2n} \sum_{j=1}^n \left( \left| \mathfrak{a}_{\mathcal{H}_{ij}^k} - \mathfrak{a}_{LTID_{ij}} \right| + \left| \mathfrak{a}_{\mathcal{H}_{ij}^k} - \mathfrak{a}_{LTID_{ij}} \right| \right) \quad (62)$$

For  $i = 1, 2, 3, \dots, m$  and  $k = 1, 2, 3, \dots, e$ .

**Step 4.** In this stage, we compute the closeness indices (CINs) in accordance with the model put out by Yue [38].

$$CIN^{(k)} = \frac{\sum_{i=1}^m DRGID_i^{(k)} + \sum_{i=1}^m DLTID_i^{(k)}}{\sum_{i=1}^m DGDES_i^{(k)} + \sum_{i=1}^m DRGID_i^{(k)} + \sum_{i=1}^m DLTID_i^{(k)}} \quad (63)$$

For  $k = 1, 2, 3, \dots, e$ .

**Step 5.** DMs weights are determined in this stage using the formulas below:

$$q^{(k)} = \frac{CIN^{(k)}}{\sum_{k=1}^e CIN^{(k)}} \quad (64)$$

**Step 6.** The revised group decision (RGDIS) is computed by using the recommended IF entropy measured to establish the attribute weights.

$$RES_{ij} = \prod_{i=1}^k \left( \mathcal{H}_{ij}^{q^{(k)}} \right)^{(k)} = \left\langle 1 - \min \left( 1, \left( \sum_{i=1}^n q^{(k)} (1 - \mathcal{A}_{\mathcal{H}_{ij}})^{IO} \right)^{\frac{1}{IO}} \right), \min \left( 1, \left( \sum_{i=1}^n q^{(k)} \mathcal{A}_{\mathcal{H}_{ij}}^{IO} \right)^{\frac{1}{IO}} \right) \right\rangle. \quad (65)$$

**Step 7.** Equation (9) is used to get the IF entropy value for each attribute.

$$\overline{E\mathcal{C}}_j = E(RES_{1j}, RES_{2j}, \dots, RES_{mj}), j = 1, 2, 3, \dots, n. \quad (66)$$

**Step 8.** Calculating attribute weights is done as follows:

$$\mathcal{U}\mathcal{C}_j = \frac{1 - \overline{E\mathcal{C}}_j}{n - \prod_{j=1} \overline{E\mathcal{C}}_j}, j = 1, 2, 3, \dots, n. \quad (67)$$

**Step 9.** Utilizing the attributes weight vector, the weighted normalized decision matrices are computed as follows:

$$DM(\mathcal{H})_{ij}^{(k)} = \sum_{k=1}^e \mathcal{U}\mathcal{C}_j \mathcal{H}_{ij}^{(k)} = \left\langle \min \left( 1, \left( \mathcal{U}\mathcal{C}_j \mathcal{A}_{\mathcal{H}_{ij}}^{IO} \right)^{\frac{1}{IO}} \right), 1 - \min \left( 1, \left( \mathcal{U}\mathcal{C}_j (1 - \mathcal{A}_{\mathcal{H}_{ij}})^{IO} \right)^{\frac{1}{IO}} \right) \right\rangle, \quad (68)$$

For each  $k = 1, 2, 3, \dots, e$ .

**Step 10.** Utilizing the weight normalized decision matrices  $\overline{\mathcal{E}}(\mathcal{H}_{ij}^{(k)})$ , computed the  $POIS^{(k)}$  and  $NEIS^{(k)}$  for each DMs as follows:

$$POIS^{(k)} = \left\{ \overline{\mathcal{E}}(\mathcal{H}_{ij}^{(k)}) : \max_i \overline{SRE}(\mathcal{H}_{ij}) \right\}, j = 1, 2, 3, \dots, n \quad (69)$$

$$NEIS^{(k)} = \left\{ \overline{\mathcal{E}}(\mathcal{H}_{ij}^{(k)}) : \min_i \overline{SRE}(\mathcal{H}_{ij}) \right\}, j = 1, 2, 3, \dots, n \quad (70)$$

**Step 11.** Figured out the WGDM by using Definition 5 from  $\overline{\mathcal{E}}(\mathcal{H}_{ij})$  to  $PIS^{(k)}$  and  $NIS^{(k)}$  are designated and demarcated as follows:

$$DIS_i^{+k} = \frac{1}{2n} \sum_{j=1}^n \mathcal{U}\mathcal{C}_j \left( \left| \mu_{DM(\mathcal{H}_{ij}^k)} - \mu_{POIS_j^k} \right| + \left| \gamma_{\mathcal{H}_{ij}^k} - \gamma_{POIS_j^k} \right| \right) \quad (71)$$

$$NEIS_i^{+k} = \frac{1}{2n} \sum_{j=1}^n \mathcal{C}\mathcal{E}_j \left( \left| \mu_{DM(\mathcal{X}_j^k)} - \mu_{NEIS_j^k} \right| + \left| \gamma_{\mathcal{X}_j^k} - \gamma_{NEIS_j^k} \right| \right) \quad (72)$$

For each  $i = 1, 2, 3, \dots, m$ .

**Step 12.** The following formula is used to calculate each DM's revised closeness indices (RCIs):

$$RCI_i^{(k)} = \frac{DIS_i^{-(k)}}{DIS_i^{+(k)} + DIS_i^{-(k)}} \quad (73)$$

**Step 13.** Using the DMs weights, the following formula is used to construct the final revised closeness indices (FRCINs):

$$FRCIN_i = \sum_{k=1}^e q^{(k)} RCI_i^{(k)} \quad (74)$$

The option with the highest value is our best choice when the calculated FRCINs values are ranked in descending order.

## 6. Illustrative example

Many governmental and non-governmental organizations have recently concentrated their efforts on the promotion of resources that are sustainable. Due to growing environmental concerns, numerous companies have started making green products or choosing green suppliers that maximize corporate performance while reducing pollutants in the environment, emissions, hazardous waste, and energy usage. The decision-makers consider several different providers as they try to choose the best green source of information. The suggested MAGDM approach evaluates the green supplier selection problem.

In this section, the planned MAGDM approach is first demonstrated using a numerical example concerning a company's choice of green suppliers that was taken from [18]. The characteristics and advantages of the suggested approach are then demonstrated by a comparison of the current decision-making technique with the existing decision-making strategies employing intuitionistic fuzzy information.

**Example 1.** In this problem, a company situated in Ankara must choose the best green supplier from a group of four companies/options  $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4)$ . To solve the issue of choosing the best green supplier, a senior executive of this organization put together a group of three specialists/decision makers (DMs)  $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ . These options are assessed using the ten selected criteria, which are (1) Quality  $\overline{\mathcal{A}}_1$ , (2) Cost  $\overline{\mathcal{A}}_2$ , (3) Service and delivery  $\overline{\mathcal{A}}_3$ , (4) Sustainability  $\overline{\mathcal{A}}_4$ , (5) Technology  $\overline{\mathcal{A}}_5$ , (6) Green Manufacturing system  $\overline{\mathcal{A}}_6$ , (7) Green supplier image  $\overline{\mathcal{A}}_7$ , (8) Cooperation  $\overline{\mathcal{A}}_8$ , (9) Green application  $\overline{\mathcal{A}}_9$ , (10) Environmental management and control  $\overline{\mathcal{A}}_{10}$ . The weights of the criteria and DMs are completely unknown. The three DMs evaluate these companies using the above criteria and provide their evaluation in the form of linguistic terms listed in Tables 2, 3, and 4. To solve this MAGDM problem, firstly, the linguistic data should be converted into intuitionistic fuzzy information using Table 1. The intuitionistic fuzzy decision matrices are listed in Table 5, 6 and 7 respectively.

Table 2. linguistic information provided by DM<sub>1</sub>

| Alternative/<br>Criteria | $\overline{\mathcal{A}}_1$ | $\overline{\mathcal{A}}_2$ | $\overline{\mathcal{A}}_3$ | $\overline{\mathcal{A}}_4$ | $\overline{\mathcal{A}}_5$ | $\overline{\mathcal{A}}_6$ | $\overline{\mathcal{A}}_7$ | $\overline{\mathcal{A}}_8$ | $\overline{\mathcal{A}}_9$ | $\overline{\mathcal{A}}_{10}$ |
|--------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|-------------------------------|
| $\mathcal{C}_1$          | VS                         | VS                         | S                          | S                          | M                          | S                          | M                          | ES                         | S                          | VS                            |
| $\mathcal{C}_2$          | VS                         | ES                         | VS                         | VS                         | VS                         | VS                         | S                          | VS                         | VS                         | ES                            |
| $\mathcal{C}_3$          | S                          | S                          | S                          | M                          | IS                         | VIS                        | VIS                        | VS                         | VS                         | VS                            |
| $\mathcal{C}_4$          | ES                         | VS                         | VS                         | ES                         | VS                         | ES                         | VS                         | ES                         | ES                         | ES                            |

Table 3. Linguistic information provided by DM<sub>2</sub>

| Alternative/<br>Criteria | $\overline{\mathcal{A}}_1$ | $\overline{\mathcal{A}}_2$ | $\overline{\mathcal{A}}_3$ | $\overline{\mathcal{A}}_4$ | $\overline{\mathcal{A}}_5$ | $\overline{\mathcal{A}}_6$ | $\overline{\mathcal{A}}_7$ | $\overline{\mathcal{A}}_8$ | $\overline{\mathcal{A}}_9$ | $\overline{\mathcal{A}}_{10}$ |
|--------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|-------------------------------|
| $\mathcal{C}_1$          | S                          | VS                         | VS                         | M                          | S                          | VS                         | S                          | VS                         | M                          | VS                            |
| $\mathcal{C}_2$          | ES                         | VS                         | VS                         | ES                         | VS                         | VS                         | VS                         | ES                         | S                          | VS                            |
| $\mathcal{C}_3$          | VS                         | S                          | M                          | IS                         | IS                         | M                          | M                          | VS                         | M                          | VS                            |
| $\mathcal{C}_4$          | ES                         | ES                         | VS                         | ES                         | ES                         | VS                         | VS                         | S                          | VS                         | ES                            |

Table 4. Linguistic information provided by DM<sub>3</sub>

| Alternative/<br>Criteria | $\overline{\mathcal{A}}_1$ | $\overline{\mathcal{A}}_2$ | $\overline{\mathcal{A}}_3$ | $\overline{\mathcal{A}}_4$ | $\overline{\mathcal{A}}_5$ | $\overline{\mathcal{A}}_6$ | $\overline{\mathcal{A}}_7$ | $\overline{\mathcal{A}}_8$ | $\overline{\mathcal{A}}_9$ | $\overline{\mathcal{A}}_{10}$ |
|--------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|-------------------------------|
| $\mathcal{C}_1$          | S                          | VS                         | VS                         | S                          | S                          | M                          | M                          | ES                         | VS                         | ES                            |
| $\mathcal{C}_2$          | VS                         | ES                         | VS                         | VS                         | ES                         | VS                         | VS                         | S                          | VS                         | ES                            |
| $\mathcal{C}_3$          | S                          | S                          | IS                         | M                          | VIS                        | IS                         | S                          | ES                         | M                          | VS                            |
| $\mathcal{C}_4$          | ES                         | ES                         | S                          | VS                         | ES                         | VS                         | ES                         | VS                         | ES                         | ES                            |

Table 5.  $\overset{=1}{\Xi}$  IF information

| Alternat<br>ive/<br>Criteria | $\overline{\mathcal{A}}_1$  | $\overline{\mathcal{A}}_2$  | $\overline{\mathcal{A}}_3$  | $\overline{\mathcal{A}}_4$  | $\overline{\mathcal{A}}_5$  | $\overline{\mathcal{A}}_6$  | $\overline{\mathcal{A}}_7$  | $\overline{\mathcal{A}}_8$  | $\overline{\mathcal{A}}_9$  | $\overline{\mathcal{A}}_{10}$ |
|------------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-------------------------------|
| $\mathcal{C}_1$              | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$   |
| $\mathcal{C}_2$              | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$    |
| $\mathcal{C}_3$              | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.4, 0.55 \rangle$ | $\langle 0.2, 0.75 \rangle$ | $\langle 0.2, 0.75 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$   |
| $\mathcal{C}_4$              | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$    |

Step 1. Utilizing formulae (37) and (38), to obtain GDES. Which is listed in Table 8 below.

**Table 6.**  $\Xi^2$  IF information

| Alternati<br>ve/<br>Criteria | $\overline{\mathcal{A}}_1$  | $\overline{\mathcal{A}}_2$  | $\overline{\mathcal{A}}_3$  | $\overline{\mathcal{A}}_4$  | $\overline{\mathcal{A}}_5$  | $\overline{\mathcal{A}}_6$  | $\overline{\mathcal{A}}_7$  | $\overline{\mathcal{A}}_8$  | $\overline{\mathcal{A}}_9$  | $\overline{\mathcal{A}}_{10}$ |
|------------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-------------------------------|
| $\mathcal{CE}_1$             | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.8, 0.15 \rangle$   |
| $\mathcal{CE}_2$             | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$   |
| $\mathcal{CE}_3$             | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.4, 0.55 \rangle$ | $\langle 0.4, 0.55 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.8, 0.15 \rangle$   |
| $\mathcal{CE}_4$             | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$    |

**Step 2.** Utilizing formulae (39) and (40) to calculate GRGID and GLTID. Which are listed in Tables 9 and 10.

**Table 7.**  $\Xi^3$  IF information

| Alternati<br>ve/<br>Criteria | $\overline{\mathcal{A}}_1$  | $\overline{\mathcal{A}}_2$  | $\overline{\mathcal{A}}_3$  | $\overline{\mathcal{A}}_4$  | $\overline{\mathcal{A}}_5$  | $\overline{\mathcal{A}}_6$  | $\overline{\mathcal{A}}_7$  | $\overline{\mathcal{A}}_8$  | $\overline{\mathcal{A}}_9$  | $\overline{\mathcal{A}}_{10}$ |
|------------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-------------------------------|
| $\mathcal{CE}_1$             | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$    |
| $\mathcal{CE}_2$             | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$    |
| $\mathcal{CE}_3$             | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.4, 0.55 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.2, 0.75 \rangle$ | $\langle 0.4, 0.55 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.5, 0.45 \rangle$ | $\langle 0.8, 0.15 \rangle$   |
| $\mathcal{CE}_4$             | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$    |

**Step 3.** Utilizing formulae (41), (42) and (43), to calculate DGDES, DGRGID and DGLTID. Which are listed in Tables 11, 12 and 13.

**Table 8.** GDES matrix

| Alternati<br>ve/<br>Criteria | $\overline{\mathcal{A}}_1$       | $\overline{\mathcal{A}}_2$       | $\overline{\mathcal{A}}_3$       | $\overline{\mathcal{A}}_4$       | $\overline{\mathcal{A}}_5$       | $\overline{\mathcal{A}}_6$       | $\overline{\mathcal{A}}_7$       | $\overline{\mathcal{A}}_8$       | $\overline{\mathcal{A}}_9$       | $\overline{\mathcal{A}}_{10}$    |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{CE}_1$             | $\langle 0.7349, 0.2152 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.7681, 0.1820 \rangle$ | $\langle 0.6403, 0.3102 \rangle$ | $\langle 0.6403, 0.3102 \rangle$ | $\langle 0.6782, 0.2726 \rangle$ | $\langle 0.5745, 0.3762 \rangle$ | $\langle 0.8679, 0.1164 \rangle$ | $\langle 0.6782, 0.2726 \rangle$ | $\langle 0.8347, 0.1330 \rangle$ |
| $\mathcal{CE}_2$             | $\langle 0.8347, 0.1330 \rangle$ | $\langle 0.8679, 0.1164 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.8347, 0.1330 \rangle$ | $\langle 0.8347, 0.1330 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.7681, 0.1820 \rangle$ | $\langle 0.8042, 0.2646 \rangle$ | $\langle 0.7681, 0.1820 \rangle$ | $\langle 0.8679, 0.1164 \rangle$ |
| $\mathcal{CE}_3$             | $\langle 0.7348, 0.2152 \rangle$ | $\langle 0.7000, 0.2500 \rangle$ | $\langle 0.5477, 0.4035 \rangle$ | $\langle 0.4690, 0.4812 \rangle$ | $\langle 0.3464, 0.6052 \rangle$ | $\langle 0.3873, 0.5651 \rangle$ | $\langle 0.5099, 0.4440 \rangle$ | $\langle 0.8347, 0.1330 \rangle$ | $\langle 0.6164, 0.3348 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ |
| $\mathcal{CE}_4$             | $\langle 0.9000, 0.1000 \rangle$ | $\langle 0.8679, 0.1164 \rangle$ | $\langle 0.7681, 0.1820 \rangle$ | $\langle 0.8679, 0.1164 \rangle$ | $\langle 0.8679, 0.1164 \rangle$ | $\langle 0.8347, 0.1330 \rangle$ | $\langle 0.8347, 0.1330 \rangle$ | $\langle 0.8042, 0.1643 \rangle$ | $\langle 0.8679, 0.1164 \rangle$ | $\langle 0.9000, 0.1000 \rangle$ |

**Step 4.** Utilizing formula (44), to calculate the CINs. Which is listed below:

$$CIN_1 = 4.702, CIN_2 = 4.7216, CIN_3 = 4.8170.$$

**Step 5.** Utilizing formula (45), to calculate the weights of the decision makers. Which is given below:

$$\mathcal{E}^1 = 0.3302, \mathcal{E}^2 = 0.3316, \mathcal{E}^3 = 0.3383$$

Table 9. GRGIS matrix

| Alternati<br>ve/<br>Criteria | $\overline{A}_1$            | $\overline{A}_2$            | $\overline{A}_3$            | $\overline{A}_4$            | $\overline{A}_5$            | $\overline{A}_6$            | $\overline{A}_7$            | $\overline{A}_8$           | $\overline{A}_9$            | $\overline{A}_{10}$         |
|------------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|----------------------------|-----------------------------|-----------------------------|
| $\mathcal{CE}_1$             | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.9, 0.1 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  |
| $\mathcal{CE}_2$             | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  |
| $\mathcal{CE}_3$             | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.4, 0.55 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.9, 0.1 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ |
| $\mathcal{CE}_4$             | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$ | $\langle 0.9, 0.1 \rangle$  | $\langle 0.9, 0.1 \rangle$  |

Step 6. Utilizing formula (46), to calculate RGDES. Which is listed in Table 14.

Table 10. GLTID matrix

| Alternati<br>ve/<br>Criteria | $\overline{A}_1$            | $\overline{A}_2$            | $\overline{A}_3$            | $\overline{A}_4$            | $\overline{A}_5$            | $\overline{A}_6$            | $\overline{A}_7$            | $\overline{A}_8$            | $\overline{A}_9$            | $\overline{A}_{10}$         |
|------------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $\mathcal{CE}_1$             | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.8, 0.15 \rangle$ |
| $\mathcal{CE}_2$             | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ |
| $\mathcal{CE}_3$             | $\langle 0.7, 0.25 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.4, 0.55 \rangle$ | $\langle 0.4, 0.55 \rangle$ | $\langle 0.2, 0.75 \rangle$ | $\langle 0.2, 0.75 \rangle$ | $\langle 0.2, 0.75 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.5, 0.45 \rangle$ | $\langle 0.8, 0.15 \rangle$ |
| $\mathcal{CE}_4$             | $\langle 0.9, 0.1 \rangle$  | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.7, 0.25 \rangle$ | $\langle 0.8, 0.15 \rangle$ | $\langle 0.9, 0.1 \rangle$  |

Step 7. Utilizing formula (47), to calculate the IF entropy measure corresponding to each attribute. Which given below:

$\overline{E\mathcal{CE}}_1 = 0.3939, \overline{E\mathcal{CE}}_2 = 0.3754, \overline{E\mathcal{CE}}_3 = 0.5593, \overline{E\mathcal{CE}}_4 = 0.5993, \overline{E\mathcal{CE}}_5 = 0.5295, \overline{E\mathcal{CE}}_6 = 0.5649$   
 $\overline{E\mathcal{CE}}_7 = 0.6673, \overline{E\mathcal{CE}}_8 = 0.3350, \overline{E\mathcal{CE}}_9 = 0.6111, \overline{E\mathcal{CE}}_{10} = 0.2894.$

Step 8. Utilizing formula (48), to calculate weights of the criteria. Which is given below:

$\mathfrak{u}\mathcal{CE}_1 = 0.0606, \mathfrak{u}\mathcal{CE}_2 = 0.0625, \mathfrak{u}\mathcal{CE}_3 = 0.0441, \mathfrak{u}\mathcal{CE}_4 = 0.0401, \mathfrak{u}\mathcal{CE}_5 = 0.0471, \mathfrak{u}\mathcal{CE}_6 = 0.0435,$   
 $\mathfrak{u}\mathcal{CE}_7 = 0.0333, \mathfrak{u}\mathcal{CE}_8 = 0.0665, \mathfrak{u}\mathcal{CE}_9 = 0.0389, \mathfrak{u}\mathcal{CE}_{10} = 0.0711.$

Table 11. DGDES matrix

| DMs/Alternatives   | $\mathcal{CE}_1$ | $\mathcal{CE}_2$ | $\mathcal{CE}_3$ | $\mathcal{CE}_4$ |
|--------------------|------------------|------------------|------------------|------------------|
| $\overline{\Xi}^1$ | 0.2509           | 0.1426           | 0.4886           | 0.1685           |
| $\overline{\Xi}^2$ | 0.3846           | 0.2280           | 0.2496           | 0.1509           |
| $\overline{\Xi}^3$ | 0.3171           | 0.1362           | 0.3647           | 0.1379           |

Step 9. Utilizing formula (49), and the attributes weight vector, to calculate the weighted normalized decision matrices. Which are listed in Tables 15, 16 and 17.

Table 12. DGRGIS matrix

| DMs/Alternatives | $\mathcal{CE}_1$ | $\mathcal{CE}_2$ | $\mathcal{CE}_3$ | $\mathcal{CE}_4$ |
|------------------|------------------|------------------|------------------|------------------|
|------------------|------------------|------------------|------------------|------------------|

|             |        |        |        |        |
|-------------|--------|--------|--------|--------|
| $\bar{E}_1$ | 0.3875 | 0.2000 | 0.4875 | 0.1125 |
| $\bar{E}_2$ | 0.3750 | 0.1625 | 0.4375 | 0.2000 |
| $\bar{E}_3$ | 0.3250 | 0.1625 | 0.5000 | 0.1625 |

**Table 13. DGLTIS matrix**

| DMs/Alternatives | $\mathcal{E}_1$ | $\mathcal{E}_2$ | $\mathcal{E}_3$ | $\mathcal{E}_4$ |
|------------------|-----------------|-----------------|-----------------|-----------------|
| $\bar{E}_1$      | 0.3875          | 0.1750          | 0.4500          | 0.2500          |
| $\bar{E}_2$      | 0.4000          | 0.2125          | 0.5000          | 0.1625          |
| $\bar{E}_3$      | 0.4750          | 0.2125          | 0.4375          | 0.2000          |

**Step 10.** Utilizing formulae (50) and (51), to calculate POIS and NEIS. Which is listed in Tables 18 and 19.

**Table 14. RGDES matrix**

| Alternative/<br>Criteria | $\bar{A}_1$                             | $\bar{A}_2$                             | $\bar{A}_3$                             | $\bar{A}_4$                             | $\bar{A}_5$                             | $\bar{A}_6$                             | $\bar{A}_7$                             | $\bar{A}_8$                             | $\bar{A}_9$                             | $\bar{A}_{10}$                          |
|--------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|
| $\mathcal{E}_1$          | $\langle \frac{0.7345}{0.2156} \rangle$ | $\langle \frac{0.8000}{0.1500} \rangle$ | $\langle \frac{0.7684}{0.1817} \rangle$ | $\langle \frac{0.6406}{0.3099} \rangle$ | $\langle \frac{0.6409}{0.3096} \rangle$ | $\langle \frac{0.6772}{0.2736} \rangle$ | $\langle \frac{0.5741}{0.3765} \rangle$ | $\langle \frac{0.8681}{0.1163} \rangle$ | $\langle \frac{0.6791}{0.2717} \rangle$ | $\langle \frac{0.8352}{0.1328} \rangle$ |
| $\mathcal{E}_2$          | $\langle \frac{0.8345}{0.1331} \rangle$ | $\langle \frac{0.8681}{0.1163} \rangle$ | $\langle \frac{0.8000}{0.1500} \rangle$ | $\langle \frac{0.8345}{0.1331} \rangle$ | $\langle \frac{0.8352}{0.1328} \rangle$ | $\langle \frac{0.8000}{0.1500} \rangle$ | $\langle \frac{0.7684}{0.1817} \rangle$ | $\langle \frac{0.8035}{0.1649} \rangle$ | $\langle \frac{0.7683}{0.1818} \rangle$ | $\langle \frac{0.8681}{0.1163} \rangle$ |
| $\mathcal{E}_3$          | $\langle \frac{0.7347}{0.2154} \rangle$ | $\langle \frac{0.7000}{0.2500} \rangle$ | $\langle \frac{0.5466}{0.4046} \rangle$ | $\langle \frac{0.4692}{0.4810} \rangle$ | $\langle \frac{0.3456}{0.6061} \rangle$ | $\langle \frac{0.3876}{0.5648} \rangle$ | $\langle \frac{0.5117}{0.4421} \rangle$ | $\langle \frac{0.8352}{0.1328} \rangle$ | $\langle \frac{0.6154}{0.3358} \rangle$ | $\langle \frac{0.8}{0.15} \rangle$      |
| $\mathcal{E}_4$          | $\langle \frac{0.9000}{0.1000} \rangle$ | $\langle \frac{0.8683}{0.1162} \rangle$ | $\langle \frac{0.7676}{0.1825} \rangle$ | $\langle \frac{0.8675}{0.1166} \rangle$ | $\langle \frac{0.8683}{0.1162} \rangle$ | $\langle \frac{0.8343}{0.1332} \rangle$ | $\langle \frac{0.8352}{0.1328} \rangle$ | $\langle \frac{0.8040}{0.1643} \rangle$ | $\langle \frac{0.8835}{0.1163} \rangle$ | $\langle 0.9, 0.1 \rangle$              |

**Step 11.** Utilizing formulae (52) and (53), to calculate distances from POIS and NEIS of each alternative. Which are listed in Tables 20 and 21.

Table 15. DM<sup>1</sup> Normalized weighted decision matrix

| Alternati<br>ve/<br>Criteria | $\overline{\mathcal{A}}_1$       | $\overline{\mathcal{A}}_2$       | $\overline{\mathcal{A}}_3$       | $\overline{\mathcal{A}}_4$       | $\overline{\mathcal{A}}_5$       | $\overline{\mathcal{A}}_6$       | $\overline{\mathcal{A}}_7$       | $\overline{\mathcal{A}}_8$       | $\overline{\mathcal{A}}_9$       | $\overline{\mathcal{A}}_{10}$    |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{E}_1$              | $\langle 0.1970, 0.7907 \rangle$ | $\langle 0.1999, 0.7876 \rangle$ | $\langle 0.1469, 0.8426 \rangle$ | $\langle 0.1401, 0.8499 \rangle$ | $\langle 0.1085, 0.8807 \rangle$ | $\langle 0.1460, 0.8436 \rangle$ | $\langle 0.0912, 0.8997 \rangle$ | $\langle 0.2321, 0.7679 \rangle$ | $\langle 0.1380, 0.8521 \rangle$ | $\langle 0.2133, 0.7734 \rangle$ |
| $\mathcal{E}_2$              | $\langle 0.1970, 0.7907 \rangle$ | $\langle 0.2249, 0.7751 \rangle$ | $\langle 0.1679, 0.8216 \rangle$ | $\langle 0.1602, 0.8298 \rangle$ | $\langle 0.1735, 0.8156 \rangle$ | $\langle 0.1669, 0.8227 \rangle$ | $\langle 0.1277, 0.8632 \rangle$ | $\langle 0.2063, 0.7808 \rangle$ | $\langle 0.1578, 0.8324 \rangle$ | $\langle 0.2399, 0.7601 \rangle$ |
| $\mathcal{E}_3$              | $\langle 0.1723, 0.8154 \rangle$ | $\langle 0.1750, 0.8126 \rangle$ | $\langle 0.1469, 0.8426 \rangle$ | $\langle 0.1001, 0.8899 \rangle$ | $\langle 0.0868, 0.9024 \rangle$ | $\langle 0.0417, 0.9479 \rangle$ | $\langle 0.0365, 0.9544 \rangle$ | $\langle 0.2063, 0.7808 \rangle$ | $\langle 0.1578, 0.8324 \rangle$ | $\langle 0.2133, 0.7734 \rangle$ |
| $\mathcal{E}_4$              | $\langle 0.2216, 0.7784 \rangle$ | $\langle 0.1999, 0.7876 \rangle$ | $\langle 0.1679, 0.8216 \rangle$ | $\langle 0.1802, 0.8198 \rangle$ | $\langle 0.1735, 0.8156 \rangle$ | $\langle 0.1877, 0.8123 \rangle$ | $\langle 0.1459, 0.8450 \rangle$ | $\langle 0.2321, 0.7679 \rangle$ | $\langle 0.1775, 0.8225 \rangle$ | $\langle 0.2399, 0.7601 \rangle$ |

Table 16. DM<sup>2</sup> Normalized weighted decision matrix

| Alternati<br>ve/<br>Criteria | $\overline{\mathcal{A}}_1$       | $\overline{\mathcal{A}}_2$       | $\overline{\mathcal{A}}_3$       | $\overline{\mathcal{A}}_4$       | $\overline{\mathcal{A}}_5$       | $\overline{\mathcal{A}}_6$       | $\overline{\mathcal{A}}_7$       | $\overline{\mathcal{A}}_8$       | $\overline{\mathcal{A}}_9$       | $\overline{\mathcal{A}}_{10}$    |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{E}_1$              | $\langle 0.1723, 0.8154 \rangle$ | $\langle 0.1999, 0.7876 \rangle$ | $\langle 0.1679, 0.8216 \rangle$ | $\langle 0.1001, 0.8899 \rangle$ | $\langle 0.1518, 0.8373 \rangle$ | $\langle 0.1669, 0.8227 \rangle$ | $\langle 0.1277, 0.8632 \rangle$ | $\langle 0.2063, 0.7808 \rangle$ | $\langle 0.0986, 0.8915 \rangle$ | $\langle 0.2133, 0.7734 \rangle$ |
| $\mathcal{E}_2$              | $\langle 0.2216, 0.7784 \rangle$ | $\langle 0.1999, 0.7876 \rangle$ | $\langle 0.1679, 0.8216 \rangle$ | $\langle 0.1802, 0.8198 \rangle$ | $\langle 0.1735, 0.8156 \rangle$ | $\langle 0.1669, 0.8227 \rangle$ | $\langle 0.1459, 0.8450 \rangle$ | $\langle 0.2321, 0.7679 \rangle$ | $\langle 0.1380, 0.8521 \rangle$ | $\langle 0.2133, 0.7734 \rangle$ |
| $\mathcal{E}_3$              | $\langle 0.1970, 0.7907 \rangle$ | $\langle 0.1750, 0.8126 \rangle$ | $\langle 0.1050, 0.8845 \rangle$ | $\langle 0.0801, 0.9099 \rangle$ | $\langle 0.0868, 0.9024 \rangle$ | $\langle 0.1043, 0.8853 \rangle$ | $\langle 0.0912, 0.8997 \rangle$ | $\langle 0.2063, 0.7808 \rangle$ | $\langle 0.0986, 0.8915 \rangle$ | $\langle 0.2133, 0.7734 \rangle$ |
| $\mathcal{E}_4$              | $\langle 0.2216, 0.7784 \rangle$ | $\langle 0.2249, 0.7751 \rangle$ | $\langle 0.1679, 0.8216 \rangle$ | $\langle 0.1802, 0.8198 \rangle$ | $\langle 0.1952, 0.8048 \rangle$ | $\langle 0.1669, 0.8227 \rangle$ | $\langle 0.1459, 0.8450 \rangle$ | $\langle 0.1805, 0.8066 \rangle$ | $\langle 0.1578, 0.8324 \rangle$ | $\langle 0.2399, 0.7601 \rangle$ |

Table 17. DM<sup>3</sup> Normalized weighted decision matrix

| Alternati<br>ve/<br>Criteria | $\overline{\mathcal{A}}_1$       | $\overline{\mathcal{A}}_2$       | $\overline{\mathcal{A}}_3$       | $\overline{\mathcal{A}}_4$       | $\overline{\mathcal{A}}_5$       | $\overline{\mathcal{A}}_6$       | $\overline{\mathcal{A}}_7$       | $\overline{\mathcal{A}}_8$       | $\overline{\mathcal{A}}_9$       | $\overline{\mathcal{A}}_{10}$    |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{E}_1$              | $\langle 0.1723, 0.8154 \rangle$ | $\langle 0.1999, 0.7876 \rangle$ | $\langle 0.1679, 0.8216 \rangle$ | $\langle 0.1401, 0.8499 \rangle$ | $\langle 0.1518, 0.8373 \rangle$ | $\langle 0.1043, 0.8853 \rangle$ | $\langle 0.0912, 0.8997 \rangle$ | $\langle 0.2321, 0.7679 \rangle$ | $\langle 0.1578, 0.8324 \rangle$ | $\langle 0.2399, 0.7601 \rangle$ |
| $\mathcal{E}_2$              | $\langle 0.1970, 0.7907 \rangle$ | $\langle 0.2249, 0.7751 \rangle$ | $\langle 0.1679, 0.8216 \rangle$ | $\langle 0.1602, 0.8298 \rangle$ | $\langle 0.1952, 0.8048 \rangle$ | $\langle 0.1669, 0.8227 \rangle$ | $\langle 0.1459, 0.8450 \rangle$ | $\langle 0.1805, 0.8066 \rangle$ | $\langle 0.1578, 0.8324 \rangle$ | $\langle 0.2399, 0.7601 \rangle$ |
| $\mathcal{E}_3$              | $\langle 0.1723, 0.8154 \rangle$ | $\langle 0.1750, 0.8126 \rangle$ | $\langle 0.0840, 0.9055 \rangle$ | $\langle 0.1001, 0.8899 \rangle$ | $\langle 0.0434, 0.9458 \rangle$ | $\langle 0.0834, 0.9061 \rangle$ | $\langle 0.1277, 0.8632 \rangle$ | $\langle 0.2321, 0.7679 \rangle$ | $\langle 0.0986, 0.8915 \rangle$ | $\langle 0.2133, 0.7734 \rangle$ |
| $\mathcal{E}_4$              | $\langle 0.2216, 0.7784 \rangle$ | $\langle 0.2249, 0.7751 \rangle$ | $\langle 0.1469, 0.8426 \rangle$ | $\langle 0.1602, 0.8298 \rangle$ | $\langle 0.1952, 0.8048 \rangle$ | $\langle 0.1669, 0.8227 \rangle$ | $\langle 0.1642, 0.8358 \rangle$ | $\langle 0.2063, 0.7808 \rangle$ | $\langle 0.1775, 0.8225 \rangle$ | $\langle 0.2399, 0.7601 \rangle$ |

Table 18. POIS for each DMs

| DMs/<br>Criteria | $\bar{A}_1$                                      | $\bar{A}_2$                                      | $\bar{A}_3$                                      | $\bar{A}_4$                                      | $\bar{A}_5$                                      | $\bar{A}_6$                                      | $\bar{A}_7$                                      | $\bar{A}_8$                                      | $\bar{A}_9$                                      | $\bar{A}_{10}$                                   |
|------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $\bar{A}_1$      | $\langle 0.2216, \langle 0.7784 \rangle \rangle$ | $\langle 0.2249, \langle 0.7751 \rangle \rangle$ | $\langle 0.1679, \langle 0.8216 \rangle \rangle$ | $\langle 0.1802, \langle 0.8198 \rangle \rangle$ | $\langle 0.1735, \langle 0.8156 \rangle \rangle$ | $\langle 0.1877, \langle 0.8123 \rangle \rangle$ | $\langle 0.1459, \langle 0.8450 \rangle \rangle$ | $\langle 0.2321, \langle 0.7679 \rangle \rangle$ | $\langle 0.1775, \langle 0.8225 \rangle \rangle$ | $\langle 0.2399, \langle 0.7601 \rangle \rangle$ |
| $\bar{A}_2$      | $\langle 0.2216, \langle 0.7784 \rangle \rangle$ | $\langle 0.2249, \langle 0.7751 \rangle \rangle$ | $\langle 0.1679, \langle 0.8216 \rangle \rangle$ | $\langle 0.1802, \langle 0.8198 \rangle \rangle$ | $\langle 0.1952, \langle 0.8048 \rangle \rangle$ | $\langle 0.1669, \langle 0.8227 \rangle \rangle$ | $\langle 0.1459, \langle 0.8450 \rangle \rangle$ | $\langle 0.2321, \langle 0.7679 \rangle \rangle$ | $\langle 0.1578, \langle 0.8324 \rangle \rangle$ | $\langle 0.2399, \langle 0.7601 \rangle \rangle$ |
| $\bar{A}_3$      | $\langle 0.2216, \langle 0.7784 \rangle \rangle$ | $\langle 0.2249, \langle 0.7751 \rangle \rangle$ | $\langle 0.1679, \langle 0.8216 \rangle \rangle$ | $\langle 0.1602, \langle 0.8298 \rangle \rangle$ | $\langle 0.1952, \langle 0.8048 \rangle \rangle$ | $\langle 0.1669, \langle 0.8227 \rangle \rangle$ | $\langle 0.1642, \langle 0.0912 \rangle \rangle$ | $\langle 0.2321, \langle 0.7679 \rangle \rangle$ | $\langle 0.1775, \langle 0.8225 \rangle \rangle$ | $\langle 0.2399, \langle 0.7601 \rangle \rangle$ |

Table 19. NEIS for each DMs

| DMs/<br>Criteria | $\bar{A}_1$                                      | $\bar{A}_2$                                      | $\bar{A}_3$                                      | $\bar{A}_4$                                      | $\bar{A}_5$                                      | $\bar{A}_6$                                      | $\bar{A}_7$                                      | $\bar{A}_8$                                      | $\bar{A}_9$                                      | $\bar{A}_{10}$                                   |
|------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $\bar{A}_1$      | $\langle 0.1723, \langle 0.8154 \rangle \rangle$ | $\langle 0.1750, \langle 0.8126 \rangle \rangle$ | $\langle 0.1469, \langle 0.8426 \rangle \rangle$ | $\langle 0.1001, \langle 0.8899 \rangle \rangle$ | $\langle 0.0868, \langle 0.9024 \rangle \rangle$ | $\langle 0.0417, \langle 0.9479 \rangle \rangle$ | $\langle 0.0365, \langle 0.9544 \rangle \rangle$ | $\langle 0.2063, \langle 0.7808 \rangle \rangle$ | $\langle 0.1380, \langle 0.8521 \rangle \rangle$ | $\langle 0.2399, \langle 0.7601 \rangle \rangle$ |
| $\bar{A}_2$      | $\langle 0.1723, \langle 0.8154 \rangle \rangle$ | $\langle 0.1750, \langle 0.8126 \rangle \rangle$ | $\langle 0.1050, \langle 0.8845 \rangle \rangle$ | $\langle 0.0801, \langle 0.9099 \rangle \rangle$ | $\langle 0.0868, \langle 0.9024 \rangle \rangle$ | $\langle 0.1043, \langle 0.8853 \rangle \rangle$ | $\langle 0.0912, \langle 0.8997 \rangle \rangle$ | $\langle 0.1805, \langle 0.8066 \rangle \rangle$ | $\langle 0.0986, \langle 0.8915 \rangle \rangle$ | $\langle 0.2399, \langle 0.7601 \rangle \rangle$ |
| $\bar{A}_3$      | $\langle 0.1723, \langle 0.8154 \rangle \rangle$ | $\langle 0.1750, \langle 0.8126 \rangle \rangle$ | $\langle 0.0840, \langle 0.9055 \rangle \rangle$ | $\langle 0.1001, \langle 0.8899 \rangle \rangle$ | $\langle 0.0434, \langle 0.9458 \rangle \rangle$ | $\langle 0.0834, \langle 0.9061 \rangle \rangle$ | $\langle 0.0912, \langle 0.8997 \rangle \rangle$ | $\langle 0.1805, \langle 0.8066 \rangle \rangle$ | $\langle 0.0986, \langle 0.8915 \rangle \rangle$ | $\langle 0.2399, \langle 0.7601 \rangle \rangle$ |

Table 20. Distance from POIS

| DMs/Alternatives | $\mathcal{E}_1$ | $\mathcal{E}_2$ | $\mathcal{E}_3$ | $\mathcal{E}_4$ |
|------------------|-----------------|-----------------|-----------------|-----------------|
| $\bar{A}_1$      | 0.1520          | 0.0507          | 0.2870          | 0.0094          |
| $\bar{A}_2$      | 0.1458          | 0.0374          | 0.2695          | 0.0226          |
| $\bar{A}_3$      | 0.3189          | 0.2322          | 0.4794          | 0.2063          |

Table 21. Distance from NEIS

| DMs/Alternatives | $\mathcal{E}_1$ | $\mathcal{E}_2$ | $\mathcal{E}_3$ | $\mathcal{E}_4$ |
|------------------|-----------------|-----------------|-----------------|-----------------|
| $\bar{A}_1$      | 0.1548          | 0.2361          | 0.0199          | 0.2775          |
| $\bar{A}_2$      | 0.1590          | 0.2674          | 0.0352          | 0.2622          |
| $\bar{A}_3$      | 0.3752          | 0.4346          | 0.2164          | 0.4559          |

**Step 12.** Utilizing formula (54), to calculate the revised closeness indices (RCINs) for each DM's. Which is listed in Table 22.

Table 22. RCINs

| DMs/Alternatives | $\mathcal{E}_1$ | $\mathcal{E}_2$ | $\mathcal{E}_3$ | $\mathcal{E}_4$ |
|------------------|-----------------|-----------------|-----------------|-----------------|
| $\overline{A}_1$ | 0.5046          | 0.8231          | 0.0647          | 0.9673          |
| $\overline{A}_2$ | 0.5216          | 0.8774          | 0.1155          | 0.9208          |
| $\overline{A}_3$ | 0.5405          | 0.6518          | 0.3110          | 0.6884          |

**Step 13.** Utilizing formula (55), to calculate the FRCINs. Which is listed below.  
 $FRCIN_1 = 0.5224, FRCIN_2 = 0.7831, FRCIN_3 = 0.1649, FRCIN_4 = 0.8575.$

Hence,  $\mathcal{E}_4$  is best option according to suggested attributes.

Now, we solve the same illustrative Example 3, by utilizing the TOPSIS method based on IFYWG operator. The following steps should be taken.

**Step 1.** Utilizing formulae (56) and (57), to obtain GDES. Which is listed in Table 23 below.

Table 23. GDES matrix

| Alternati<br>ve/<br>Criteria | $\overline{A}_1$                 | $\overline{A}_2$                 | $\overline{A}_3$                 | $\overline{A}_4$                 | $\overline{A}_5$                 | $\overline{A}_6$                 | $\overline{A}_7$                 | $\overline{A}_8$                 | $\overline{A}_9$                 | $\overline{A}_{10}$              |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{E}_1$              | $\langle 0.7292, 0.2217 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.2620, 0.1893 \rangle$ | $\langle 0.6214, 0.3304 \rangle$ | $\langle 0.6214, 0.3304 \rangle$ | $\langle 0.6441, 0.3096 \rangle$ | $\langle 0.5565, 0.3948 \rangle$ | $\langle 0.8586, 0.1190 \rangle$ | $\langle 0.6441, 0.3096 \rangle$ | $\langle 0.8268, 0.1354 \rangle$ |
| $\mathcal{E}_2$              | $\langle 0.8268, 0.1354 \rangle$ | $\langle 0.8586, 0.1190 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.8268, 0.1354 \rangle$ | $\langle 0.8268, 0.1354 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.7620, 0.1893 \rangle$ | $\langle 0.7840, 0.1780 \rangle$ | $\langle 0.7620, 0.1893 \rangle$ | $\langle 0.8586, 0.1190 \rangle$ |
| $\mathcal{E}_3$              | $\langle 0.7292, 0.2217 \rangle$ | $\langle 0.7000, 0.2500 \rangle$ | $\langle 0.5170, 0.4349 \rangle$ | $\langle 0.4646, 0.4856 \rangle$ | $\langle 0.3267, 0.6238 \rangle$ | $\langle 0.3545, 0.5965 \rangle$ | $\langle 0.4285, 0.5252 \rangle$ | $\langle 0.8268, 0.1354 \rangle$ | $\langle 0.5757, 0.3775 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ |
| $\mathcal{E}_4$              | $\langle 0.9000, 0.1000 \rangle$ | $\langle 0.7000, 0.1190 \rangle$ | $\langle 0.7620, 0.1893 \rangle$ | $\langle 0.8586, 0.1190 \rangle$ | $\langle 0.8586, 0.1190 \rangle$ | $\langle 0.3545, 0.1354 \rangle$ | $\langle 0.8268, 0.1354 \rangle$ | $\langle 0.7840, 0.1780 \rangle$ | $\langle 0.8586, 0.1190 \rangle$ | $\langle 0.9000, 0.1000 \rangle$ |

**Step 2.** Utilizing formulae (58) and (59) to calculate GRGID and GLTID. Which are listed in Tables 9 and 10. (The GRID matrix and GLTID matrix remain the same as obtained above.)

**Step 3.** Utilizing formulae (60), (61) and (62), to calculate DGDES, DGRGID and DGLTID. Which are listed in Tables 24, 25 and 26.

Table 24. DGDES matrix

| DMs/Alternatives | $\mathcal{E}_1$ | $\mathcal{E}_2$ | $\mathcal{E}_3$ | $\mathcal{E}_4$ |
|------------------|-----------------|-----------------|-----------------|-----------------|
| $\overline{A}_1$ | 0.2774          | 0.1223          | 0.4745          | 0.3088          |
| $\overline{A}_2$ | 0.3885          | 0.2079          | 0.2698          | 0.2911          |
| $\overline{A}_3$ | 0.3334          | 0.1557          | 0.3773          | 0.2912          |

**Table 25. DGRGID matrix**

| DMs/Alternatives           | $\mathcal{CE}_1$ | $\mathcal{CE}_2$ | $\mathcal{CE}_3$ | $\mathcal{CE}_4$ |
|----------------------------|------------------|------------------|------------------|------------------|
| $\overline{\mathcal{A}}_1$ | 0.3875           | 0.2000           | 0.4875           | 0.1125           |
| $\overline{\mathcal{A}}_2$ | 0.3750           | 0.1625           | 0.4375           | 0.2000           |
| $\overline{\mathcal{A}}_3$ | 0.3250           | 0.1625           | 0.5000           | 0.1625           |

**Table 26. DGLTID matrix**

| DMs/Alternatives           | $\mathcal{CE}_1$ | $\mathcal{CE}_2$ | $\mathcal{CE}_3$ | $\mathcal{CE}_4$ |
|----------------------------|------------------|------------------|------------------|------------------|
| $\overline{\mathcal{A}}_1$ | 0.3875           | 0.1756           | 0.4532           | 0.2531           |
| $\overline{\mathcal{A}}_2$ | 0.4532           | 0.2125           | 0.5432           | 0.1625           |
| $\overline{\mathcal{A}}_3$ | 0.4753           | 0.2125           | 0.4375           | 0.2234           |

**Step 4.** Utilizing formula (63), to calculate the CINs. Which is listed below:

$$CIN_1 = 4.702, CIN_2 = 4.7090, CIN_3 = 4.77721.$$

**Step 5.** Utilizing formula (64), to calculate the weights of the decision makers. Which is given below:

$$\mathcal{E}^1 = 0.3313, \mathcal{E}^2 = 0.3319, \mathcal{E}^3 = 0.3367.$$

**Step 6.** Utilizing formula (65), to calculate RGDES. Which is listed in Table 27.

**Table 27. RGDES matrix**

| Alternati<br>ve/<br>Criteria | $\overline{\mathcal{A}}_1$       | $\overline{\mathcal{A}}_2$       | $\overline{\mathcal{A}}_3$       | $\overline{\mathcal{A}}_4$       | $\overline{\mathcal{A}}_5$       | $\overline{\mathcal{A}}_6$       | $\overline{\mathcal{A}}_7$       | $\overline{\mathcal{A}}_8$       | $\overline{\mathcal{A}}_9$       | $\overline{\mathcal{A}}_{10}$    |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{CE}_1$             | $\langle 0.7290, 0.2219 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.7622, 0.1891 \rangle$ | $\langle 0.6217, 0.3301 \rangle$ | $\langle 0.6218, 0.3300 \rangle$ | $\langle 0.6432, 0.3104 \rangle$ | $\langle 0.5563, 0.3950 \rangle$ | $\langle 0.8587, 0.1190 \rangle$ | $\langle 0.6447, 0.3090 \rangle$ | $\langle 0.8271, 0.1352 \rangle$ |
| $\mathcal{CE}_2$             | $\langle 0.8267, 0.1355 \rangle$ | $\langle 0.8587, 0.1190 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.8267, 0.1355 \rangle$ | $\langle 0.8271, 0.1352 \rangle$ | $\langle 0.8000, 0.1500 \rangle$ | $\langle 0.7622, 0.1891 \rangle$ | $\langle 0.1835, 0.1784 \rangle$ | $\langle 0.7621, 0.1891 \rangle$ | $\langle 0.8587, 0.1190 \rangle$ |
| $\mathcal{CE}_3$             | $\langle 0.7291, 0.2219 \rangle$ | $\langle 0.7000, 0.2500 \rangle$ | $\langle 0.5162, 0.4356 \rangle$ | $\langle 0.4647, 0.4855 \rangle$ | $\langle 0.3260, 0.6245 \rangle$ | $\langle 0.3548, 0.5962 \rangle$ | $\langle 0.4296, 0.5241 \rangle$ | $\langle 0.8271, 0.1352 \rangle$ | $\langle 0.5752, 0.3780 \rangle$ | $\langle 0.8023, 0.1532 \rangle$ |
| $\mathcal{CE}_4$             | $\langle 0.9000, 0.1000 \rangle$ | $\langle 0.8588, 0.1189 \rangle$ | $\langle 0.7616, 0.1897 \rangle$ | $\langle 0.8582, 0.1192 \rangle$ | $\langle 0.8588, 0.1189 \rangle$ | $\langle 0.8266, 0.1355 \rangle$ | $\langle 0.8271, 0.1352 \rangle$ | $\langle 0.7840, 0.1779 \rangle$ | $\langle 0.8587, 0.1190 \rangle$ | $\langle 0.9000, 0.1000 \rangle$ |

**Step 7.** Utilizing formula (66), to calculate the IF entropy measure corresponding to each attribute. Which given below:

$$\overline{E}\mathcal{CE}_1 = 0.4037, \overline{E}\mathcal{CE}_2 = 0.3834, \overline{E}\mathcal{CE}_3 = 0.5828, \overline{E}\mathcal{CE}_4 = 0.6116, \overline{E}\mathcal{CE}_5 = 0.5376, \overline{E}\mathcal{CE}_6 = 0.5698 \\ \overline{E}\mathcal{CE}_7 = 0.6775, \overline{E}\mathcal{CE}_8 = 0.3630, \overline{E}\mathcal{CE}_9 = 0.6823, \overline{E}\mathcal{CE}_{10} = 0.2971.$$

**Step 8.** Utilizing formula (67), to calculate weights of the criteria. Which is given below:

$$u\mathcal{CE}_1 = 0.0596, u\mathcal{CE}_2 = 0.0617, u\mathcal{CE}_3 = 0.0417, u\mathcal{CE}_4 = 0.0385, u\mathcal{CE}_5 = 0.0462, u\mathcal{CE}_6 = 0.0430, \\ u\mathcal{CE}_7 = 0.0323, u\mathcal{CE}_8 = 0.0637, u\mathcal{CE}_9 = 0.0318, u\mathcal{CE}_{10} = 0.0703.$$

**Step 9.** Utilizing formula (68), and the attributes weight vector, to calculate the weighted normalized decision matrices. Which are listed in Tables 28, 29 and 30.

**Table 28. Normalized decision matrix DM<sub>1</sub>**

| Alternati<br>ve/<br>Criteria | $\overline{\overline{A}}_1$      | $\overline{\overline{A}}_2$      | $\overline{\overline{A}}_3$      | $\overline{\overline{A}}_4$      | $\overline{\overline{A}}_5$      | $\overline{\overline{A}}_6$      | $\overline{\overline{A}}_7$      | $\overline{\overline{A}}_8$      | $\overline{\overline{A}}_9$      | $\overline{\overline{A}}_{10}$   |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{CE}_1$             | $\langle 0.1954, 0.7924 \rangle$ | $\langle 0.1987, 0.7889 \rangle$ | $\langle 0.1430, 0.8468 \rangle$ | $\langle 0.1374, 0.8528 \rangle$ | $\langle 0.1075, 0.8817 \rangle$ | $\langle 0.1452, 0.8444 \rangle$ | $\langle 0.0898, 0.9012 \rangle$ | $\langle 0.2272, 0.7728 \rangle$ | $\langle 0.1248, 0.8663 \rangle$ | $\langle 0.2121, 0.7746 \rangle$ |
| $\mathcal{CE}_2$             | $\langle 0.1954, 0.7924 \rangle$ | $\langle 0.2235, 0.7765 \rangle$ | $\langle 0.1634, 0.8264 \rangle$ | $\langle 0.1571, 0.8331 \rangle$ | $\langle 0.1720, 0.8172 \rangle$ | $\langle 0.1659, 0.8237 \rangle$ | $\langle 0.1257, 0.8653 \rangle$ | $\langle 0.2019, 0.7855 \rangle$ | $\langle 0.1426, 0.8485 \rangle$ | $\langle 0.2386, 0.7614 \rangle$ |
| $\mathcal{CE}_3$             | $\langle 0.1709, 0.8168 \rangle$ | $\langle 0.1738, 0.8138 \rangle$ | $\langle 0.1430, 0.8468 \rangle$ | $\langle 0.0982, 0.8920 \rangle$ | $\langle 0.0860, 0.9032 \rangle$ | $\langle 0.0415, 0.9481 \rangle$ | $\langle 0.0359, 0.9551 \rangle$ | $\langle 0.2019, 0.7855 \rangle$ | $\langle 0.1426, 0.8485 \rangle$ | $\langle 0.2121, 0.7714 \rangle$ |
| $\mathcal{CE}_4$             | $\langle 0.2198, 0.7802 \rangle$ | $\langle 0.1987, 0.7889 \rangle$ | $\langle 0.1634, 0.8264 \rangle$ | $\langle 0.1767, 0.8233 \rangle$ | $\langle 0.1720, 0.8172 \rangle$ | $\langle 0.1867, 0.8133 \rangle$ | $\langle 0.1437, 0.8473 \rangle$ | $\langle 0.2272, 0.7728 \rangle$ | $\langle 0.1604, 0.8396 \rangle$ | $\langle 0.2386, 0.7614 \rangle$ |

**Table 29. Normalized decision matrix DM<sub>2</sub>**

| Alternati<br>ve/<br>Criteria | $\overline{\overline{A}}_1$      | $\overline{\overline{A}}_2$      | $\overline{\overline{A}}_3$      | $\overline{\overline{A}}_4$      | $\overline{\overline{A}}_5$      | $\overline{\overline{A}}_6$      | $\overline{\overline{A}}_7$      | $\overline{\overline{A}}_8$      | $\overline{\overline{A}}_9$      | $\overline{\overline{A}}_{10}$   |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{CE}_1$             | $\langle 0.1709, 0.8168 \rangle$ | $\langle 0.1987, 0.7889 \rangle$ | $\langle 0.1634, 0.8264 \rangle$ | $\langle 0.0982, 0.8920 \rangle$ | $\langle 0.1505, 0.8387 \rangle$ | $\langle 0.1659, 0.8237 \rangle$ | $\langle 0.1257, 0.8653 \rangle$ | $\langle 0.2019, 0.7885 \rangle$ | $\langle 0.0891, 0.9020 \rangle$ | $\langle 0.2121, 0.7746 \rangle$ |
| $\mathcal{CE}_2$             | $\langle 0.2198, 0.7802 \rangle$ | $\langle 0.1987, 0.7889 \rangle$ | $\langle 0.1634, 0.8264 \rangle$ | $\langle 0.1767, 0.8233 \rangle$ | $\langle 0.1720, 0.8172 \rangle$ | $\langle 0.1659, 0.8237 \rangle$ | $\langle 0.1437, 0.8473 \rangle$ | $\langle 0.2272, 0.7728 \rangle$ | $\langle 0.1248, 0.8663 \rangle$ | $\langle 0.2121, 0.7746 \rangle$ |
| $\mathcal{CE}_3$             | $\langle 0.1954, 0.7924 \rangle$ | $\langle 0.1738, 0.8138 \rangle$ | $\langle 0.1021, 0.8877 \rangle$ | $\langle 0.0785, 0.9117 \rangle$ | $\langle 0.0860, 0.9032 \rangle$ | $\langle 0.1037, 0.8859 \rangle$ | $\langle 0.0898, 0.9012 \rangle$ | $\langle 0.2019, 0.7855 \rangle$ | $\langle 0.0891, 0.9020 \rangle$ | $\langle 0.2121, 0.7746 \rangle$ |
| $\mathcal{CE}_4$             | $\langle 0.2198, 0.7802 \rangle$ | $\langle 0.2235, 0.7765 \rangle$ | $\langle 0.1634, 0.8264 \rangle$ | $\langle 0.1767, 0.8233 \rangle$ | $\langle 0.1935, 0.8065 \rangle$ | $\langle 0.1659, 0.8237 \rangle$ | $\langle 0.1437, 0.8473 \rangle$ | $\langle 0.1767, 0.8107 \rangle$ | $\langle 0.1426, 0.8485 \rangle$ | $\langle 0.2386, 0.7614 \rangle$ |

**Step 10.** Utilizing formulae (69) and (70), to calculate POIS<sup>k</sup> and NEiS<sup>k</sup>. Which is listed in Tables 31 and 32.

**Table 30. Normalized decision matrix DM<sub>3</sub>**

| Alternati<br>ve/<br>Criteria | $\overline{\overline{A}}_1$      | $\overline{\overline{A}}_2$      | $\overline{\overline{A}}_3$      | $\overline{\overline{A}}_4$      | $\overline{\overline{A}}_5$      | $\overline{\overline{A}}_6$      | $\overline{\overline{A}}_7$      | $\overline{\overline{A}}_8$      | $\overline{\overline{A}}_9$      | $\overline{\overline{A}}_{10}$   |
|------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\mathcal{CE}_1$             | $\langle 0.1709, 0.8168 \rangle$ | $\langle 0.1987, 0.7889 \rangle$ | $\langle 0.1634, 0.8264 \rangle$ | $\langle 0.1374, 0.8528 \rangle$ | $\langle 0.1505, 0.8387 \rangle$ | $\langle 0.1037, 0.8859 \rangle$ | $\langle 0.0898, 0.9012 \rangle$ | $\langle 0.2272, 0.7728 \rangle$ | $\langle 0.1426, 0.8485 \rangle$ | $\langle 0.2386, 0.7614 \rangle$ |
| $\mathcal{CE}_2$             | $\langle 0.1954, 0.7924 \rangle$ | $\langle 0.2235, 0.7765 \rangle$ | $\langle 0.1634, 0.8264 \rangle$ | $\langle 0.1571, 0.8331 \rangle$ | $\langle 0.1935, 0.8065 \rangle$ | $\langle 0.1659, 0.8237 \rangle$ | $\langle 0.1437, 0.8473 \rangle$ | $\langle 0.1767, 0.8107 \rangle$ | $\langle 0.1426, 0.8485 \rangle$ | $\langle 0.2386, 0.7614 \rangle$ |
| $\mathcal{CE}_3$             | $\langle 0.1709, 0.8168 \rangle$ | $\langle 0.1738, 0.8138 \rangle$ | $\langle 0.0817, 0.9081 \rangle$ | $\langle 0.0982, 0.8920 \rangle$ | $\langle 0.0430, 0.9462 \rangle$ | $\langle 0.0830, 0.9067 \rangle$ | $\langle 0.1257, 0.8653 \rangle$ | $\langle 0.2272, 0.7728 \rangle$ | $\langle 0.0891, 0.9020 \rangle$ | $\langle 0.2121, 0.7746 \rangle$ |
| $\mathcal{CE}_4$             | $\langle 0.2198, 0.7802 \rangle$ | $\langle 0.2235, 0.7765 \rangle$ | $\langle 0.1430, 0.8468 \rangle$ | $\langle 0.1571, 0.8331 \rangle$ | $\langle 0.1935, 0.8065 \rangle$ | $\langle 0.1659, 0.8237 \rangle$ | $\langle 0.1616, 0.8384 \rangle$ | $\langle 0.2019, 0.7855 \rangle$ | $\langle 0.1604, 0.8396 \rangle$ | $\langle 0.2386, 0.7614 \rangle$ |

**Table 31. POIS matrix**

| DMs/<br>Criter<br>ia | $\bar{\bar{A}}_1$                                | $\bar{\bar{A}}_2$                                | $\bar{\bar{A}}_3$                                | $\bar{\bar{A}}_4$                                | $\bar{\bar{A}}_5$                                | $\bar{\bar{A}}_6$                                | $\bar{\bar{A}}_7$                                | $\bar{\bar{A}}_8$                                | $\bar{\bar{A}}_9$                                | $\bar{\bar{A}}_{10}$                             |
|----------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $\bar{\bar{I}}^1$    | $\langle 0.2198, \langle 0.7802 \rangle \rangle$ | $\langle 0.2235, \langle 0.7765 \rangle \rangle$ | $\langle 0.1634, \langle 0.8264 \rangle \rangle$ | $\langle 0.1767, \langle 0.8233 \rangle \rangle$ | $\langle 0.1720, \langle 0.8172 \rangle \rangle$ | $\langle 0.1867, \langle 0.8133 \rangle \rangle$ | $\langle 0.1437, \langle 0.8473 \rangle \rangle$ | $\langle 0.2272, \langle 0.7728 \rangle \rangle$ | $\langle 0.1604, \langle 0.8396 \rangle \rangle$ | $\langle 0.2386, \langle 0.7614 \rangle \rangle$ |
| $\bar{\bar{I}}^2$    | $\langle 0.2198, \langle 0.7802 \rangle \rangle$ | $\langle 0.2235, \langle 0.7765 \rangle \rangle$ | $\langle 0.1634, \langle 0.8264 \rangle \rangle$ | $\langle 0.1767, \langle 0.8233 \rangle \rangle$ | $\langle 0.1935, \langle 0.8065 \rangle \rangle$ | $\langle 0.1659, \langle 0.8237 \rangle \rangle$ | $\langle 0.1437, \langle 0.8473 \rangle \rangle$ | $\langle 0.2272, \langle 0.7728 \rangle \rangle$ | $\langle 0.1426, \langle 0.8485 \rangle \rangle$ | $\langle 0.2386, \langle 0.7614 \rangle \rangle$ |
| $\bar{\bar{I}}^3$    | $\langle 0.2198, \langle 0.7802 \rangle \rangle$ | $\langle 0.2235, \langle 0.7765 \rangle \rangle$ | $\langle 0.1634, \langle 0.8264 \rangle \rangle$ | $\langle 0.1571, \langle 0.8331 \rangle \rangle$ | $\langle 0.1935, \langle 0.8065 \rangle \rangle$ | $\langle 0.1659, \langle 0.8237 \rangle \rangle$ | $\langle 0.1616, \langle 0.0898 \rangle \rangle$ | $\langle 0.2272, \langle 0.7728 \rangle \rangle$ | $\langle 0.1604, \langle 0.8396 \rangle \rangle$ | $\langle 0.2386, \langle 0.7614 \rangle \rangle$ |

**Step 11.** Utilizing formulae (71) and (72), to calculate distances from POIS and NEIS of each alternative. Which are listed in Tables 33 and 34.

**Table 32. NEIS matrix**

| DMs/<br>Criter<br>ia | $\bar{\bar{A}}_1$                                | $\bar{\bar{A}}_2$                                | $\bar{\bar{A}}_3$                                | $\bar{\bar{A}}_4$                                | $\bar{\bar{A}}_5$                                | $\bar{\bar{A}}_6$                                | $\bar{\bar{A}}_7$                                | $\bar{\bar{A}}_8$                                | $\bar{\bar{A}}_9$                                | $\bar{\bar{A}}_{10}$                             |
|----------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $\bar{\bar{I}}^1$    | $\langle 0.1709, \langle 0.8168 \rangle \rangle$ | $\langle 0.1738, \langle 0.8138 \rangle \rangle$ | $\langle 0.1430, \langle 0.8468 \rangle \rangle$ | $\langle 0.0982, \langle 0.8920 \rangle \rangle$ | $\langle 0.0860, \langle 0.9032 \rangle \rangle$ | $\langle 0.0415, \langle 0.9481 \rangle \rangle$ | $\langle 0.0359, \langle 0.9551 \rangle \rangle$ | $\langle 0.2019, \langle 0.7855 \rangle \rangle$ | $\langle 0.1248, \langle 0.8663 \rangle \rangle$ | $\langle 0.2386, \langle 0.7614 \rangle \rangle$ |
| $\bar{\bar{I}}^2$    | $\langle 0.1709, \langle 0.8168 \rangle \rangle$ | $\langle 0.1738, \langle 0.8138 \rangle \rangle$ | $\langle 0.1021, \langle 0.8877 \rangle \rangle$ | $\langle 0.0785, \langle 0.9117 \rangle \rangle$ | $\langle 0.0860, \langle 0.9032 \rangle \rangle$ | $\langle 0.1037, \langle 0.8859 \rangle \rangle$ | $\langle 0.0898, \langle 0.9012 \rangle \rangle$ | $\langle 0.1767, \langle 0.8107 \rangle \rangle$ | $\langle 0.0891, \langle 0.9020 \rangle \rangle$ | $\langle 0.2386, \langle 0.7614 \rangle \rangle$ |
| $\bar{\bar{I}}^3$    | $\langle 0.1709, \langle 0.8168 \rangle \rangle$ | $\langle 0.1738, \langle 0.8138 \rangle \rangle$ | $\langle 0.0817, \langle 0.9081 \rangle \rangle$ | $\langle 0.0982, \langle 0.8920 \rangle \rangle$ | $\langle 0.0430, \langle 0.9462 \rangle \rangle$ | $\langle 0.0830, \langle 0.9067 \rangle \rangle$ | $\langle 0.0898, \langle 0.1616 \rangle \rangle$ | $\langle 0.1767, \langle 0.8107 \rangle \rangle$ | $\langle 0.0891, \langle 0.9020 \rangle \rangle$ | $\langle 0.2386, \langle 0.7614 \rangle \rangle$ |

**Step 12.** Utilizing formula (73), to calculate the revised closeness indices (*RCINs*) for each *DM's*. Which is listed in Table 35.

**Step 13.** Utilizing formula (74), to calculate the *FRCINs*. Which is listed below.

$$FRCIN_1 = 0.5239, FRCIN_2 = 0.7837, FRCIN_3 = 0.1652, FRCIN_4 = 0.8568.$$

Hence,  $\bar{\bar{C}}_4$  is best option according to suggested attributes.

**Table 33. Distance from POIS**

| DMs/Alternatives  | $\bar{\bar{C}}_1$ | $\bar{\bar{C}}_2$ | $\bar{\bar{C}}_3$ | $\bar{\bar{C}}_4$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| $\bar{\bar{I}}^1$ | 0.1487            | 0.0494            | 0.2831            | 0.0093            |
| $\bar{\bar{I}}^2$ | 0.1414            | 0.0362            | 0.2634            | 0.0220            |
| $\bar{\bar{I}}^3$ | 0.3179            | 0.2318            | 0.4736            | 0.2068            |

**Table 34. Distance from NEIS**

| DMs/Alternatives                                           | $\mathcal{CE}_1$ | $\mathcal{CE}_2$ | $\mathcal{CE}_3$ | $\mathcal{CE}_4$ |
|------------------------------------------------------------|------------------|------------------|------------------|------------------|
| $\begin{smallmatrix} \equiv \\ \sqcup \end{smallmatrix} 1$ | 0.1532           | 0.2326           | 0.0188           | 0.2727           |
| $\begin{smallmatrix} \equiv \\ \sqcup \end{smallmatrix} 2$ | 0.1567           | 0.2619           | 0.0347           | 0.2562           |
| $\begin{smallmatrix} \equiv \\ \sqcup \end{smallmatrix} 3$ | 0.3707           | 0.4299           | 0.2169           | 0.4504           |

**Table 35. RCINS matrix**

| DMs/Alternatives                                           | $\mathcal{CE}_1$ | $\mathcal{CE}_2$ | $\mathcal{CE}_3$ | $\mathcal{CE}_4$ |
|------------------------------------------------------------|------------------|------------------|------------------|------------------|
| $\begin{smallmatrix} \equiv \\ \sqcup \end{smallmatrix} 1$ | 0.5074           | 0.8247           | 0.0624           | 0.9669           |
| $\begin{smallmatrix} \equiv \\ \sqcup \end{smallmatrix} 2$ | 0.5257           | 0.8785           | 0.1166           | 0.9206           |
| $\begin{smallmatrix} \equiv \\ \sqcup \end{smallmatrix} 3$ | 0.5383           | 0.6497           | 0.3141           | 0.6853           |

## 6.1 Effect of parameter P on Final ranking results

In this subsection, effects of the parameter P on the final ranking results have been investigated, while utilizing IFWA and IFYWG operators.

Figure 1 illustrates how changing values of the parameter  $\mathcal{IO}$  might lead to different score values, but the ranking order does not change. Additionally, it is evident that as the value of the parameter  $\mathcal{IO}$  rises, so do the score values. Like Figure 2, one can see that using the IFYWG operator, different values of the parameter P might result in different score values, but the ranking order stays the same as determined above. Additionally, it is evident that the score values rise in tandem with the parameter P's value. The initial TOPSIS technique, which is utilized to address MAGDM issues, is more adaptable because of the generalized parameter. According to the real requirements of the circumstance, the decision-makers select the parameter's value.

## 6.2 Comparison with different approaches

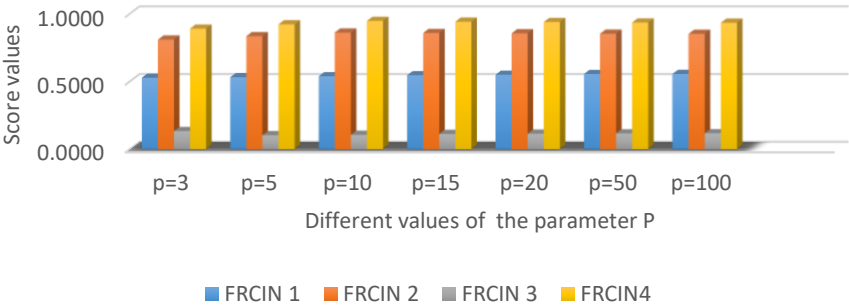
In this subsection the initiated approach is compared with some existing approaches and existing TOPSIS methods for dealing with IF information.

From figure 3, we can see that by solving the same Example 1, with different MAGDM approaches, we can get the same ranking order as obtained from the initiated TOPSIS approach based on the newly initiated IFYWA and IFYWG operators.

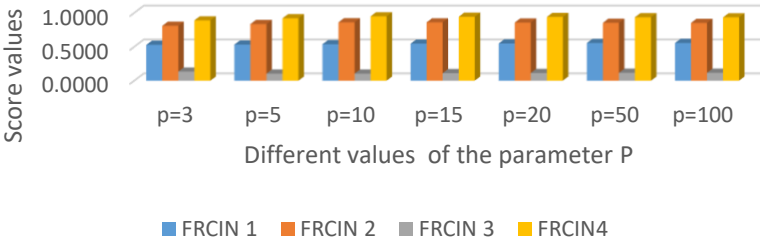
This shows that the initiated approach is valid. The existing IF approaches are based on the IFWA and IFWG operators, while the initiated MAGDM approach is based on TOPSIS and IFY aggregation operators. The importance degrees of both DMs and criterion are completely known in the existing MAGDM approaches. Similarly, the existing MADM/MAGDM approaches based on TOPSIS method, the importance degrees of DMs and the criterion are known in terms of linguistic terms. Therefore, the proposed MAGDM approach has several advantages over the existing approaches based on simple aggregation

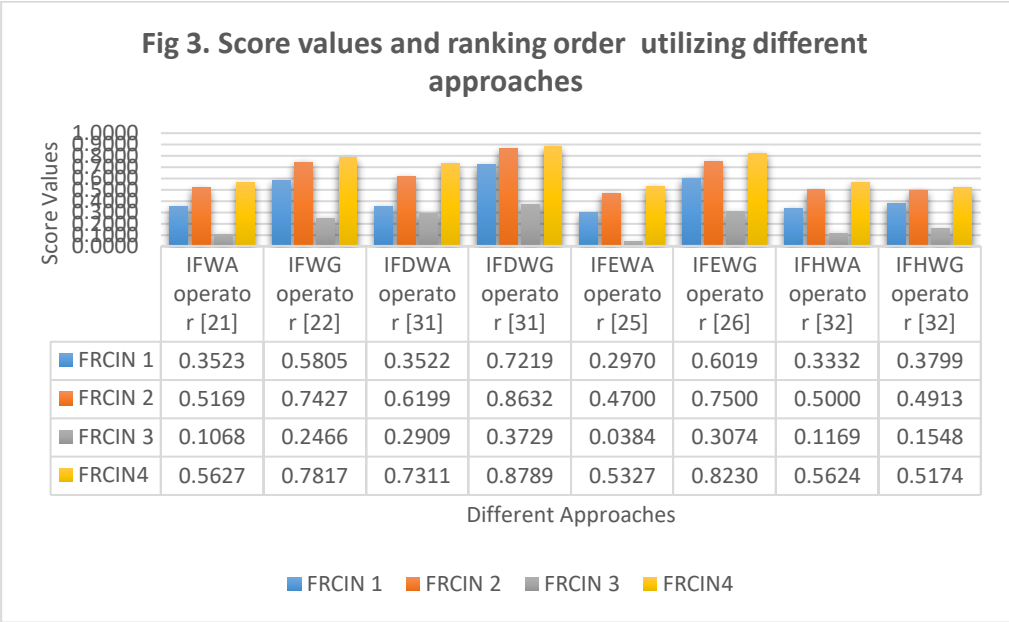
operators or TOPSIS. The developed MAGDM approach is based on TOPSIS method and the TOPSIS method is based on Yager aggregation operators and Yager operational laws. The proposed TOPSIS method is flexible as it consists of generalized parameters, which makes the MAGDM process flexible. The decision makers may select the value of the parameter according to the actual need of the situation. So, due to these characteristics the developed MAGDM approach is more reliable and flexible to be utilized in real MAGDM problems.

**Fig 1. Effect of the parameter P on final ranking orders utilizing IFYWA operators**



**Fig 2. Effect of the parameter P on final ranking results utilizing IFYWG operator**





7. Conclusions

Decision makers feel more at ease choosing to employ IFS over FS, since IFS is a developing and effective generalized idea that has been used as strategic instruments to combat the uncertainties and vagueness data linked with MAGDM difficulties. The MAGDM difficulties in the IF environment are dealt with in this study using an innovative, updated TOPSIS-based decision-making procedure using entirely unknowledgeable information on the decision makers and criteria weights. Firstly, the Yager operational laws are initiated for IFNs have been initiated and various core properties of these operational laws are investigated. Secondly, utilizing these operational laws various weighted aggregation operators and their core characteristics are investigated. Thirdly, an enhanced fuzzy TOPSIS-based method for dealing with MAGDM problems under IF information is initiated. In this MAGDM approach the weights of the DMs and attributes are completely unknown. The weights of DMs and attributes are obtained utilizing the IF entropy measure. Instead of only depending on the personal preferences of experts, entropy based IF TOPSIS employs the entropy procedure to calculate realistic importance of attributes. The extent of incorrectness of the selected data is measured by entropy. Attributes with less variety are specified as less importance, whereas attributes with significant variation across options comprise more details and hence receive a bigger importance. The accuracy, reliability, and resilience of selection are enhanced by entropy weighting in combination with enhanced IF TOPSIS, especially in challenging circumstances, where ambiguity and inaccuracy are inevitable. Lastly, a numerical example related to the selection of suitable green suppliers is provided to show the efficacy and practicality of the initiated MAGDM approach, along with the comparison to some existing MAGDM approaches.

In future, we will try to extend the developed approach for some other fuzzy structures such as, neutrosophic sets, picture fuzzy sets, cubic IFS sets, etc.

**-Ethical and informed consent for data used:** *The authors state that this is their original work, and it is neither submitted nor under consideration in any other journal simultaneously*

**-Consent to Publish:** *Not applicable.*

**-Authors Contributions:** *All authors are Equal.*

**-Funding:** *No funding.*

**-Competing Interests:** *The authors declare that they have no competing interests.*

**Data availability and access:** *The data utilized in this manuscript are hypothetical and artificial, and one can use these data before prior permission by just citing this manuscript.*

## References

- [1] Aggarwal A., Gupta A., Verma R., Kumari R., Ranking of efficient fuzzy portfolios by hybrid MSBM-TOPSIS technique, *Foundations of Computing and Decision Sciences*, 50, 1, 2025, 1–24.
- [2] Ahmadi H.B., Hashemi S., Petrudi S.H., and Wang X., Integrating sustainability into supplier selection with analytical hierarchy process and improved grey relational analysis: a case of telecom industry, *The International Journal of Advanced Manufacturing Technology*, 90, 2017, 2413-2427.
- [3] Ali H., Zhang J., Liu S. Shoaib M., An integrated decision-making approach for global supplier selection and order allocation to create an environmentally friendly supply chain, *Kybernetes*, 52, 8, 2023, 2649–2671.
- [4] Atanassov K. T., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20, 1, 1986, 87–96.
- [5] Burillo P., Bustince H., Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, *Fuzzy Sets and Systems*, 78, 3, 1996, 305-316.
- [6] Büyüközkan G., and Çifçi G., A novel hybrid MCDM approach based on fuzzy DEMATEL, fuzzy ANP and fuzzy TOPSIS to evaluate green suppliers, *Expert Systems with Applications*, 39, 3, 2012, 3000-3011.
- [7] Çakır S., Selecting the best supplier at a steel-producing company under fuzzy environment, *The International Journal of Advanced Manufacturing Technology*, 88, 2017, 1345-1361.
- [8] Chen C.T., Extensions of the TOPSIS for group decision-making under fuzzy environment, *Fuzzy Sets and Systems*, 114, 1, 2000, 1–9.
- [9] Chen S. M., Chen S.W., Fuzzy forecasting based on two-factors second-order fuzzy-trend logical relationship groups and the probabilities of trends of fuzzy logical relationships, *IEEE Transactions on Cybernetics*, 45, 3, 2014, 391-403.

- [10] De S. K., Biswas R., Roy A. R., An application of intuitionistic fuzzy sets in medical diagnosis, *Fuzzy sets and Systems*, 117, 2, 2001, 209-213.
- [11] Ecer F., Multi-criteria decision making for green supplier selection using interval type-2 fuzzy AHP: a case study of a home appliance manufacturer, *Operational Research*, 22, 1, 2022, 199-233.
- [12] Garg H., Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making, *Computers & Industrial Engineering*, 101, 2016, 53–69.
- [13] Garg H., Novel intuitionistic fuzzy decision-making method based on an improved operation laws and its application, *Engineering Applications of Artificial Intelligence*, 60, 2017, 164–174.
- [14] Huang J. Y., Intuitionistic fuzzy Hamacher aggregation operators and their application to multiple attribute decision making, *Journal of Intelligent & Fuzzy Systems*, 27, 1, 2014, 505–513.
- [15] Humphreys P., Huang G., Cadden T., McIvor R., Integrating design metrics within the early supplier selection process, *Journal of Purchasing and Supply Management*, 13, 1, 2007, 42-52.
- [16] Hwang C. L., Yoon K., Methods for multiple attribute decision making, in *Multiple Attribute Decision Making: Methods and Applications—A State-of-the-Art Survey*, *Lecture Notes in Economics and Mathematical Systems*, vol. 186, Berlin, Germany: Springer, pp. 58–191, 1981.
- [17] Jin J., Garg H., Intuitionistic fuzzy three-way ranking-based TOPSIS approach with a novel entropy measure and its application to medical treatment selection, *Advances in Engineering Software*, 180, 2023, Article no.103459.
- [18] Joshi D., Kumar S., Intuitionistic fuzzy entropy and distance measure based TOPSIS method for multi-criteria decision making, *Egyptian Informatics Journal*, 15, 2, 2014, 97-104.
- [19] Li D. F., Multi-attribute decision making method based on generalized OWA operators with intuitionistic fuzzy sets, *Expert Systems with Applications*, 37, 12, 2010, 8673-8678.
- [20] Medić N., Anišić Z., Lalić B., Marjanović U., Brezocnik M., Hybrid fuzzy multi-attribute decision making model for evaluation of advanced digital technologies in manufacturing: Industry 4.0 perspective, *Advances in Production Engineering and Management*, 14, 4, 2019, 483-493.
- [21] Mousakhani S., Nazari-Shirkouhi S., Bozorgi-Amiri A., A novel interval type-2 fuzzy evaluation model-based group decision analysis for green supplier selection problems: A case study of battery industry, *Journal of Cleaner Production*, 168, 2017, 205-218.
- [22] Onat N. C., Gumus S., Kucukvar M., Tatari O., Application of the TOPSIS and intuitionistic fuzzy set approaches for ranking the life cycle sustainability performance

- of alternative vehicle technologies, *Sustainable Production and Consumption*, 6, 2016, 12–25.
- [23] Rahman K., Abdullah S., Jamil M., Khan M. Y., Some generalized intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute group decision making, *International Journal of Fuzzy Systems*, 20, 2018, 1567–1575.
- [24] Rouyendegh B. D., Yildizbasi A., Üstünyer P., Intuitionistic fuzzy TOPSIS method for green supplier selection problem, *Soft Computing*, 24, 2020, 2215–2228.
- [25] Roy M. K., Shivakoti I., Phipon R., Sharma A., A holistic approach to polymeric material selection for laser beam machining using methods of DEA and TOPSIS, *Foundations of Computing and Decision Sciences*, 45, 4, 2020, 339–357.
- [26] Shen F., Ma X., Li Z., Xu Z., Cai D., An extended intuitionistic fuzzy TOPSIS method based on a new distance measure with an application to credit risk evaluation, *Information Sciences*, 428, 2018, 105–119.
- [27] Seikh M. R., Mandal U., Intuitionistic fuzzy Dombi aggregation operators and their application to multiple attribute decision-making, *Granular Computing*, 6, 2021, 473–488.
- [28] Tan A., Shi S., Wu W.Z., Li J., Pedrycz W., Granularity and entropy of intuitionistic fuzzy information and their applications, *IEEE Transactions on Cybernetics*, 52, 1, 2022, 192–204.
- [29] Verma R., Chandra S., Interval-valued intuitionistic fuzzy-analytic hierarchy process for evaluating the impact of security attributes in fog-based internet of things paradigm, *Computer Communications*, 175, 2021, 35–46.
- [30] Wang W., Liu X., Intuitionistic fuzzy geometric aggregation operators based on Einstein operations, *International Journal of Intelligent Systems*, 26, 11, 2011, 1049–1075.
- [31] Wang W., Liu X., Intuitionistic fuzzy information aggregation using Einstein operations, *IEEE Transactions on Fuzzy Systems*, 20, 5, 2012, 923–938.
- [32] Wei G., Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making, *Applied Soft Computing*, 10, 2, 2010, 423–431.
- [33] Xu Z., Intuitionistic fuzzy aggregation operators, *IEEE Transactions on Fuzzy Systems*, 15, 6, 2007, 1179–1187.
- [34] Xu Z., Yager R. R., Some geometric aggregation operators based on intuitionistic fuzzy sets, *International Journal of General Systems*, 35, 4, 2006, 417–433.
- [35] Yager R. R., On a general class of fuzzy connectives, *Fuzzy Sets and Systems*, 4, 1980, 235–242.

- [36] Yang M. S., Hussain Z., Mehboob A., Belief and plausibility measures on intuitionistic fuzzy sets with construction of belief–plausibility TOPSIS, *Complexity*, 2020, 2020, Article ID 8895546.
- [37] Yazdani M., Chatterjee P., Zavadskas E.K., Zolfani S.H., Integrated QFD-MCDM framework for green supplier selection, *Journal of Cleaner Production*, 142, 2017, 3728–3740.
- [38] Yue Z., An extended TOPSIS for determining weights of decision makers with interval numbers, *Knowledge-Based Systems*, 24, 1, 2011, 146–153.
- [39] Zhao X., Wei G., Some intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making, *Knowledge-Based Systems*, 37, 2013, 472–479.
- [40] Zhu Y. J., Li D. F., A new definition and formula of entropy for intuitionistic fuzzy sets, *Journal of Intelligent & Fuzzy Systems*, 30, 6, 2016, 3057–3066.

*Received 19.03.2025, Accepted 23.12.2025*