



Machine Learning for Multi Objective Convex Separable Programming (MOCSP) with Aggregation of Linear Approximations and Portfolio Optimization

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Abstract. A novel technique is developed for nonlinear optimization problem which is convex, separable and having multiple objective functions. In the development of the model all the objectives and the constraints of the multi objective model are linearly approximated over suitable intervals. The linear approximations are then aggregated to account for the original problem. The developed technique has been utilized for portfolio optimization problem. Firstly, the minimum variance model has been formulated and solved with machine learning techniques. Secondly, the risk aversion model has been formulated and solved. The results obtained are combined into a multi objective framework of convex separable programming problem. All the three problems have been solved with the help of the XGBoost, neural network, and decision forest regression models. The renowned Python machine libraries of scikit-learn and keras have been utilized. The results identified portfolios that can return more financial benefits to the investors while investing in the capital market. The results of the proposed MOCSP approach are 22.5% improved in case of risk aversion model. Additionally, 17% improvement has been recorded in case of the minimum risk model. The MAE and RMSE for both XGBoost and decision forest regression have a fair value 0.0001. MAE and RMSE for the neural network regression have been recorded 1% and 2%, respectively. Both Accuracy and F1 score for XGBoost are 91%, for neural network regression are 98%, and for decision forest are 92%, respectively.

Keywords: Multi objective Convex Separable Programming (MOCSP), Capital Market, Convex Optimization, Machine Learning, Data Science.

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1. Introduction

Multi-objective optimization is concerned with multiple criteria decision making having mathematical optimization problems with more than one objective functions simultaneously.

It has been successfully applied to problems in science, engineering, business, economics, transportation and logistics. In such problems it is required to make optimal decisions when there is a trade-offs between two or more conflicting objectives.

For example, to minimize the cost and maximize the level of comfort when buying a new car. Another example may be to maximize the performance while minimize the fuel cost and pollution from emission of hazardous gases of a transport vehicle. Practically there can be more than three objectives. For nontrivial multi-objective problem, there exists no simultaneous solution optimizing all the objectives. Thus the objectives are called conflicting. A solution of the multi-objective problem is called Pareto optimal, nondominated, or non-inferior, if it is such that none of the objective can be improved without degrading some other objectives. If there are no subjective constraints, there exist infinitely many number of Pareto optimal solutions. Researchers therefore formulate their multi-objective problems wisely with various goals. These goals may be to get a representative Pareto optimal solutions, and/or evaluate the trade-offs within different objective functions, and/or to get a solution satisfying the subjective constraints and preferences in decision making (DM).

Convex optimization refers to optimization problems that minimize convex objectives functions over convex sets. Convex optimization is applicable to various disciplines, for example, control systems, estimation, communications, signal processing, networks, circuit design, data analysis, statistics, finance, and structural optimization. Convexity ensure a global optimal solution of an optimization problem in a feasible constrained region.

Separable programming is a approximate mathematical programming technique to solve non linear programming problems. It can be readily applied to problems when all the nonlinear functions appearing in the problem are separable. In this method a separable function is expressed as the sum of sub-functions and each sub-function is a function of one variable only. To solve separable programming restricted simplex method is used. This method restricts only a choice of variables to enter the basis. Moreover, it avoids some variables not to enter the basis. In other words it restrict some illegal combination of variables not to enter the basis.

Convex separable programming has been studied previously, but multi objective convex separable programming has not been studied. Thus, in this research multi objective convex separable programming technique is studied. Starting with the problem formulation a solution methodology is proposed. The proposed methodology is then implemented to portfolio optimization problems in the capital market structure.

The statement of the proposed research problem is “How convex separable programming technique can be applied to multi objective separable programming

problems". In the development of the proposed approach some important structural properties of the multi objective convex separable programming model are studied.

The model proposed is supposed to address the following objectives:

1. How convex separable programming approach can be implemented in case of multi objective separable programming?
2. What will be the revised form of the convex separable programming in case of multi objective case?
3. How to convert objective functions to separable convex programming form?
4. How to convert constraints of the problem to separable convex programming form?
5. How to implement the convex separable approach for the minimum variance model in portfolio optimization in the capital market structure using scikit-learn, XGBoost, neural network, and decision forest regression?
6. How to implement the convex separable approach for the risk aversion model for portfolio optimization in the capital market structure using scikit-learn, XGBoost, neural network, and decision forest regression?
7. How to implement proposed multi objective MOCSP approach for highlighting optimal portfolios in the capital market structure using scikit-learn, XGBoost, neural network, and decision forest regression?
8. How to draw conclusions and witty suggestions to investors in capital market structure?

The convexity requirement ensure the fact that the separable programming technique guarantees an estimated global optima if it minimize a convex objective function or it maximize a concave objective function with convex constraints.

When convexity conditions hold, the accuracy of the technique is impeccable to the coarseness of the piece-wise linear estimations used in the calculations. Moreover, the superior advantage of the proposed MOCSP technique is that the problem is addressed using the linear programming techniques. There is no need to go for the restrictive simplex method to enforce the adjacency conditions. The novel contributions of the study are presented below.

1. Theoretically develop the convex separable programming approach for multi objective convex separable programming problem.
2. Propose a method of solution for multi objective convex separable programming problem using scikit-learn, XGBoost, neural network, and decision forest regression.
3. Implementing the convex separable approach for the minimum variance model in portfolio optimization?
4. Implement the separable programming technique for the risk aversion model for portfolio optimization?
5. From (2) and (3) develop and solve the multi objective convex separable programming problem for portfolio optimization.
6. The results of the proposed MOSCP are improved 22.5% in case of the risk aversion model and 17% for minimum risk model.
7. The main contribution is that the problem is addressed utilizing linear programming techniques and do not require the conditions of adjacency to hold.

The research is organized as follows. In section (2), literature review is presented. The new concept of the MOCSP technique is developed in section 3. This is followed by section 4 in which the methodology and research flow has been presented. The proposed methodology has been implemented on portfolio optimization problem in section 5. This

section has three subsections. In subsection 5.1, the concept of separable programming technique is applied for the minimum variance technique. In subsection 5.2, technique of convex separable programming is applied to the aversion model of the portfolio optimization model. The results of the subsection 5.1 and 5.2 are then combined to formulate the proposed multi-objective convex separable programming model in subsection 5.3. In section 6, the results obtained are comprehensively discussed. The research is concluded in section 7.

2. Literature Review

The literature review has been grouped, so that first of all studies have been presented addressing uncertainty modeling in mathematical programming problems. Afterwards, review of the separable programming, convex separable programming in linear and non linear form and their applications have been presented. Finally, the techniques implemented for portfolio optimization problems have been presented. From these studies, research gap has been identified for developing Multi Objective Convex Separable Programming (MOCSP) utilizing machine learning approach together with the technique of aggregation of linear approximations and its implementation to portfolio optimization problems.

Large number of problems fall into the category of large-scale optimization problems (Taha [57]; Charnes & Lemke [17]; Miller [48]). To address such large scale problems it is required to divide it into small problems. The solutions of the individual problems are then accumulated in one problem to account for solution of the original problem. Although considered a nonlinear programming technique, separable programming can be commonly used in a linear programming environment Hadley [33]. The author furthered that the separable programming formulation yields a linear program whenever concave objective function is maximized subject to a feasible convex set.

The incorporation of fuzzy uncertainty in mathematical programming problems were addressed in (Ebrahimnejad et al., [25]; Ebrahimnejad & Nasseri [26]). Transportation problems were solved using fuzzy linear programming approach in Ebrahimnejad [27] and using data envelopment analysis in (Bagheri et al., [7]; Bagheri et al., [8]). Ali et al., [5] addressed multiobjective shortest path problem by using fuzzy inference technique. A reference ideal technique for Pythagorean fuzzy numbers were proposed in Cables et al., [15]. In study Al-Qudaimi [4], Ishita technique was developed to address intuitionistic fuzzy linear programming. Ebrahimnejad et al., [28] addressed fuzzy linear programming problems using trapezoidal fuzzy numbers.

Falk and Soland [30] presented an algorithm for non-convex separable programming problem. Thomas et al., [58] presented a strategy for converting a quadratic objective to a lower bound separable convex objective. Kochenberger [40] solved geometric programming problem with the help of separable programming technique. Bard and Falk [9] used separable programming technique to address linear complementarity problem. Spingarn [52] used partial inverses to solve convex separable programming problems. The study focused on single objective convex separable programming problem. Ali et al., [6]

used a two-stage approximation method to solve convex separable programming problems. The authors addressed single objective separable programming problem.

Ellis & Revelle [29] implemented separable linear programming algorithm to maximize aggregate hydro power generation problem. The authors solved the problem by using standard linear programming software called MINOS. Fukushima [31] applied intermittent order method of multipliers to solve convex separable problem. Furthermore, the research addressed single objective separable programming problems. Dahiya et al., [23] solved convex optimization problem with single conveyable constraint. He & Yuan [35] used a splitting algorithm together with Gaussian back substitution method to solve convex separable problems to solve single objective separable programming problems. He et al., [34] proposed a splitting algorithm to solve convex separable problem having a single objective. Li et al, [42] used method of periodic direction of multipliers to solve convex separable problems with complex variables. The authors addressed single objective convex separable programming problems. Chen et al., [18] used an inertial proximal approach to solve separable convex problems. In the study, the authors assumed linear constraints for the separable programming problems. Li et al., [42] used two-block alternating direction technique to solve separable convex problem. Their problem consists of with multiple blocks, which were then approximated using two block direction technique. Sun et al., [56] used a proximal atomization algorithmic to solve convex separable problem. The problem consisted of single objective convex programming problem. The technique was then applied for compressive sensing problem. Akhil & Sundaresan [2] presented a survey of algorithms used to solve convex separable programming problems with linear increasing constraints. Chen & Zhou [19] studied the characteristic properties of convex fuzzy optimization problems. The study Abrahm & Arri [1] used polygonal approximations for the convex separable convex programming. Zwick [60] commented to device approximations for the convex functions. Stefanov [53] proved the necessary and sufficient conditions for convex separable objective function subjected to convex separable inequality constraints and bounded variables. Problems with convex separable functions subjected to linear inequality and equality constraints, and box constraints were addressed in Stefanov [54]. Box constraints have lower and upper bounds for variables in formulation have practical applications in production planning and resource allocation problems. Sui & Peng [55] addressed separable convex programming problems using Lagrangian multipliers and dual objective function formulations.

The proposed methodology is implemented to portfolio optimization problems in the capital market structure. Firstly implemented on the minimum variance model and the on the risk aversion model. From literature, Markowitz [46], initially introduced the idea of the portfolio optimization in his seminal paper. He further elaborated the concept of optimal portfolio selection. He explained that expected returns of the portfolio are desirable phenomena whereas portfolio variance is an undesirable one. Moreover, a portfolio is efficient if it increase the expected return. It is also efficient if it minimize the risk of expected return. The mean-variance model promisingly optimizes a portfolio, although sometime suggesting portfolio that are not financially optimal Michaud [47]. The study pointed that the widely used Mean-variance fails to address the concept of hedging. The advent of the advanced computational robotic advisors, artificial intelligent and machine learning tools have brought about a revolutionized paradigm in the field of portfolio optimization. Nevertheless, impressive tools have been developed; the mean-variance

model of the Markowitz has significant importance in the area of economics and finance (Bera et al., [10]; Brodie et al., [12]).

The nonlinear portfolio optimization problem can be readily solved by quadratic programming (Cottle & Infanger, [22]; Bruder et al., [13]; Choueifaty et al., [20]; Bruder et al., [14]). The diversification strategy of indexing by quadratic entropy of Carmichael et al., ([16] helps investors to get a clear idea of portfolio optimization. Many portfolio optimization problems and technique were developed in (Bourgeron et al., [11]; de-Jong [24]; Lezmi et al., [41]; Gonzalez et al., [32]; Ragsdale [50]; Jurczenko, [36]). Constraints satisfaction methods play important role in solving portfolio optimization problems. To solve mean variance problems, it is required to formulate correctly distributed constraints. Moehle et al., [49] used the method of alternating direction multipliers to construct optimal portfolios. The authors applied Linear constrained separable optimization for portfolio optimization. Khan et al., [37] used penalty function method to address portfolio optimization problem. The authors utilized single objective quadratic programming for portfolio optimization problems. Khan and Aftab [38] utilized fuzzy linear programming techniques to educational planning problems. Khan et al., [39] adopted back propagation method in fuzzy neural network to solve profitable portfolio identification problem. In the study, Lagrangian optimization techniques have been adopted for single objective quadratic programming for portfolio optimization problems. Shi et al., [51] used the method of alternating direction multipliers to highlight profitable portfolios. The study further identified profitable portfolios to inject capital into it based on performance.

A thorough review of literature depicts that although convex separable programming has been studied previously, multi objective convex separable programming has not been studied. Thus there is a promising methodological gap to address multi objective convex separable programming problems. Thus, in this research multi objective convex separable programming technique is studied. First of all, the problem is formulated and then a solution methodology is proposed. The proposed methodology is then implemented to portfolio optimization problems in the capital market structure. Table 1 summarizes the novelty, contribution and gap of the proposed study with the existing literature. From the Table 1, it is evident that the proposed study has methodological, evidence and empirical types of gaps with the existing literature. It can be clearly seen that no one in the literature addressed the multi objective convex separable programming problem and thus there is a methodological gap. There is an evidence gap because of its application to portfolio optimization problem. Finally due to performing new experiments the study has empirical gap with the existing literature.

Table 1. Gap in the literature for multi objective convex separable programming.

Research	Problem	Technique	Applied To	Gap
Proposed Research	Multi objective Convex Separable Programming	Aggregation of Linear Approximation using ML approaches of XGboosting	Portfolio Optimization Problem (Theory +Implementation)	Proposed
Spingarn [52]	Single objective convex	Partial inverses	Developed Theory	Methodological Gap (New Methods)

	separable programming problem			Evidence Gap as it is applied to portfolio optimization problem. Empirical Gap (New Experiment)
Ali et al., [6]	Single objective convex separable programming problem	Two-stage approximation method	Developed Theory	Methodological Gap (New Methods) Evidence Gap as it is applied to portfolio optimization problem. Empirical Gap (New Experiment)
Fukushima [31]	Single objective convex separable programming problem	Method of multipliers	Developed Theory	Methodological Gap (New Methods) Evidence Gap as it is applied to portfolio optimization problem. Empirical Gap (New Experiment)
He & Yuan [35]	Single objective convex separable programming problem	Splitting algorithm together with Gaussian back substitution method	Developed Theory	Methodological Gap (New Methods) Evidence Gap as it is applied to portfolio optimization problem. Empirical Gap (New Experiment)
He et al., [34]	Single objective convex	Splitting algorithm	Developed Theory	Methodological Gap (New Methods)

	separable programming problem			Evidence Gap as it is applied to portfolio optimization problem. Empirical Gap (New Experiment)
Chen et al., [18]	Single objective convex separable programming problem	Inertial proximal approach	Developed Theory	Methodological Gap (New Methods) Evidence Gap as it is applied to portfolio optimization problem. Empirical Gap (New Experiment)
Li et al., [42]	Single objective convex separable programming problem	Two-block alternating direction	Developed Theory	Methodological Gap (New Methods) Evidence Gap as it is applied to portfolio optimization problem. Empirical Gap (New Experiment)
Sun et al., [56]	Single objective convex separable programming problem	Proximal atomization algorithmic	Compressive sensing	Methodological Gap (New Methods) Evidence Gap as it is applied to portfolio optimization problem. Empirical Gap (New Experiment)
Moehle et al., [49]	Linear constrained separable optimzaton	Alternating direction method of multipliers	Theory and implementation	Methodological gap Empirical gap
Khan et al., [37]	Single objective	Penalty function	Theory and implementation	Methodological gap

	quadratic programming	method using fuzzy neural network		Empirical gap
Khan et al., [39]	Single objective quadratic programming	Back propagation method using fuzzy neural network	Theory and implementation	Methodological gap Empirical gap
Shi et al., [51]	Single objective quadratic programming	Alternating direction method of multipliers	Theory and implementation	Methodological gap Empirical gap

From review of literature it is clear there is a possible gap to address multi objective convex separable programming problems. Thus, in this research multi objective convex separable programming technique is studied. First of all, the problem is formulated and then a solution methodology is proposed. The proposed methodology is then implemented to portfolio optimization problems in the capital market structure using scikit-learn and XGBoost Python machine learning libraries.

3. Proposed Methodology of Multi Objective Convex Separable Programming Approach

In this section the proposed MOCSP technique is developed. Multi objective linear programming problem is presented as (1). We follow the formulation of Hamady A. Taha pp. 699-706 (2007), where, a single objective convex separable programming problem is considered. Thus the formulation of (1) is based on Taha [57]. Thus, the formulation of (1) is based on the worldwide accepted model of Taha [57], page 699.

In the model (1), the z_l , $l=1,2,3,...,n$, denote “ n ” objectives to be either maximized or minimized. The functions f_i^l , $i,l=1,2,3,...,n$, denote the “ n ” separable parts of the objective function z_l , $l=1,2,3,...,n$. There are total “ m ” constraints. The functions g_i^j , $i=1,2,3,...,n$, $j=1,2,3,...,m$, denote the i^{th} separable part of the constraint g_i , $i=1,2,3,...,n$. The numbers b_j , $j=1,2,3,...,m$, denote the total available resources of the constraints g_i^j , $i=1,2,3,...,n$, $j=1,2,3,...,m$. Finally, x_i , $i=1,2,3,...,n$, denote the decision variables to be found.

$$\begin{aligned}
\text{Min } z_1 &= f_1^1(x_1) + f_1^2(x_2) + f_1^3(x_3) + \dots + f_1^n(x_n) \\
\text{Max } z_2 &= f_2^1(x_1) + f_2^2(x_2) + f_2^3(x_3) + \dots + f_2^n(x_n) \\
\text{Min } z_3 &= f_3^1(x_1) + f_3^2(x_2) + f_3^3(x_3) + \dots + f_3^n(x_n) \\
&\cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\
&\cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\
&\cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\
\text{Min } z_n &= f_n^1(x_1) + f_n^2(x_2) + f_n^3(x_3) + \dots + f_n^n(x_n) \\
\text{s.t.} \\
g_1^1(x_1) + g_1^2(x_2) + g_1^3(x_3) + \dots + g_1^m(x_n) &\leq b_1, \\
g_2^1(x_1) + g_2^2(x_2) + g_2^3(x_3) + \dots + g_2^m(x_n) &\leq b_2, \\
g_3^1(x_1) + g_3^2(x_2) + g_3^3(x_3) + \dots + g_3^m(x_n) &\leq b_3, \\
&\cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\
&\cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\
&\cdot \quad \quad \quad \cdot \quad \quad \quad \cdot \\
g_n^1(x_1) + g_n^2(x_1) + g_n^3(x_2) + \dots + g_n^m(x_n) &\leq b_m, \\
x_i &\geq 0.
\end{aligned}$$

(1)

The multi objective model (1), in reduced form can be (2).

$$\text{Min / Max } z_l = \sum_{i=1}^n \sum_{l=1}^l f(x_i)$$

s.t.

$$\sum_{i=1}^n g_i^j(x_i) \leq b_j,$$

$$i, l = 1, 2, 3, \dots, n,$$

$$j = 1, 2, 3, \dots, m.$$

(2)

To solve multi objective convex separable programming problem, piece wise linear approximation method is used. In this method each non linear objective function and constraint is reduced to their piece wise linear approximations. The objective function and the constraints are divided into finite number of breaking points. Moreover, it is assumed that the objective function and the constraints behave linearly on the small breaking intervals. Thus the given problem is converted into an equivalent linear program. Furthermore, in the equivalent linear program the total effects of the individual variables are summed up to give the optimal value of the original variables.

Let us assume that the breaking points of $f_i^l(x_i)$ be $x_i = a_{ki}$, $k = 0, 1, 2, \dots, ki$. Moreover, x_{ki} define the increase of variable x_i in the range (a_{k-1i}, a_{ki}) , $k = 1, 2, \dots, ki$. Then the objective function can be converted into its equivalent convex separable form as (3).

$$f_i^l(x_i) = \sum_{k=1}^{ki} (\rho_{ki}^l) x_{ki} + f_i^l(a_{0i}) \quad (3)$$

Where $x_i = \sum_{k=1}^{ki} x_{ki}$ denote the total sum of all the component variables, provided that $0 < x_{ki} < (a_{ki} - a_{k-1i})$, $k = 1, 2, \dots, ki$. Furthermore, ρ_{ki}^l can be defined as equation (4).

$$\rho_{ki}^l = \frac{f_i^l(a_{ki}) - f_i^l(a_{k-1i})}{(a_{ki} - a_{k-1i})} \quad (4)$$

The constraints part can also be converted as (5).

$$g_i^j(x_i) = \sum_{k=1}^{ki} (\rho_{ki}^j) x_{ki} + g_i^j(a_{0i}) \quad (5)$$

Again $x_i = \sum_{k=1}^{ki} x_{ki}$ provided that $0 < x_{ki} < (a_{ki} - a_{k-1i})$, $k = 1, 2, \dots, ki$. Moreover, ρ_{ki}^j can be defined as equation (6).

$$\rho_{ki}^j = \frac{g_i^j(a_{ki}) - g_i^j(a_{k-1i})}{(a_{ki} - a_{k-1i})} \quad (6)$$

Finally the complete problem is presented as model (7).

$$\begin{aligned} \text{Min} \quad z &= \sum_{i=1}^n \left(\sum_{k=1}^{ki} (\rho_{ki}^l) x_{ki} + f_i^l(a_{0i}) \right) \\ \text{s.t.} \quad & \sum_{i=1}^n \left(\sum_{k=1}^{ki} (\rho_{ki}^j) x_{ki} + g_i^j(a_{0i}) \right) \leq b_j, \\ & 0 < x_{ki} < (a_{ki} - a_{k-1i}), \\ & k = 1, 2, \dots, ki, \\ & i = 1, 2, 3, \dots, n, \quad j = 1, 2, 3, \dots, m. \end{aligned} \quad (7)$$

The model (7) is the required formulation the proposed multi objective convex separable programming.

Now we propose a solution methodology of solving (7). To solve the multi objective convex separable model (7), each objective of the multi objective model is solved and then these solutions are used / combined into a single objective linear programming problem. Thus a final compromise linear programming model is formulated having the effect of all the multi objectives present in the given model. Since the compromise linear programming model is formulated from the solutions of the single objectives, it possesses all the effects of each objective function.

Consider a maximization linear programming problem with optimal solution z_1^* as shown in (8).

$$\begin{aligned} \text{Max } z_1 &= C_1 X \\ \text{s.t.} \\ A_1 X &\leq b_1 \\ X &\geq 0 \end{aligned} \tag{8}$$

Let another minimization linear programming problem with optimal solution z_2^* as shown in (9).

$$\begin{aligned} \text{Min } z_2 &= C_2 X \\ \text{s.t.} \\ A_2 X &\leq b_2 \\ X &\geq 0 \end{aligned} \tag{9}$$

The third maximization linear programming problem having optimal solution z_3^* as shown in (10).

$$\begin{aligned} \text{Max } z_3 &= C_3 X \\ \text{s.t.} \\ A_3 X &\leq b_3 \\ X &\geq 0 \end{aligned} \tag{10}$$

Then the optimal value obtained from (8) and (9) and (10) can be combined to formulate a multi objective linear programming problem (11).

Model (11) present the required the technique to solve the multi objective convex separable programming problem. Thus (11) is the compromise model obtained from the optimal solutions of (8), (9) and (10), having the combined effects of all the three models.

$$\begin{aligned}
 & \text{Min } Q \\
 & \text{s.t.} \\
 & w_1 \frac{(z_1^* - C_1 X)}{z_1^*} \leq Q \\
 & w_2 \frac{(C_2 X - z_2^*)}{z_2^*} \leq Q \\
 & w_3 \frac{(z_3^* - C_3 X)}{z_3^*} \leq Q \\
 & A_1 X \leq b_1 \\
 & A_2 X \leq b_2 \\
 & A_3 X \leq b_3 \\
 & X \geq 0
 \end{aligned}
 \tag{11}$$

4. Methodology and Research Workflow

The research methodology adopted in this research follow a systematic flow. The new MOCSP technique is first developed using aggregations of linear approximations. The technique is then implementing for portfolio optimization problem. This stage has two phases. In this phase 1, the concept of separable programming technique is applied for the minimum variance technique resulting in model (16). In phase 2, the technique of convex separable programming is applied to the risk aversion model resulting in model (18). In the next stage, the minimum variance model (16) and risk aversion model (18), are solved using XG-Boost, Neural Network Regression, and Decision Forest Regression, respectively. The best results of the both the models are selected. In the next stage, the results obtained for models (16) and (18) are utilized to formulate the proposed multi-objective convex separable programming model (19). The model (19), is then solved using XG-Boost, Neural Network Regression, and Decision Forest Regression, respectively. Finally, the best results of the model (19) are selected and compared with other techniques in the literature. The systematic methodology and research flow can be found in the Figure (1).

The step by step flow is summarized below.

1. Develop the MOCSP technique using aggregations of linear approximations.
2. Implement the technique for portfolio optimization problem. Firstly, the concept of separable programming technique is applied for the minimum variance technique.

Secondly, the technique of convex separable programming is implemented for the risk aversion model resulting.

3. Solve the minimum variance model and the risk aversion model using XG-Boost, Neural Network Regression, and Decision Forest Regression, respectively. Choose the best results of the both the models.

4. The results obtained for the minimum variance model and the risk aversion model are used in the formulation of the proposed multi-objective convex separable programming model.

5. The proposed multi-objective convex separable programming model is now solved with the aid of XG-Boost, Neural Network Regression, and Decision Forest Regression, respectively.

6. Finally, the best results of the proposed multi-objective convex separable programming model are picked and compared with those present in the literature.

The techniques used are Decision Forests, XGBoost, and Neural Network regression, all implemented using Scikit-learn library of the Python Machine Learning software. Decision Forest is an ensemble of many decision trees constructing a forest. The technique used for feature selection is bagging or bootstrapping. This reduces variance compared to a single decision tree. To minimize variance bootstrap technique is used. The depth is set to a level of 20 to control overfitting. For stable output prediction a minimum sample leaf size of 3 is used. XGBoost is also an ensemble of many trees. It sequentially corrects the errors in such a way so that the next tree corrects the errors of the prior tree. The use of this technique is justified due to its sequential nature to correct the errors and get high level of accuracy. The parameter L1 values are set greater than zero and L2 greater than one to control complexity. To overcome overfitting the subsample size is kept 0.8. The learning rate is kept 0.01 to improve generalization. Neural Network is made of interconnect layers transforming input features into predicted outputs via nonlinear activation functions. ReLU activation function is used for faster training. Adam optimizer adaptively change the learning rate in a range of 0.001-0.01 to achieve stable convergence. Additionally, a dropout rate of 0.3 minimizes overfitting.

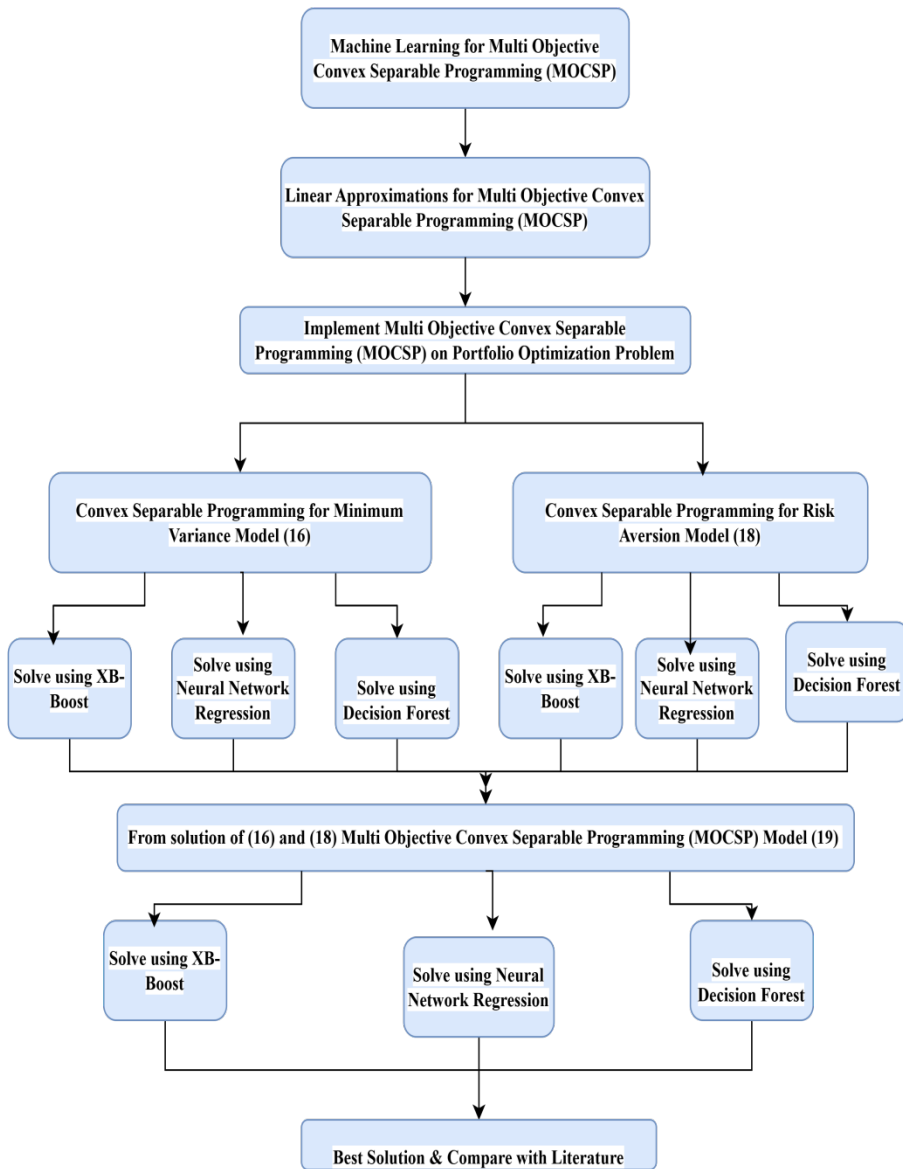


Figure 1. Methodology and Research Flow.

5. Implementation of the Proposed Multi Objective Convex Separable Programming (MOCSP) Model

In this section the proposed MOCSP technique is implemented on the non linear portfolio selection problem studied in Ragsdale [50]. First of all it is applied to the minimum variance model. After that the risk aversion model is addressed using separable programming environment. The results obtained from these models are then utilized to formulate the proposed MOCSP technique.

5.1 Convex Separable Approach for the Minimum Variance Portfolio Model

The general minimum variance model is formulated as given in (12).

$$\begin{aligned}
 \text{Min Portfolion Varaince} &= \sum_{i=1}^n \sigma_i^2 p_i^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n \sigma_{ij} p_i p_j = p^T C p \\
 \text{s.t} \\
 \sum_{i=1}^n p_i &= 1, \\
 \sum_{i=1}^n \mu_i p_i &= 1, \\
 0 \leq p_i &\leq 1.
 \end{aligned}
 \tag{12}$$

Here p_i and μ_i denote the funds in investment “ i ” and the return of the investment “ i ”, respectively. Moreover, σ_i^2 and σ_{ij} denote the variance of investment “ i ” and covariance between investments “ i ” and “ j ”, respectively. The investment matrix is $p = [p_1, p_2, \dots, p_n]$, and “ T ” denote the transpose of the matrix. Furthermore, “ C ” denote the covariance matrix defined as (13).

$$C = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1n}^2 \\ \sigma_{21}^2 & \sigma_2^2 & \dots & \sigma_{2n}^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_{n1}^2 & \sigma_{n2}^2 & & \sigma_n^2 \end{bmatrix} \quad (13)$$

Ragsdale [50] used minimum variance model to suggest optimal portfolio for an investor. In this study the model is revised as a convex separable programming problem. The assumptions of the portfolio are given in the lines below.

Three companies namely International Business Corporation (IBC), National Broadcasting System (NBS) and the national Motor Corporation (NMC) have twelve years average returns 7.64%, 13.43% and 14.93%, respectively. The variances in the annual returns for IBC, NMC, NBS are 0.00258, 0.00276, 0.03677, respectively. Moreover, the covariance between IBC and NMC is approximately -0.00025, the covariance between IBC and NBS is approximately 0.00440 and the covariance between NMC and NBS is approximately -0.00542. Furthermore, the investor desires to have at least a 12% average return with minimum risk from the investment. Let p_1 , p_2 , p_3 be the proportion of total funds invested in IBC, NMC and NBS, respectively. Based on this information the following model is formulated.

$$\text{Min: } 0.00258p_1^2 + 0.00276p_2^2 + 0.03677p_3^2 + 2(-0.00025*p_1p_2 + 0.00440*p_1p_3 - 0.00542*p_2p_3)$$

s.t.

$$p_1 + p_2 + p_3 = 1,$$

$$0.0764p_1 + 0.1343p_2 + 0.1493p_3 \geq 0.12, \quad (14)$$

$$p_1, p_2, p_3 \geq 0,$$

$$p_1, p_2, p_3 \leq 1.$$

Model (13) is translated a convex separable programming model as given in (15) with the modification that the total return should be 14%.

$$\text{Min: } 0.00258p_1^2 + 0.00276p_2^2 + 0.03677p_3^2 + 2(-0.00025*p_1p_2 +$$

$$\begin{aligned}
& 0.00440*p_1p_3 - 0.00542*p_2p_3 = 0.00258p_1^2 + 0.00276p_2^2 + 0.03677p_3^2 - 0.0005 \\
& \left[\frac{(p_1 + p_2)^2}{4} - \frac{(p_1 - p_2)^2}{4} \right] + 0.0088 \left[\frac{(p_1 + p_3)^2}{4} - \frac{(p_1 - p_3)^2}{4} \right] - 0.01084 \\
& \left[\frac{(p_2 + p_3)^2}{4} - \frac{(p_2 - p_3)^2}{4} \right] = 0.00258p_1^2 + 0.00276p_2^2 + 0.03677p_3^2 - 0.000125[p_4^2 - p_5^2] \\
& + 0.0022[p_6^2 - p_7^2] - 0.00271[p_8^2 - p_9^2] = 0.00258p_1^2 + 0.00276p_2^2 + 0.03677p_3^2 - 0.000125p_4^2 + \\
& 0.000125p_5^2 + 0.0022p_6^2 - 0.0022p_7^2 - 0.00271p_8^2 + 0.00271p_9^2
\end{aligned}$$

s.t.

$$p_1 + p_2 + p_3 = 1$$

$$0.0764p_1 + 0.1343p_2 + 0.1493p_3 \geq 0.14 \quad (15)$$

$$p_1 + p_2 - p_4 = 0$$

$$p_1 - p_2 - p_5 = 0$$

$$p_1 + p_3 - p_6 = 0$$

$$p_1 - p_3 - p_7 = 0$$

$$p_2 + p_3 - p_8 = 0$$

$$p_2 - p_3 - p_9 = 0$$

$$p_1, p_2, p_3 \geq 0, p_1, p_2, p_3 \leq 1$$

$$p_4, p_5, p_6, p_7, p_8, p_9 \geq 0$$

Table 2. Transformations of the First and Second Parts of the Objective Function over the Interval [0.1, 1].

a_{ik} $k =$ $0, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$f_1^1 =$ $0.000025p_1^2$	$\rho_{ki}^1 =$ $\frac{f_1^1(a_{ki}) - f_1^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$	$f_2^1 =$ $0.0000276p_2^2$	$\rho_{ki}^l =$ $\frac{f_2^1(a_{ki}) - f_2^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$
0.1		0.000025		0.000027	
0.2	0.1	0.0001	0.00075	0.000110	0.00082 8
0.4	0.2	0.0004	0.0015	0.000441	0.00165 6
0.6	0.2	0.0009	0.0025	0.000993	0.00276
0.8	0.2	0.0016	0.0035	0.001766	0.00386 4
1	0.2	0.0025	0.0045	0.00276	0.00496 8

Table 3. Transformations of the Third and Fourth Parts of the Objective Function over the Interval [0.1, 1].

a_{ik} $k = 0$ $, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$f_3^1 =$ $0.03677p_3^2$	$\rho_{ki}^1 =$ $\frac{f_3^1(a_{ki}) - f_3^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$	$f_4^1 = -0.000125p_4^2$	$\rho_{ki}^l =$ $\frac{f_4^1(a_{ki}) - f_4^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$
0.1		0.00036		-0.00000125	
0.2	0.1	0.00147	0.011031	-0.000005	- 0.0000375
0.4	0.2	0.00588	0.022062	-0.00002	-0.000075
0.6	0.2	0.01323	0.03677	-0.000045	-0.000125
0.8	0.2	0.023533	0.051478	-0.00008	-0.000175
1	0.2	0.03677	0.066186	-0.000125	-0.000225

Table 4. Transformations of Fifth and Sixth Parts of the Objective Function on the Interval [0.1, 1].

a_{ik} $k = 0$ $, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$f_5^1 =$ $0.000125p_5^2$	$\rho_{ki}^1 =$ $\frac{f_5^1(a_{ki}) - f_5^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$	$f_6^1 =$ $0.00226p_6^2$	$\rho_{ki}^l =$ $\frac{f_6^1(a_{ki}) - f_6^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$
0.1		0.0000012		0.000022	
0.2	0.1	0.000005	0.0000375	0.000088	0.00066
0.4	0.2	0.00002	0.000075	0.000352	0.00132
0.6	0.2	0.000045	0.000125	0.000792	0.0022
0.8	0.2	0.00008	0.000175	0.001408	0.00308
1	0.2	0.000125	0.000225	0.0022	0.00396

Table 5. Transformations of the Seventh and Eighth Parts of the Objective Function over the Interval [0.1, 1].

a_{ik} $k = 0$ $, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$f_7^1 =$ $-0.0022p_7^2$	$\rho_{ki}^1 =$ $\frac{f_7^1(a_{ki}) - f_7^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$	$f_8^1 =$ $-0.00271p_8^2$	$\rho_{ki}^l =$ $\frac{f_8^1(a_{ki}) - f_8^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$
0.1		- 0.000022		- 0.0000271	
0.2	0.1	- 0.000088	-0.00066	- 0.0001084	- 0.00081
0.4	0.2	- 0.000352	-0.00132	- 0.0004336	- 0.00163
0.6	0.2	- 0.000792	-0.0022	- 0.0009756	- 0.00271
0.8	0.2	- 0.001408	-0.00308	- 0.0017344	- 0.00379
1	0.2	- 0.000022		- 0.0000271	

Table 6. Transformations of the Ninth Parts of the Objective Function over the Interval [0.1, 1].

a_{ik} $k = 0, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$f_9^1 = 0.0022 p_9^2$	$\rho_{ki}^1 =$ $\frac{f_9^1(a_{ki}) - f_9^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$
0.1		0.0000271	
0.2	0.1	0.0001084	0.000813
0.4	0.2	0.0004336	0.001626
0.6	0.2	0.0009756	0.00271
0.8	0.2	0.0017344	0.003794
1	0.2	0.0000271	0.004878

The constraints part is transformed as shown in Tables (7-16).

Table 7. Transformations of the First and Second Parts of the First Constraint on the Interval [0.1, 1].

a_{ik} $k = 0, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$g_1^1 = p_1$	$\rho_{ki}^1 =$ $\frac{g_1^1(a_{ki}) - g_1^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$	$g_2^1 = p_2$	$\rho_{ki}^1 =$ $\frac{g_2^1(a_{ki}) - g_2^1(a_{k-1i})}{(a_{ki} - a_{k-1i})}$
0.1		0.1		0.1	
0.2	0.1	0.2	1	0.2	1
0.4	0.2	0.4	1	0.4	1
0.6	0.2	0.6	1	0.6	1
0.8	0.2	0.8	1	0.8	1
1	0.2	1	1	1	1

Table 8. Transformations of the Third Part of the First Constraint and First Part of the Second Constraint on the Interval [0.1, 1].

a_{ik} $k = 0$ $, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$g_3^1 = p_3$	$\rho_{ki}^1 =$ $\frac{g_3^1(a_{ki}) - g_3^1(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_1^2 = 0.0764 p_1$	$\rho_{ki}^1 =$ $\frac{g_1^2(a_{ki}) - g_1^2(a_{k-li})}{(a_{ki} - a_{k-li})}$
0.1		0.1		0.00764	
0.2	0.1	0.2	1	0.01528	0.0764
0.4	0.2	0.4	1	0.03056	0.0764
0.6	0.2	0.6	1	0.04584	0.0764
0.8	0.2	0.8	1	0.06112	0.0764
1	0.2	1	1	0.0764	0.0764

Table 9. Transformations of the Second and Third Parts of the Second Constraint on Interval [0.1, 1].

a_{ik} $k = 0$ $, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$g_2^2 =$ $0.1343 p_2$	$\rho_{ki}^1 =$ $\frac{g_2^2(a_{ki}) - g_2^2(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_3^2 = 0.1493 p_3$	$\rho_{ki}^1 =$ $\frac{g_1^2(a_{ki}) - g_1^2(a_{k-li})}{(a_{ki} - a_{k-li})}$
0.1		0.01343		0.01493	
0.2	0.1	0.02686	0.1343	0.02986	0.1493
0.4	0.2	0.05372	0.1343	0.05972	0.1493
0.6	0.2	0.08058	0.1343	0.08958	0.1493
0.8	0.2	0.10744	0.1343	0.11944	0.1493
1	0.2	0.1343	0.1343	0.1493	0.1493

Table 10. Transformations of the First and Second Parts of the Third Constraint on the Interval $[0.1, 1]$.

a_{ik} $k = 0$ $, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$g_3^1 = p_1$	$\rho_{ki}^1 =$ $\frac{g_3^1(a_{ki}) - g_3^1(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_3^2 = p_2$	$\rho_{ki}^1 = \frac{g_3^2(a_{ki}) - g_3^2(a_{k-li})}{(a_{ki} - a_{k-li})}$
0.1		0.1		0.1	
0.2	0.1	0.2	1	0.2	1
0.4	0.2	0.4	1	0.4	1
0.6	0.2	0.6	1	0.6	1
0.8	0.2	0.8	1	0.8	1
1	0.2	1	1	1	1

Table 11. Transformations of the Third Part of the Third Constraint and First Part of the Fourth Constraint on the Interval $[0.1, 1]$.

a_{ik} $k = 0$ $, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$g_3^3 = -p_4$	$\rho_{ki}^1 =$ $\frac{g_3^1(a_{ki}) - g_3^1(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_4^1 = p_1$	$\rho_{ki}^1 =$ $\frac{g_4^1(a_{ki}) - g_4^1(a_{k-li})}{(a_{ki} - a_{k-li})}$
0.1		0.1		0.1	
0.2	0.1	0.2	1	0.2	1
0.4	0.2	0.4	1	0.4	1
0.6	0.2	0.6	1	0.6	1
0.8	0.2	0.8	1	0.8	1
1	0.2	0.1	1	1	1

Table 12. Transformations of the Second and Third Parts of the Fourth Constraint on the Interval $[0.1, 1]$.

a_{ik} $k = 0$ $, 1, \dots, n$	$(a_{ik} - a_{ik-1})$ $k = 0, 1, \dots, n$	$g_4^2 = -p_2$	$\rho_{ki}^1 =$ $\frac{g_4^2(a_{ki}) - g_4^2(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_4^3 = -p_5$	$\rho_{ki}^1 =$ $\frac{g_4^3(a_{ki}) - g_4^3(a_{k-li})}{(a_{ki} - a_{k-li})}$
0.1		-0.1		-0.1	
0.2	0.1	-0.2	-1	-0.2	-1
0.4	0.2	-0.4	-1	-0.4	-1
0.6	0.2	-0.6	-1	-0.6	-1
0.8	0.2	-0.8	-1	-0.8	-1
1	0.2	-1	-1	-1	-1

Table 13. Transformations of the Fifth Constraint on the Interval $[0.1, 1]$.

a_{ik} $k = 0$ $, 1, \dots, n$	$g_5^1 = p_1$	$\rho_{ki}^1 =$ $\frac{g_5^1(a_{ki}) - g_5^1(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_5^2 = p_3$	$\rho_{ki}^1 =$ $\frac{g_5^2(a_{ki}) - g_5^2(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_5^3 = -p_6$	$\rho_{ki}^1 =$ $\frac{g_5^3(a_{ki}) - g_5^3(a_{k-li})}{(a_{ki} - a_{k-li})}$
0.1	0.1		0.1		-0.1	
0.2	0.2	1	0.2	1	-0.2	-1
0.4	0.4	1	0.4	1	-0.4	-1
0.6	0.6	1	0.6	1	-0.6	-1
0.8	0.8	1	0.8	1	-0.8	-1
1	1	1	1	1	-1	-1

Table 14. Transformations of the Sixth Constraint on the Interval [0.1, 1].

a_{ik} $k = 0$ $, 1, \dots, n$	$g_6^1 = p_1$	$\rho_{ki}^1 =$ $\frac{g_6^1(a_k) - g_6^1(a_{k-li})}{(a_k - a_{k-li})}$	$g_6^2 = -p_3$	$\rho_{ki}^1 =$ $\frac{g_6^2(a_k) - g_6^2(a_{k-li})}{(a_k - a_{k-li})}$	$g_6^3 = -p_7$	$\rho_{ki}^1 =$ $\frac{g_6^3(a_k) - g_6^3(a_{k-li})}{(a_k - a_{k-li})}$
0.1	0.1		-0.1		0.1	
0.2	0.2	1	-0.2	-1	0.2	1
0.4	0.4	1	-0.4	-1	0.4	1
0.6	0.6	1	-0.6	-1	0.6	1
0.8	0.8	1	-0.8	-1	0.8	1
1	1	1	-1	-1	1	1

Table 15. Transformations of the Seventh Constraint on the Interval [0.1, 1].

a_{ik} $k = 0$ $, 1, \dots, n$	$g_7^1 = p_2$	$\rho_{ki}^1 =$ $\frac{g_7^1(a_k) - g_7^1(a_{k-li})}{(a_k - a_{k-li})}$	$g_7^2 = p_3$	$\rho_{ki}^1 =$ $\frac{g_7^2(a_k) - g_7^2(a_{k-li})}{(a_k - a_{k-li})}$	$g_7^3 = -p_8$	$\rho_{ki}^1 =$ $\frac{g_7^3(a_k) - g_7^3(a_{k-li})}{(a_k - a_{k-li})}$
0.1	0.1		0.1		-0.1	
0.2	0.2	1	0.2	1	-0.2	-1
0.4	0.4	1	0.4	1	-0.4	-1
0.6	0.6	1	0.6	1	-0.6	-1
0.8	0.8	1	0.8	1	-0.8	-1
1	1	1	1	1	-1	-1

Table 16. Transformations of the Eighth Constraint on the Interval [0.1, 1].

a_{ik} $k = 0$ $, 1, \dots, n$	$g_8^1 = p_2$	$\rho_{ki}^1 =$ $\frac{g_8^1(a_{ki}) - g_8^1(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_8^2 = -p_3$	$\rho_{ki}^1 =$ $\frac{g_8^2(a_{ki}) - g_8^2(a_{k-li})}{(a_{ki} - a_{k-li})}$	$g_8^3 = -p$	$\rho_{ki}^1 =$ $\frac{g_8^3(a_{ki}) - g_8^3(a_{k-li})}{(a_{ki} - a_{k-li})}$
0.1	0.1		-0.1		-0.1	
0.2	0.2	1	-0.2	-1	-0.2	-1
0.4	0.4	1	-0.4	-1	-0.4	-1
0.6	0.6	1	-0.6	-1	-0.6	-1
0.8	0.8	1	-0.8	-1	-0.8	-1
1	1	1	-1	-1	-1	-1

Based on the Tables (2-16), the convex separable programming problem (15) is converted to the linear programming problem (16).

$$\begin{aligned} \text{Min } & 0.00075p_{11} + 0.0015p_{12} + 0.0025p_{13} + 0.0035p_{14} + 0.0045p_{15} + 0.000828p_{21} + \\ & 0.001656p_{22} + 0.00276p_{23} + 0.003864p_{24} + 0.004968p_{25} + 0.0011031p_{31} + 0.022062p_{32} + \\ & 0.03677p_{33} + 0.051478p_{34} + 0.066186p_{35} - 0.0000375p_{41} - 0.000075p_{42} - 0.000125p_{43} - \\ & 0.000175p_{44} - 0.000225p_{45} + 0.0000375p_{51} + 0.000075p_{52} + 0.000125p_{53} + 0.000175p_{54} + \\ & 0.000225p_{55} + 0.00066p_{61} + 0.00132p_{62} + 0.0022p_{63} + 0.00308p_{64} + 0.00396p_{65} - \\ & 0.00066p_{71} - 0.00132p_{72} - 0.0022p_{73} - 0.00308p_{74} - 0.00396p_{75} - 0.00081p_{81} - 0.00163p_{82} - \\ & 0.00271p_{83} - 0.00379p_{84} - 0.00488p_{85} + 0.000813p_{91} + 0.001626p_{92} + 0.00271p_{93} + \\ & 0.003794p_{94} + 0.004878p_{95} \end{aligned}$$

s.t.

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) + (p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) + (p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) \leq 1$$

$$\begin{aligned} & 0.0764(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) + 0.1343(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) + 0.1493(p_{31} + p_{32} \\ & + p_{33} + p_{34} + p_{35}) \geq 0.14 \end{aligned} \quad (16)$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) + (p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) - 1(p_{41} + p_{42} + p_{43} + p_{44} + p_{45}) \leq 0$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) - 1 (p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) - 1 (p_{51} + p_{52} + p_{53} + p_{54} + p_{55}) \leq 0$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) + (p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) - 1 (p_{61} + p_{62} + p_{63} + p_{64} + p_{65}) \leq 0$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) - 1 (p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) - 1 (p_{71} + p_{72} + p_{73} + p_{74} + p_{75}) \leq 0$$

$$(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) + (p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) - 1 (p_{81} + p_{82} + p_{83} + p_{84} + p_{85}) \leq 0$$

$$(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) - 1 (p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) - 1 (p_{91} + p_{92} + p_{93} + p_{94} + p_{95}) \leq 0$$

$$0 \leq p_{ij} \leq 1, i=1,2,\dots,9, j=1, 2,\dots,5.$$

The problem (16) has been solved with XGBoost, neural network, and decision forest regression models. The convex separable programming model (16) is solved using scikit-learn and XGBoost Python machine learning libraries. The optimal solution obtained is

$$p_1 = \sum_{i=0}^5 p_{1i} = 0.33, \quad p_2 = \sum_{i=0}^5 p_{2i} = 0.33, \quad p_3 = \sum_{i=0}^5 p_{3i} = 0.33, \quad p_4 = \sum_{i=0}^5 p_{4i} = 0.66,$$

$$p_5 = \sum_{i=0}^5 p_{5i} = 0, \quad p_6 = \sum_{i=0}^5 p_{6i} = 0.66, \quad p_7 = \sum_{i=0}^5 p_{7i} = 0, \quad p_8 = \sum_{i=0}^5 p_{8i} = 0.66,$$

$$p_9 = \sum_{i=0}^5 p_{9i} = 0, \text{ for } i = 0, 1, 2, \dots, 5. \text{ Note that variables from } p_4 \text{ to } p_9 \text{ are the}$$

requirements of the convex separable programming technique. Moreover, the portfolio variance is -0.0020. The problem solution is also obeying all the constraints of the multi objective convex separable programming techniques. Thus, the solution suggests that the investor should place 33% funds in IBC, NMC, and NBS. The suggested investment plan will minimize the portfolio variance while keeping the investment return at least 14%. Moreover, other possible features of the XGBoost simulations are presented below.

	p11	p12	p13	p14	p15	p21	p22	p23	p24	p25	...	p35
0	0.452692	0.0	0.0	0.0	0.0	0.452692	0.0	0.0	0.0	0.0	...	0.0
1	0.406220	0.0	0.0	0.0	0.0	0.406220	0.0	0.0	0.0	0.0	...	0.0
2	0.413441	0.0	0.0	0.0	0.0	0.413441	0.0	0.0	0.0	0.0	...	0.0
3	0.449052	0.0	0.0	0.0	0.0	0.449052	0.0	0.0	0.0	0.0	...	0.0
4	0.447601	0.0	0.0	0.0	0.0	0.447601	0.0	0.0	0.0	0.0	...	0.0
..	...	...	...	...	...	...	...	...	...	...	...	...
325	0.345932	0.0	0.0	0.0	0.0	0.345932	0.0	0.0	0.0	0.0	...	0.0
326	0.446526	0.0	0.0	0.0	0.0	0.446526	0.0	0.0	0.0	0.0	...	0.0

327	0.350946	0.0	0.0	0.0	0.0	0.350946	0.0	0.0	0.0	0.0	...	0.0
328	0.346233	0.0	0.0	0.0	0.0	0.346233	0.0	0.0	0.0	0.0	...	0.0
329	0.421674	0.0	0.0	0.0	0.0	0.421674	0.0	0.0	0.0	0.0	...	0.0

	P82	p83	p84	p85	p91	p92	p93	p94	p95
0	0.0	0.0	0.000000	0.547308	0.358076	0.0	0.0	0.0	0.0
1	0.0	0.0	0.593780	0.000000	0.218661	0.0	0.0	0.0	0.0
2	0.0	0.0	0.586559	0.000000	0.240322	0.0	0.0	0.0	0.0
3	0.0	0.0	0.000000	0.550948	0.347155	0.0	0.0	0.0	0.0
4	0.0	0.0	0.552399	0.000000	0.342802	0.0	0.0	0.0	0.0
...	...	...	...	...	...	...	...	...	...
325	0.0	0.0	0.000000	0.654068	0.037795	0.0	0.0	0.0	0.0
326	0.0	0.0	0.000000	0.553474	0.339577	0.0	0.0	0.0	0.0
327	0.0	0.0	0.649054	0.000000	0.052838	0.0	0.0	0.0	0.0
328	0.0	0.0	0.000000	0.653767	0.038699	0.0	0.0	0.0	0.0
329	0.0	0.0	0.000000	0.578326	0.265021	0.0	0.0	0.0	0.0

Figure (2) shows MAE of 1%, 2%, and 8% for XGBoosted regression, random forest regression, and neural network regression, respectively. RSME's reported were 7%, for both XGBoosted regression and random forest regression, and 12% for neural network regression, respectively.

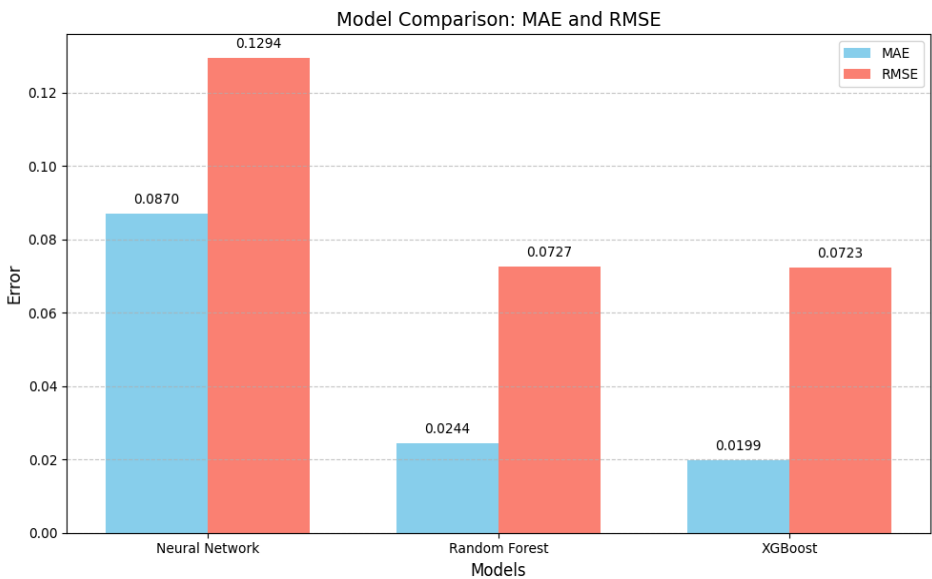


Figure 2. Errors of Neural Network, XGBoost, and Random Forest Regressions.

5.2 Convex Separable Approach for the Risk Aversion Portfolio Model

The general form of the risk aversion model is formulated as given in (17).

$$\begin{aligned}
 & \text{Max } (1-r)\mu_i - r\left(\sum_{i=1}^n \sigma_i^2 + 2\sum_{i=1}^{n-1} \sum_{j=i+1}^n \sigma_{ij}\right) \\
 & \text{s.t} \\
 & \sum_{i=1}^n p_i = 1, \\
 & 0 \leq p_i \leq 1.
 \end{aligned}
 \tag{17}$$

Where p_i and μ_i denote the funds in investment “ i ” and the return of the investment “ i ”, respectively. Moreover, σ_i^2 and σ_{ij} denote the variance of investment “ i ” and covariance between investments “ i ” and “ j ”, respectively. The investment matrix is $p = [p_1, p_2, \dots, p_n]$.

The proposed risk aversion model for the problem in Ragsdale [50] is formulated (18).

$$\begin{aligned}
 & \text{Max } (1-r)(7.64 + 13.43 + 14.93) - r(0.00258 - 0.00025 + 0.00440 - 0.00025 + \\
 & 0.00276 - 0.00542 + 0.00440 - 0.00542 + 0.03677) = 7.64 + 13.43 + 14.93 - r(7.64 + 13.43 \\
 & + 14.43) - r(0.00258 - 0.00025 + 0.00440 - 0.00025 + 0.00276 - 0.00542 + 0.00440 - \\
 & 0.00542 + 0.03677) = 36 - 36r + 0.03957r = 36 - 35.96043r
 \end{aligned}$$

s.t.

$$p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{21} + p_{22} + p_{23} + p_{24} + p_{25} + p_{31} + p_{32} + p_{33} + p_{34} + p_{35} \leq 1 \quad (18)$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) \geq 0.02$$

$$(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) \geq 0.49$$

$$(p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) \geq 0.49$$

$$0 \leq p_{ij} \leq 1, i=1,2,\dots,9, j=1, 2,\dots,5.$$

$$0 \leq r \leq 1$$

$$p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{21}, p_{22}, p_{23}, p_{24}, p_{25}, p_{31}, p_{32}, p_{33}, p_{34}, p_{35} \geq 0$$

$$p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{21}, p_{22}, p_{23}, p_{24}, p_{25}, p_{31}, p_{32}, p_{33}, p_{34}, p_{35} \leq 1$$

The problem (18) has been solved with XGBoost, neural network, and decision forest regression models. The model (18) is solved with the help of scikit-learn and XGBoost Python machine learning libraries. This solution indicates that the optimal solution is

$$p_1 = \sum_{i=0}^5 p_{1i} = 0.02, \quad p_2 = \sum_{i=0}^5 p_{2i} = 0.49, \quad p_3 = \sum_{i=0}^5 p_{3i} = 0.49, \quad \text{for}$$

$i = 0, 1, 2, \dots, 5$. Moreover, the risk aversion value is 0.36. The risk aversion value suggests that the suggested investment plan will avert risk by 36% and take it by 64%. The solution suggests that the investor should place 2% funds in IBC, and invest 49% in both NMC and NBS. The suggested investment plan bear a portfolio risk factor of 36% while keeping the investment return at least 14%.

5.3 Proposed Multi Objective Convex Separable Approach for Portfolio Optimization

The minimum variance model and the risk aversion models presented in (16) and (18) are combined to formulate the multi objective convex separable portfolio optimization model (19).

$$\text{Min } Q$$

$$\text{s.t.}$$

$$\begin{aligned} & -0.002Q + 0.00075p_{11} + 0.0015p_{12} + 0.0025p_{13} + 0.0035p_{14} + 0.0045p_{15} + 0.000828p_{21} \\ & + 0.001656p_{22} + 0.00276p_{23} + 0.003864p_{24} + 0.004968p_{25} + 0.0011031p_{31} + 0.022062p_{32} \\ & + 0.03677p_{33} + 0.051478p_{34} + 0.066186p_{35} - 0.0000375p_{41} - 0.000075p_{42} - 0.000125p_{43} - \\ & 0.000175p_{44} - 0.000225p_{45} + 0.0000375p_{51} + 0.000075p_{52} + 0.000125p_{53} + 0.000175p_{54} + \\ & 0.000225p_{55} + 0.00066p_{61} + 0.00132p_{62} + 0.0022p_{63} + 0.00308p_{64} + 0.00396p_{65} - \\ & 0.00066p_{71} - 0.00132p_{72} - 0.0022p_{73} - 0.00308p_{74} - 0.00396p_{75} - 0.00081p_{81} - 0.00163p_{82} - \\ & 0.00271p_{83} - 0.00379p_{84} - 0.00488p_{85} + 0.000813p_{91} + 0.001626p_{92} + 0.00271p_{93} + \\ & 0.003794p_{94} + 0.004878p_{95} \leq -0.002 \end{aligned}$$

$$0.36Q - 0.39957r \leq 0$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) + (p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) + (p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) \leq 1$$

$$0.0764(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) + 0.1343(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) + 0.1493(p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) \geq 0.14 \quad (19)$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) + (p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) - 1(p_{41} + p_{42} + p_{43} + p_{44} + p_{45}) \leq 0$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) - 1(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) - 1(p_{51} + p_{52} + p_{53} + p_{54} + p_{55}) \leq 0$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) + (p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) - 1(p_{61} + p_{62} + p_{63} + p_{64} + p_{65}) \leq 0$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) - 1(p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) - 1(p_{71} + p_{72} + p_{73} + p_{74} + p_{75}) \leq 0$$

$$(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) + (p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) - 1(p_{81} + p_{82} + p_{83} + p_{84} + p_{85}) \leq 0$$

$$(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) - 1(p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) - 1(p_{91} + p_{92} + p_{93} + p_{94} + p_{95}) \leq 0$$

$$(p_{11} + p_{12} + p_{13} + p_{14} + p_{15}) \geq 0.02$$

$$(p_{21} + p_{22} + p_{23} + p_{24} + p_{25}) \geq 0.49$$

$$(p_{31} + p_{32} + p_{33} + p_{34} + p_{35}) \geq 0.49$$

$$0 \leq p_{ij} \leq 1, i=1,2,\dots,9, j=1,2,\dots,5.$$

$$0 \leq r \leq 1$$

$$0 \leq Q \leq 1$$

The problem (19) has been solved with XGBoost, neural network, and decision forest regression models. The solution of the problem (19) obtained with the help of scikit-learn and XGBoost Python machine learning libraries depicts that the minimum of the objective

is $Q=0.00087$, at this level the values of $p_1 = \sum_{i=0}^5 p_{1i} = 0.40$, $p_2 = \sum_{i=0}^5 p_{2i} = 0.30$,

$$p_3 = \sum_{i=0}^5 p_{3i} = 0.30, \quad p_4 = \sum_{i=0}^5 p_{4i} = 0, \quad p_5 = \sum_{i=0}^5 p_{5i} = 0, \quad p_6 = \sum_{i=0}^5 p_{6i} = 0.0, \\ p_7 = \sum_{i=0}^5 p_{7i} = 0, \quad p_8 = \sum_{i=0}^5 p_{8i} = 0, \quad p_9 = \sum_{i=0}^5 p_{9i} = 0, \quad \text{for } i = 0, 1, 2, \dots, 5. \quad \text{The}$$

variables from p_4 to p_9 are the requirements of the convex separable programming technique. Furthermore, the risk aversion value is 0.40. Thus, the suggested investment will be 60% risk taking and 40% risk averting. The solution suggests that the investor should place 40% funds in IBC, and invest 30% in both NMC and NBS. The suggested investment plan bear a portfolio risk factor of 40% while keeping the investment return at least 14%. Besides these optimal solutions, the following solutions have been obtained at iterative passes of XGboost optimization process.

	p11	p12	p13	p14	p15	p21	p22
0	0.400000	0.000000	0.000000	0.000000	0.000000	0.300000	0.000000
1	0.000584	0.051496	0.006942	0.022215	0.031649	0.011401	0.088786
2	0.005485	0.122563	0.003084	0.011925	0.005061	0.233406	0.103750
3	0.009694	0.005252	0.010589	0.058567	0.043687	0.003546	0.076514
4	0.030595	0.006100	0.012569	0.071880	0.005913	0.127405	0.141444
	p23	p24	p25	p31	p32	p33	p34
0	0.000000	0.000000	0.000000	0.300000	0.000000	0.000000	0.000000
1	0.268467	0.126776	0.030701	0.002774	0.024854	0.106233	0.221877
2	0.008321	0.022585	0.016136	0.187437	0.083295	0.109014	0.007266
3	0.041517	0.015207	0.302475	0.116825	0.100506	0.029578	0.121853
4	0.029842	0.047675	0.173385	0.080204	0.013509	0.106852	0.112160
	p35	objective					
0	0.000000	0.000879					

1 0.005247 0.003480

2 0.080673 0.002178

3 0.064190 0.003760

4 0.040467 0.003166

Figure (3) and Table (17) depict that MAE and RMSE for both XGBoost and decision forest regression have a frail value 0.0001. MAE and RMSE for the neural network regression have been recorded 1% and 2%. Accuracy and F1 score for XGBoost are 91%, for neural network regression are 98%, for decision forest are 92%, respectively.

Table 17. Performances of the Models.

Model	MA E	RMS E	Accur acy	F1
XGBoost	0.000 147	0.000 183	0.916	0.917 647
Neural Network	0.016 238	0.020 606	0.984	0.984 496
Decision Forest	0.000 143	0.000 179	0.920	0.923 077

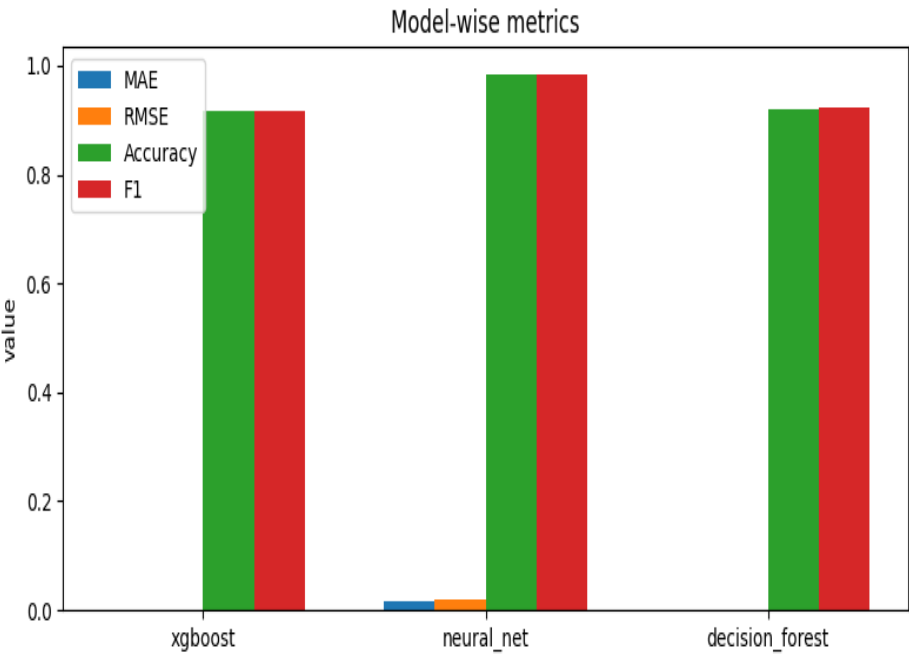


Figure 3. Graph of performances of the Models.

From the confusion matrices shown in figure (4), it is clear that the XGBoost regression model predicted 10 false responses as true, and 11 true responses as false. Likewise, the neural network regression predicted 3 false responses as true, and 1 true responses as false. Moreover, the decision forest regression predicted 12 false responses as true, and 8 true responses as false. Thus, the neural network regression committed less errors in prediction.

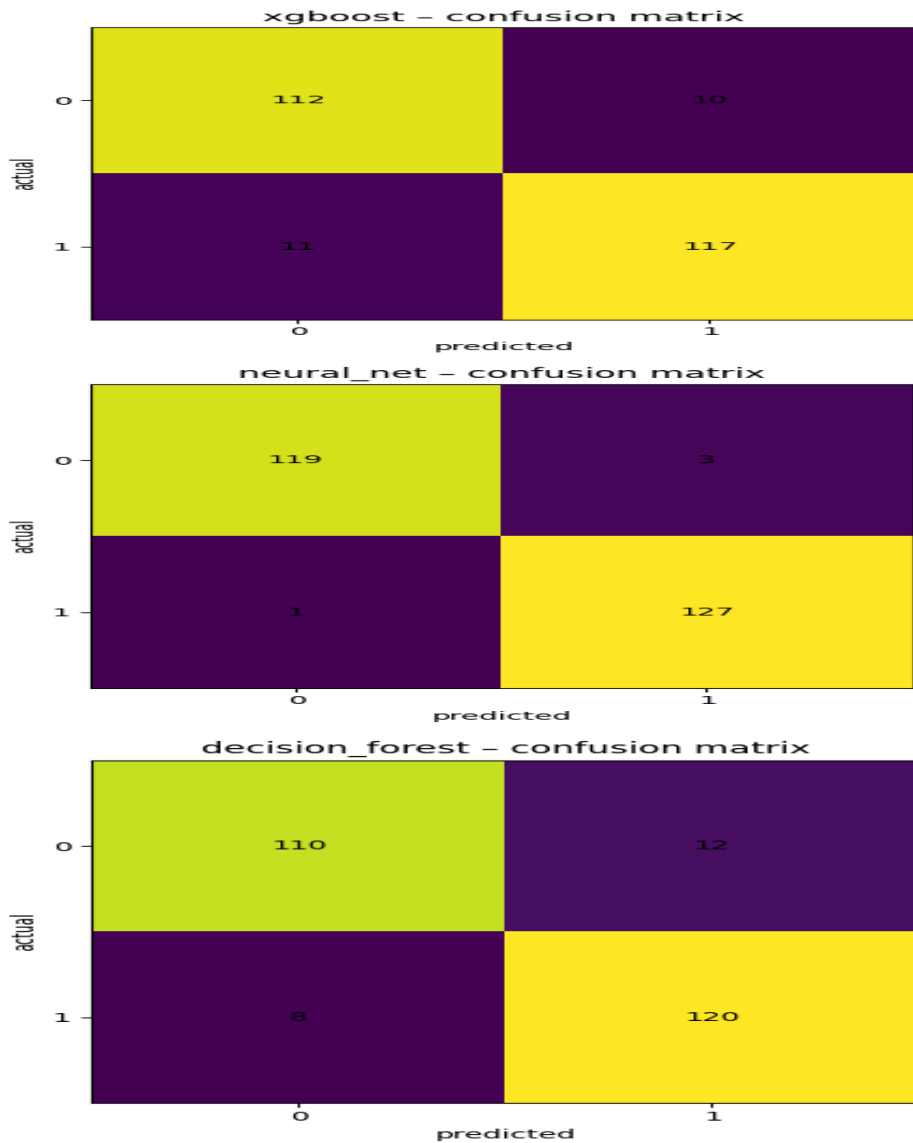


Figure 4. Confusion Matrices of the Methods.

6. Analysis and Discussion

In this section, the results obtained from solving the proposed multi objective convex separable technique are discussed and compared with the non linear model Ragsdale [50] and the related studies in literature (Jurczenko [36]; Mansoori et al., [45]; Wu et al., [59];

Malek & Oskoei [43]; Mansoori et al., [44]; Coelho, [21]). Table (18) show the results of the proposed technique and the results of the non linear minimum variance model and risk aversion models of Ragsdale [50]. From figure (5), it is clear that the proposed technique suggest promoting investment in IBC, and equal investment in NMC and NBS, to maintain a balance between risk and returns.

Table 18. The results of the proposed multi objective convex separable model and the models of Ragsdale [50].

Model	Percent Investment IBC	Percent Investment NMC	Percent Investment NBS	Risk Aversion Value	Minimum Q
Proposed Multi objective Convex Separable Model	40	30	30	0.4	0.00087
Minimum Variance Model Ragsdale [50]	27.2	63.4	9.4		
Risk Aversion Model Ragsdale [50]	36	57	7.1	0.36	

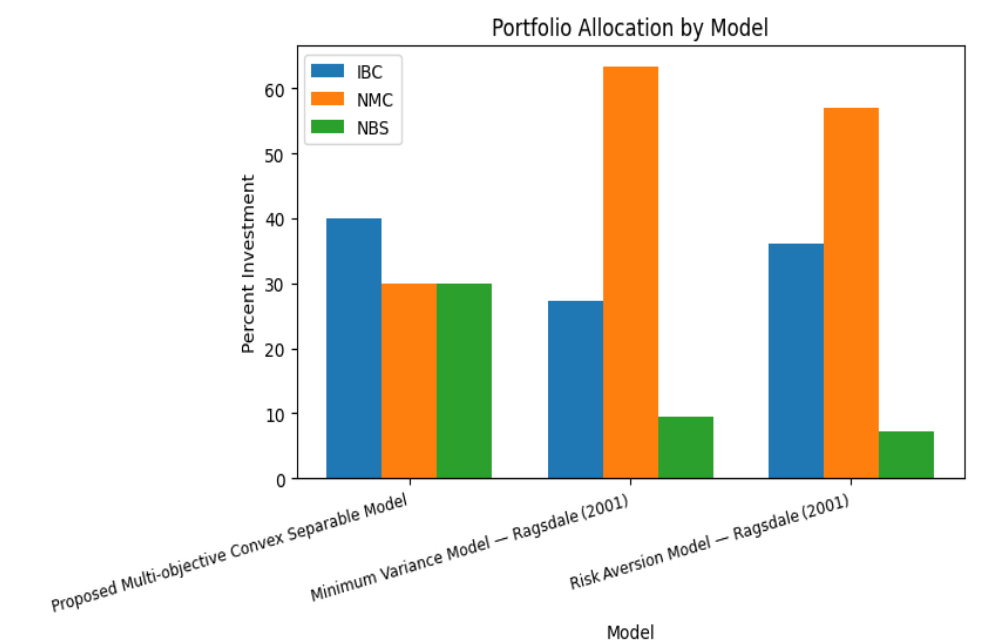


Figure 5. Graph of the proposed multi objective convex separable model and the models of Ragsdale [50].

Compare the proposed MOCSP technique with the risk aversion model of Ragsdale [50]. It is clear that the proposed results are more risk avoiding with greater returns. This situation is shown in Table (19) and figure (6). Table (20) show that percent return of the proposed model is 0.225% greater than the returns of risk aversion model of Ragsdale [50].

Table 19. Risk taking and risk aversion of proposed multi objective convex separable model and the models of Ragsdale [50].

	Percent Risk Aversion	Percent Risk Taking	Percent Return
Proposed Multi objective Convex Separable Model	40	60	14.04
Risk Aversion Model Ragsdale [50]	36	64	11.46

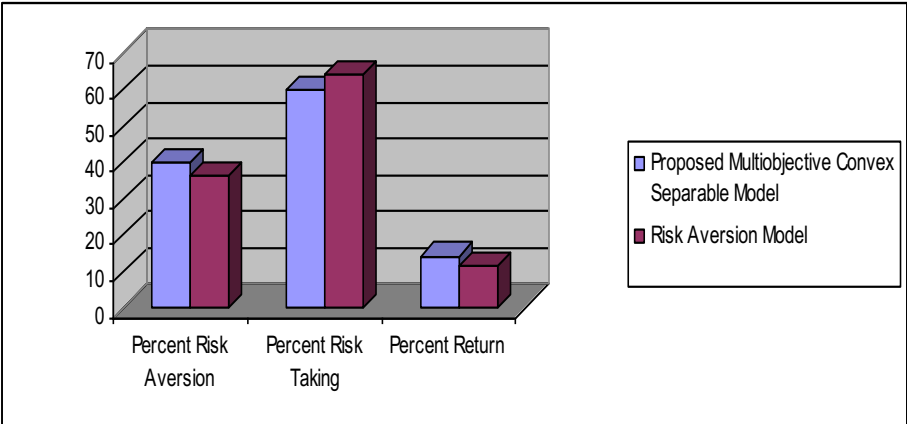


Figure 6. Risk aversion and risk taking of the proposed multi objective convex separable model and the risk aversion model of Ragsdale [50].

Table 20. Comparison and contribution of proposed multi objective convex separable model with the solutions of existing methods in literature.

		Difference	Percent Difference
Return of the Proposed Multi Objective Separable Programming Model	14.04		
Return of the Minimum Variance Model Ragsdale [50]	11.99	2.05	0.1709
Return of the Proposed Multi objective Separable Programming Model	14.04		
Return of the Risk Aversion Model Ragsdale [50]	11.46	2.58	0.225
Mansoori et al., [45] at uncertainty level $\alpha = 0.2, w_1 = \frac{1}{4}, w_2 = \frac{3}{4}$	-1.5559		
Mansoori et al., [45] at uncertainty level $\alpha = 0.5, w_1 = \frac{1}{4}, w_2 = \frac{3}{4}$	-1.8000	-0.2441	0.156
Primal of the Wu et al., [59] model 5, and Malek & Oskoei [43] New 3 model, Table 5	-224.99		
Dual of the Wu et al., [59] model 5, and Malek & Oskoei [43] New 3 model, Table 5	-225.02	-0.01	0.00004
Mansooriet al., [44] at uncertainty level $\alpha = 0.0, w_1 = \frac{1}{2}, w_2 = \frac{1}{2}$, Table 1	-18.97		
Mansooriet al., [44] at uncertainty level $\alpha = 1, w_1 = \frac{1}{2}, w_2 = \frac{1}{2}$, Table 1	-21.00	-2.03	0.10
Coelho [21] at uncertainty level $\alpha = 0.0$	439.57		
Coelho [21] at uncertainty level $\alpha = 1$	504.11	64.54	0.1468
Khan et al., [37] Adadelta Starting value	199		
Khan et al., [37] Adadelta Ending Value	163	36	0.18
Khan et al., [37] Adam Starting value	94		
Khan et al., [37] Adam Ending Value	85	9	0.09
Khan et al., [37] RMSprop Starting value	122		
Khan et al., [37] RMSprop Ending Value	118	3	0.03

In Table (20) we compare the results of the proposed study with the solutions in the existing literature. Firstly, we compare the returns of the proposed MOCSP technique with the minimum variance model of Ragsdale [50]. Table (20) show that the return of the proposed multi objective convex separable programming technique is 17% more than the returns of the minimum variance model of Ragsdale [50]. Figure (7) show the difference in the returns of the proposed multi objective convex programming approach with the minimum variance model studied in Ragsdale [50], that is 17% improved.

Furthermore, we compare the performance of the proposed method with some related studies (Mansoori et al., [45]; Wu et al., [59]; Malek & Oskoei [43]; Mansoori et al., [44]; Coelho, [21]). Table (20) shows that the performance of the technique adopted in Mansoori et al., [45] improve the objective function by 15% with uncertainty level of 0.2 to 0.5, while the proposed models improve 22.5% and 17%.

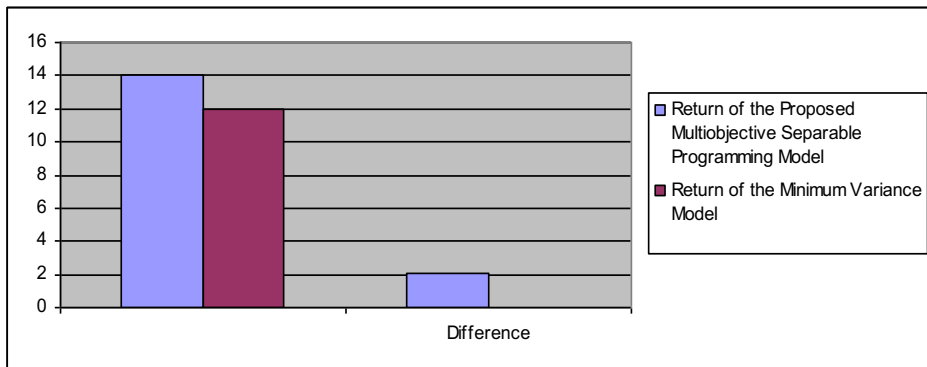


Figure 7. Difference in returns of proposed multi objective convex separable model with the minimum variance model of the model of Ragsdale [50].

Table (20) shows that the Primal of the Wu et al., [59] model 5, and Malek & Oskoei [43] New 3 model, Table 5, minutely improves the objective function by 0.004%. Again the proposed techniques improve the objectives by 22.5% and 17% for the risk aversion and minimum variance models, respectively. Mansooriet al., [44] Table 1, at uncertainty level $\alpha = 0.0$, to $\alpha = 1$, reported 10% improvement in objective function. Coelho [21], at uncertainty level $\alpha = 0.0$, to $\alpha = 1$, reported 14% improvement in objective function. The proposed techniques reported 22.5% and 17% improvement in the objective function. Thus it is better than (Mansoori et al., [44]; Coelho, [21]). The Adadelta, Adam, and RMSprop models of the Khan et al., [37] brings improvements of 18%, 9%, and 3%, respectively, in the objective function. Thus, the current results are superior to Khan et al, [37].

From the above discussion it can be easily deducted that the returns of the proposed MOCSP approach are 22.5% and 17% improved from the minimum variance model and risk aversion model studied in (Ragsdale [50]; Mansoori et al., [45]; Wu et al., [59]; Malek

& Oskoei [43]; Mansoori et al., [44]; Coelho, [21]; Khan et al., [37]). Thus, the proposed method is promising, credible and more beneficial for portfolio optimization in capital market structure. Convex separable programming problems having promising applications in farm planning Thomas [58], hydro power generation Ellis & Revelle [29] and production planning Stefanov [54], can easily be adopted in multi objective convex separable case in future, besides the theory development (Spingarn [52]; Ali et al., 1986; Fukushima [31]; Chen et al., [18]; Li et al., [42], Sun et al., [56]) and (Akhil & Sundaresan [3]; Abrham & Arri [1]; Zwick [60]; Stefanov, [53-54]; Sui & Peng [55]).

The key performance indicators are listed below.

1. The performance evaluation indicator is the optimization of the objective functions.
2. Avoiding the restrictive simplex basis method for solve multi objective convex separable programming problem.
3. Convex optimization ensure a global optimal solution within the constrained resources.
4. Python is a popular software in data science. It allows data analysts and professionals to carrying complex statistical computations, visualizing the data, building machine learning models, manipulating and analyzing the data.

The study have some limitations. The main limitations are listed below.

1. The main limitation is that the study requires many computational iterations. The effects of many objective functions are included into one model. This process takes many iterative solutions and then combining into one representative model.
2. The accuracy is impeccable by the coarseness of the piece wise linear estimations utilized in the solution process.

7. Conclusions

In this research, a new technique namely MOCSP has been developed and solved using scikit-learn and keras libraries of the Python. The machine learning techniques of XGBoost, neural network, decision forest have been utilized to solve the three MOCSP problems formulated. The minimum variance and risk aversion models have been developed. These models have been combined as a multi objective problem. The results can highlight balanced portfolios to the investors that can be a diverse combination of optimal portfolio returns and risk aversion at the same time. The MAE and RMSE for both XGBoost and decision forest regression have a minute value of 0.0001. For the neural network regression MAE and RMSE are 1% and 2%, respectively. F1 score and accuracy for XGBoost are 91%, while these values for neural network regression are 98%, and for decision forest are 92%, respectively. The main advantage is that the problems is converted to a linear programming problem and then easily solved. It does not require the adjacency requirements. The main disadvantage is that it requires many computational iterations. The accuracy may be compromised by the choice of the piece wise linear estimations. In future the proposed method can be applied to farm planning problem Thomas [58], hydro power generation Ellis & Revelle [29] and production planning Stefanov [54], besides the theory

development (Spingarn [52]; Ali et al., [6]; Fukushima [31]; Chen et al., [18]; Li et al., [42], Sun et al., [56]) and (Akhil & Sundaresan [3]; Abraham & Arri [1]; Zwick [60]; Stefanov [53-54]; Sui & Peng [55]).

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This article does not contain any studies with human participants performed by any of the authors.

Conflict of Interest:

The authors declare that they have no conflict of interest.

Authors Credit:

Izaz Ullah Khan contributed the main idea, conceptualization, writing, software, and supervision. **Zahoor Ahmad** performed data curation, resources, writing, and investigation. **Mehran Ullah** performed formal analysis, validation, reviewing, editing, and supervision. **Muhammad Shahbaz Shah** contributed to graphics, resources, software, and visualization.

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