Control and Cybernetics
vol. **54** (2025) No. 3
pages: 289-308
DOI: 10.2478/candc-2025-0010

Optimality conditions for ε-weak local quasi-efficient solutions of D.C. vector optimization problems*

by

Nazih Abderrazzak Gadhi and Imad Amrani Zerrifi

LIMAS, FSDM, Sidi Mohamed Ben Abdellah University, Fez, Morocco.
abderrazzak.gadhinazih@usmba.ac.ma
izerrifiamrani@gmail.com

Abstract: In this paper, we address a non-convex vector optimization problem, in which the objective function and constraints are defined as differences of convex vector-valued maps. By employing a separation argument, we derive necessary optimality conditions, expressed in terms of ε-regularized subdifferentials, for a point to be an ε-weak local quasi-efficient solution. To ensure the paper is self-contained, we also present sufficient optimality conditions and provide examples to illustrate the results.

Keywords: D.C. mappings (difference-of-convex-mappings); ε-weak quasi-efficient solutions; ε-regularized subdifferentials; optimality conditions.

1. Introduction

In optimization programming, finding an exact minimizer is not always possible, leading to the development of approximate solutions, such as ε-solutions and ε-quasi-solutions, which are essential in algorithmic analysis. This concept, introduced by Loridan (1982), is notably inspired by Ekeland's Variational Principle (Ekeland, 1974). Subsequently, many researchers explored approximate solutions in convex and nonconvex optimization problems, establishing necessary conditions for these solutions under various constraint qualifications (Attouch and Rishi, 1993; Dutta, 2005; Hamel, 2001; Mordukovich and Wang, 2002).

Over the past decades, the concept of approximate solutions for multiobjective optimization problems (MOPs) has garnered significant attention from researchers, due to several reasons. On the one hand, iterative and heuristic

*Submitted: September 2024; Accepted: November 2025

algorithms inevitably produce approximate solutions. On the other hand, while the set of efficient solutions may be empty, under relatively mild conditions, the set of approximately efficient points can still be nonempty. The concept of ε-approximate solutions was extended to MOPs by Loridan (1984). Since then, several researchers have investigated approximate solutions in the context of multiobjective optimization (Gadhi, 2024; Ghaznavi-Ghosoni and Khorran, 2011; Li and Wang, 1998; Liu, 1999). For example, Li and Wang ((1998) introduced the concept of ε-proper efficiency and established the necessary conditions for ε-properly efficient points in a general MOP, using a scalarization technique. Later, Liu ((1999) derived a necessary and sufficient condition for ε-properly efficient points in a convex MOP. Then, Ghaznavi-Ghosoni and Khorram (2011) established the necessary and sufficient conditions for approximate weakly efficient points in an MOP.

In this paper, we study the optimality conditions for ε-weak quasi-efficient solutions in the context of difference-of-convex (D.C.) vector optimization problems

$$(P): \quad \begin{cases} K_Y - \text{Min} & G(x) - H(x) \\ \text{subject to}: & x \in C \text{ and } G_1(x) - H_1(x) \in -K_Z, \end{cases}$$

where X is a normed linear space, Y and Z are topological vector spaces, $K_Y \subset Y$ and $K_Z \subset Z$ are pointed, closed, and convex cones with nonempty interiors, $G,\ H : X \to Y$ are K_Y-convex mappings, $G_1,\ H_1 : X \to Z$ are K_Z-convex mappings and C is a closed and convex subset of X.

The abbreviation D.C. refers to cases, in which the objective and/or constraint functions are expressed as differences of convex functions. Notably, many nonconvex mappings encountered in nonsmooth optimization fall into this category. Over the past few decades, extensive research has been conducted on nonconvex vector optimization problems (see Benslimane, Gadhi and Zerrifi, 2024; Gadhi, 2024; Gadhi and Metrane, 2004; Laghdir and Metrane, 2005; Guo and Li, 2014; Khazayed and Farajzadeh, 2022; Shafie and Bozorgnia, 2019; Su and Hang, 2022). For instance, Su and Hang (2022) established Kuhn-Tucker and Fritz John necessary optimality conditions for local weak minimizers in fractional multiobjective programming problems involving set, cone, and equality constraints, along with sufficient optimality conditions for these minimizers. Gadhi and Metrane (2004) employed the Diff-Max concept and ε-subdifferentials for convex mappings to derive sufficient optimality conditions for a D.C. vector optimization problem of type (P). Meanwhile, Gadhi, Laghdir and Metrane (2005) provided global necessary and sufficient optimality conditions for D.C. vector optimization problems with reverse convex constraints. Guo and Li (2014) formulated necessary and sufficient conditions for an ε-weak Pareto minimal point and an ε-proper Pareto minimal point, using an alternative theorem in conjunction with the concept of approximate pseudodissipativity (Penot, 2011). Having challenged some of Guo and Li's results using counterexamples, Shafie and Bozorgnia (2019) and Khazayel and Farajzadeh (2022) proposed alternative optimality conditions. Recently, Gadhi (2024) established sufficient optimality conditions for (P) using ε-approximate pseudo-dissipativity, and, in the process, corrected certain existing results from Guo and Li (2024), Khazayed and Farajzadeh (2022) and Shafie and Bozorgnia (2019)

All the results cited above were formulated using ε-subdifferentials for convex mappings, where the perturbation $\varepsilon \in K_Y$ is assumed to be constant. But what if this perturbation were not constant and, instead, decreased as we approached the point under consideration? Such a technique could lead to more precise and refined optimality conditions, taking into account the local behavior of the function. Additionally, given the convex nature of the functions defining the problem, we wonder if it is possible to directly derive the optimality conditions, without relying on alternative theorems for the necessary conditions or approximate pseudo-dissipativity for the sufficient ones.

To address the questions that have just been raised, we have undertaken this work. Our approach consists of using the Hahn-Banach separation theorem and ε-regularized subdifferentials to establish the necessary conditions for a point to be an ε-weak local quasi-efficient solution of problem (P). With these results established, and using a result of Dür (2003), related to local minima of D.C. functions, we provide sufficient conditions for ε-weak quasi-efficiency and illustrate our findings with examples. When $\varepsilon = 0_Y$, we recover certain results that have already been published in the literature. To the best of our knowledge, no previous work has addressed optimality conditions for ε-weak local quasi-efficiency in D.C. vector optimization problems using ε-regularized subdifferentials.

The remainder of the paper is organized as follows: Section 2 presents the basic definitions and preliminary material from nonsmooth variational analysis. Sections 3 and 4 discuss the main results, focusing on the optimality conditions. Finally, Section 5 provides the conclusion.

2. Preliminaries

In this paper, we consider X, Y and Z as topological vector spaces, where the origins of Y and Z are denoted by 0_Y and 0_Z, respectively. The dual spaces of Y and Z are denoted by Y^* and Z^*, respectively, with the duality pairing represented by $\langle .,. \rangle$. Let $L(X,Y)$ denote the set of all linear continuous operators from X to Y. We define $K_Y \subset Y$ (and similarly $K_Z \subset Z$) as a proper, convex, and closed cone with a nonempty interior (i.e., $intK_Y \neq \emptyset$), which induces a partial order in Y (and in Z), defined by

$$y \leq_K z \Longleftrightarrow z - y \in K_Y .$$

The positive dual cone K_Y^* of K_Y is defined by

$$K_Y^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \text{ for all } y \in K_Y\}.$$

Note that when $Y = \mathbb{R}$, the cone K_Y is defined as $[0, +\infty[$, and consequently, its dual cone K_Y^* is also $[0, +\infty[$. For a subset Ξ of X, δ_Ξ stands for the indicator function, defined by

$$\delta_\Xi(x) := \begin{cases} 0 & \text{if} \quad x \in \Xi, \\ +\infty & \text{if} \quad x \notin \Xi. \end{cases}$$

The following property, established by Craven (1995), will be used throughout the rest of the paper.

Proposition 1 (Craven, 1995) *If $intK_Y \neq \emptyset$, then*

$$y \in intK_Y \Longleftrightarrow \langle y^*, y\rangle > 0, \ \forall y^* \in K_Y^* \setminus \{0\}.$$

Let C be a nonempty convex subset of X. Given that convexity is crucial to the following investigations, let us revisit the following concept, see Zowe (1974). A vector-valued map $F : X \to Y$ is said to be K_Y-convex on C if for all $x_1, \ x_2 \in C$ and $0 \leq \lambda \leq 1$, one has

$$\lambda F(x_1) + (1-\lambda) F(x_2) - F(\lambda x_1 + (1-\lambda) x_2) \in K_Y.$$

Given $\varepsilon \in K_Y$ and $\overline{x} \in X$, the ε-subdifferential of F at $\overline{x}$ and the ε-regularized subdifferential of F at $\overline{x}$ are defined by

$$\partial_\varepsilon F(\overline{x}) := \{T \in L(X,Y) : F(x) - F(\overline{x}) - T(x - \overline{x}) + \varepsilon \in K_Y, \ \forall x \in X\}$$

and

$$\partial_\varepsilon^r F(\overline{x}) :=$$
$$\{T \in L(X,Y) : F(x) - F(\overline{x}) - T(x - \overline{x}) + \varepsilon \|x - \overline{x}\| \in K_Y, \ \forall x \in X\},$$

respectively. The key distinction between ε-subdifferentials and ε-regularized subdifferentials lies in the perturbation: in the ε-subdifferential, the perturbation ε is constant, while in the ε-regularized subdifferential, ε is scaled by the norm $\|x - \overline{x}\|$, making it more sensitive to the distance between x and $\overline{x}$. This implies that the perturbation's effect diminishes as x approaches $\overline{x}$. Consequently, $\partial_\varepsilon^r F(\overline{x})$ refines $\partial_\varepsilon F(\overline{x})$, offering a potentially smaller and more precise set of subgradients by adapting the perturbation to the local geometry around x. Observe that both $\partial_\varepsilon F(\overline{x})$ and $\partial_\varepsilon^r F(\overline{x})$ reduce to the strong subdifferential $\partial F(\overline{x})$ (Zowe, 1974), given by

$$\partial F(\overline{x}) := \{T \in L(X,Y) : F(x) - F(\overline{x}) - T(x - \overline{x}) \in K_Y, \ \forall x \in X\},$$

whenever $\varepsilon = 0_Y$. It is worth mentioning that $\partial_\varepsilon^r F(\overline{x}) = \partial (F + \varepsilon \|. - \overline{x}\|)(\overline{x})$ and that $\partial F(\overline{x}) \subseteq \partial_\varepsilon^r F(\overline{x})$ for all $\varepsilon \in K_Y$, which implies that Zălinescu's conditions (Zălinescu, 2008, Corollary 4) for ensuring the nonemptiness of the strong subdifferential $\partial F(\overline{x})$ are also sufficient to guarantee the nonemptiness of the ε-regularized subdifferential $\partial_\varepsilon^r F(\overline{x})$.

Example 1 *Let $\overline{x} := 0$ and consider the function $F(x) := e^{x^2}$. In this case, we have*

$$\partial_\varepsilon F(\overline{x}) = \left[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\right] \text{ and } \partial_\varepsilon^r F(\overline{x}) = [-\varepsilon, \varepsilon], \ \forall \varepsilon > 0,$$

which implies that

$$\partial_\varepsilon^r F(\overline{x}) \subsetneqq \partial_\varepsilon F(\overline{x}), \ \forall 4 > \varepsilon > 0.$$

Clearly, this example shows that necessary optimality conditions expressed in terms of ε-regularized subdifferentials, rather than ε-subdifferentials, can provide sharper conditions.

To progress, we need the following definition.

Definition 1 *Let $\varepsilon \in K_Y$ and let Ω be the set of all feasible solutions of (P), defined by*

$$\Omega := \{x \in C \mid G_1(x) - H_1(x) \in -K_Z\}.$$

A feasible vector $\overline{x} \in \Omega$ is said to be a ε-weak local quasi efficient solution of problem (P) iff there exists a neighborhood U of $\overline{x}$ such that for all $x \in U \cap \Omega$, we have

$$(G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) + \varepsilon \|x - \overline{x}\| \notin -intK_Y. \tag{1}$$

A feasible vector $\overline{x} \in \Omega$ is said to be a weak local efficient solution of problem (P) iff there exists a neighborhood U of $\overline{x}$ such that for all $x \in U \cap \Omega$, we have

$$(G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) \notin -intK_Y. \tag{2}$$

It is worth noting that any weak local efficient solution of problem (P) is also an ε-weak local quasi-efficient solution. However, the converse does not necessarily hold, as demonstrated below.

Example 2 *Let $\overline{x} = 0$, $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = [-2, 2]$, $K_Y = \mathbb{R}^2_+$, $K_Z = \mathbb{R}_+$ and $\varepsilon = (1, 0)$. Consider the functions G, H, G_1, $H_1 : \mathbb{R} \to \mathbb{R}^2$, defined by*

$$G(x) = \left(x^2, -\sqrt{x}\right), \ H(x) = \left(|x|, x^2\right), \ G_1(x) = x^2 \text{ and } H_1(x) = x.$$

On the one hand, we have $\Omega = [0,1]$, *with* $G,\ H$ *being* K_Y*-convex on* Ω, *and* $G_1,\ H_1$ *being* K_Z*-convex on* Ω. *As*

$$(G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) + \varepsilon \|x - \overline{x}\| = \left(x^2, -\sqrt{x} - x^2\right) \notin -int\mathbb{R}^2_+,$$

we conclude that $\overline{x} \in \Omega$ *is an* ε*-weak local quasi efficient solution of problem* (P). *On the other hand,* $\overline{x}$ *is not a weak local efficient solution of problem* (P). *Indeed, for any neighbourhood* U *of* $\overline{x}$ *and for sufficiently large* n, *we have* $\frac{1}{n} \in U \cap \Omega$, *and*

$$\left(G\left(\tfrac{1}{n}\right) - H\left(\tfrac{1}{n}\right)\right) - (G(\overline{x}) - H(\overline{x})) = \left(\tfrac{1}{n^2} - \tfrac{1}{n}, \tfrac{1}{\sqrt{n}} - \tfrac{1}{n^2}\right) \in -intK_Y.$$

The following example illustrates an important property of the sequence of approximate weak efficient solutions, namely, that it converges to a weak efficient solution of problem (P).

EXAMPLE 3 *Let* $\{\varepsilon_n\}_{n\geq 1}$ *and* $\{h_n\}_{n\geq 1}$ *be sequences such that* $0 \leq h_n \leq \frac{\varepsilon_n}{2}$ *and* $\varepsilon_n \underset{n\longmapsto\infty}{\rightarrow} 0$. *Define the functions* $F,\ G : \mathbb{R} \to \mathbb{R}^2$ *by*

$$G(x) = (G_1(x), G_2(x)) := \left(x^2, x^4\right) \text{ and } H(x) = (H_1(x), H_2(x)) := \left(2|x|, x^2\right), \quad \forall x \in \mathbb{R}.$$

It can be verified that $\widetilde{x} = 1$ *and* $\widehat{x} = -1$ *are weak global efficient solutions of the problem*

$$(Q): \quad \begin{cases} \mathbb{R}^2_+ - Min & G(x) - H(x) \\ subject\ to: & x \in \mathbb{R} \end{cases}$$

with

$$G(\widetilde{x}) - H(\widetilde{x}) = (-1, 0).$$

The sequence $x_n := 1 + h_n$, *which converges to* $\widetilde{x}$, *is an* $(\varepsilon_n, \varepsilon_n)$*-weak global quasi-efficient solution of* (Q). *Indeed, let*

$$\Psi(x) = (\Psi_1(x), \Psi_2(x)) \text{ where } \Psi_1(x) := G_1(x) - H_1(x) \text{ and}$$
$$\Psi_2(x) := G_2(x) - H_2(x), \forall x \in \mathbb{R}.$$

Then,

$$\Psi(x) = \left(x^2 - 2|x|, x^4 - x^2\right), \ \Psi_1(x) := x^2 - 2|x| \text{ and } \Psi_2(x) = x^4 - x^2,$$
$$\forall x \in \mathbb{R}.$$

For n large enough and x sufficiently close to x_n, *we have* $x > 0$ *and* $\varepsilon_n \leq 1$. *As*

$$\Psi_1(x) - \Psi_1(x_n) + \varepsilon_n |x - x_n| = x^2 - 2x + 1 - h_n^2 + \varepsilon_n |x_n - x|$$

we can write

$$\Psi_1(x) - \Psi_1(x_n) + \varepsilon_n |x - x_n| = (x-1)^2 - h_n^2 + \varepsilon_n \ |x_n - x| .$$

Since

$$(x-1)^2 = (x - x_n + x_n - 1)^2 = (x - x_n)^2 + 2(x - x_n) h_n + {h_n}^2$$

we obtain

$$\Psi_1(x) - \Psi_1(x_n) + \varepsilon_n |x - x_n| = (x - x_n)^2 + (x - x_n)(2h_n + \varepsilon_n \ sign(x - x_n)) .$$

As $0 \leq h_n \leq \frac{\varepsilon_n}{2}$, *we have*

$$(x - x_n)(2h_n + \varepsilon_n \ sign(x - x_n)) = \begin{cases} (x - x_n)(2h_n + \varepsilon_n) \geq 0 & if \quad x - x_n \geq 0, \\ (x - x_n)(2h_n - \varepsilon_n) \geq 0 & if \quad x - x_n < 0, \end{cases}$$

and thus

$$\Psi_1(x) - \Psi_1(x_n) + \varepsilon_n |x - x_n| \geq 0.$$

Consequently,

$$\Psi(x) - \Psi(x_n) + \varepsilon_n |x - x_n| \notin -int\mathbb{R}^2_+ .$$

3. Necessary optimality conditions

The following theorem provides the necessary optimality conditions for an ε-weak local quasi-efficient solution to the cone-constrained vector optimization problem (P), where the objective function and constraint set are represented by differences of two vector-valued maps. Since problem (P) encompasses various models of vector optimization problems, several corollaries are presented to establish necessary optimality conditions for an ε-weak local efficient solution in each case. Additionally, when $\varepsilon = 0_Y$, these conditions reduce to the necessary optimality conditions for a weak local efficient solution for the respective vector optimization problems.

Theorem 1 *Let $\overline{x} \in \Omega$ and $\varepsilon \in K_Y$. Suppose that $\overline{x}$ is an ε-weak local quasi-efficient solution of problem (P). Then, for all $T \in \partial_\varepsilon^r H(\overline{x})$ and $T_1 \in \partial H_1(\overline{x})$, there exist $y^* \in K_Y^*$ and $z^* \in K_Z^*$ such that $(y^*, z^*) \neq (0_{Y^*}, 0_{Z^*})$ and*

$$\left\{ \begin{array}{c} y^* \circ T + z^* \circ T_1 \in \partial_\rho^r (y^* \circ G + z^* \circ G_1 + \delta_C)(\overline{x}), \\ \langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle = 0, \end{array} \right. \tag{3}$$

where $\rho := \langle y^*, 2\varepsilon \rangle$.

Proof The proof of this theorem is structured in three steps: the initial setup and construction of a key set Ξ, the establishment of the main properties of Ξ, and the derivation of the multiplier rule, using the Hahn-Banach separation theorem.

A - **Initial setup**
Since $\overline{x}$ is an ε-weak local quasi-efficient solution of problem (P), we can find a neighborhood U of $\overline{x}$ such that

$$G(x) - H(x) - G(\overline{x}) + H(\overline{x}) + \varepsilon \|x - \overline{x}\| \notin -intK_Y, \ \forall x \in U \cap \Omega. \tag{4}$$

Let $T \in \partial_\varepsilon^r H(\overline{x})$ and $T_1 \in \partial H_1(\overline{x})$ be arbitrarily chosen. Consider the set

$$\Xi := \{(p, q) \in Y \times Z : \exists x \in C \cap U\}$$
such that
$$\left. \begin{array}{c} G(x) - G(\overline{x}) - T(x - \overline{x}) - p + 2\varepsilon \|x - \overline{x}\| \in -intK_Y \\ G_1(x) - H_1(\overline{x}) - T_1(x - \overline{x}) - q \in -intK_Z \end{array} \right\}.$$

B - **Properties of the set Ξ**
Let us prove that Ξ is a nonempty, convex, and open subset of $Y \times Z$ that does not contain the origin.

(i) Ξ is nonempty, because $(\nu, s + G_1(\overline{x}) - H_1(\overline{x})) \in \Xi$, for all $\nu \in intK_Y$ and $s \in intK_Z$.

(ii) The origin is not an element of Ξ. On the contrary, assume there exists $x \in U \cap C$ such that

$$\left\{ \begin{array}{c} G(x) - G(\overline{x}) - T(x - \overline{x}) + 2\varepsilon \|x - \overline{x}\| \in -intK_Y \\ G_1(x) - H_1(\overline{x}) - T_1(x - \overline{x}) \in -intK_Z. \end{array} \right. \tag{5}$$

Since $T \in \partial_\varepsilon^r H(\overline{x})$ and $T_1 \in \partial H_1(\overline{x})$, one has

$$H(x) - H(\overline{x}) - T(x - \overline{x}) + \varepsilon \|x - \overline{x}\| \in K_Y \text{ and } H_1(x) - H_1(\overline{x}) - T_1(x - \overline{x}) \in K_Z, \tag{6}$$

for all $x \in X$.
By combining (5) and (6), we obtain

$$G(x) - G(\overline{x}) + \varepsilon \|x - \overline{x}\| \in H(x) - H(\overline{x}) - K_Y - intK_Y$$

and

$$G_1(x) - H_1(\overline{x}) \in H_1(x) - H_1(\overline{x}) - K_Z - intK_Z.$$

This simplifies to:

$$\begin{cases} (G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) + \varepsilon \|x - \overline{x}\| \in -intK_Y, \\ G_1(x) - H_1(x) \in G_1(\overline{x}) - H_1(\overline{x}) - intK_Z \subset -K_Z - intK_Z \subset -K_Z. \end{cases}$$

And this leads to a contradiction with (4), since $x \in \Omega$.

(iii) Ξ is a convex set. Indeed, let $\lambda \in [0,1]$, $(p_1, q_1) \in \Xi$ and $(p_2, q_2) \in \Xi$. By definition, there exist $x_1 \in U \cap C$ and $x_2 \in U \cap C$ such that

$$\begin{cases} G(x_1) - G(\overline{x}) - T(x_1 - \overline{x}) - p_1 + 2\varepsilon \|x_1 - \overline{x}\| \in -intK_Y \\ G_1(x_1) - H_1(\overline{x}) - T_1(x_1 - \overline{x}) - q_1 \in -intK_Z, \end{cases}$$

and

$$\begin{cases} G(x_2) - G(\overline{x}) - T(x_2 - \overline{x}) - p_2 + 2\varepsilon \|x_2 - \overline{x}\| \in -intK_Y \\ G_1(x_2) - H_1(\overline{x}) - T_1(x_2 - \overline{x}) - q_2 \in -intK_Z. \end{cases}$$

Consequently, there exist $r_1,\ r_2 \in intK_Y$ and $t_1,\ t_2 \in intK_Z$ such that

$$\begin{cases} -r_1 = G(x_1) - G(\overline{x}) - T(x_1 - \overline{x}) - p_1 + 2\varepsilon \|x_1 - \overline{x}\|, \\ -r_2 = G(x_2) - G(\overline{x}) - T(x_2 - \overline{x}) - p_2 + 2\varepsilon \|x_2 - \overline{x}\|, \\ -t_1 = G_1(x_1) - H_1(\overline{x}) - T_1(x_1 - \overline{x}) - q_1, \\ -t_2 = G_1(x_2) - H_1(\overline{x}) - T_1(x_2 - \overline{x}) - q_2 \end{cases}.$$

Fix

$$x := \lambda x_1 + (1-\lambda) x_2,\ r := \lambda r_1 + (1-\lambda) r_2,\ t := \lambda t_1 + (1-\lambda) t_2,$$

$$p := \lambda p_1 + (1-\lambda) p_2 \text{ and } q := \lambda q_1 + (1-\lambda) q_2.$$

Using the convexity assumption of G and G_1, the convexity of $u \mapsto \|u - \overline{x}\|$, and the properties $K_Y + K_Y \subseteq K_Y$ and $K_Z + K_Z \subseteq K_Z$, we obtain:

$$p - r + T(x - \overline{x}) \in G(x) - G(\overline{x}) + 2\varepsilon \|x - \overline{x}\| + K_Y$$

and

$$q - t + T_1(x - \overline{x}) \in G_1(x) - H_1(\overline{x}) + K_Z.$$

Thus, there exist $r_0 \in K_Y$ and $t_0 \in K_Z$ such that

$$G(\overline{x}) - G(x) + p + T(x-\overline{x}) = r + r_0 + 2\varepsilon\,\|x-\overline{x}\|$$
and
$$H_1(\overline{x}) - G_1(x) + q + T_1(x-\overline{x}) = t + t_0.$$

Since

$$\begin{cases} r + r_0 \in intK_Y + K_Y \subset intK_Y, \\ t + t_0 \in intK_Z + K_Z \subset intK_Z, \end{cases}$$

it follows that

$$G(x) - G(\overline{x}) - p - T(x-\overline{x}) + 2\varepsilon\,\|x-\overline{x}\| \in -intK_Y$$
and
$$G_1(x) - H_1(\overline{x}) - q - T_1(x-\overline{x}) \in -intK_Z.$$

Therefore, $(p,q) \in \Xi$.

(iv) Ξ is an open set. Indeed, consider $(p,q) \in \Xi$. From the definition, there exists $x \in U \cap C$ such that

$$\begin{cases} G(x) - G(\overline{x}) - T(x-\overline{x}) - p + 2\varepsilon\,\|x-\overline{x}\| \in -intK_Y \\ G_1(x) - H_1(\overline{x}) - T_1(x-\overline{x}) - q \in -intK_Z. \end{cases}$$

Consequently, there exist $r \in intK_Y$ and $t \in intK_Z$ such that

$$\begin{cases} G(x) - G(\overline{x}) - T(x-\overline{x}) - p + 2\varepsilon\,\|x-\overline{x}\| = -r \\ G_1(x) - H_1(\overline{x}) - T_1(x-\overline{x}) - q = -t. \end{cases}$$

Since $intK_Y$ and $intK_Z$ are open sets, there exists $\delta > 0$ such that

$$\begin{cases} r + a \in intK_Y,\ G_1(\overline{x}),\ t + b \in intK_Z, \\ G(x) - T(x-\overline{x}) - (p+a) + 2\varepsilon\,\|x-\overline{x}\| = G(\overline{x}) - r - a, \\ G_1(x) - T_1(x-\overline{x}) - (q+b) = H_1(\overline{x}) - t - b, \end{cases}$$

for all $(a,b) \in \mathbb{B}_{Y\times Z}(0,\delta)$.
Consequently,

$$\begin{cases} G(x) - T(x-\overline{x}) - (p+a) + 2\varepsilon\,\|x-\overline{x}\| \in G(\overline{x}) - intK_Y \\ H(x) - T_1(x-\overline{x}) - (q+b) \in H_1(\overline{x}) - intK_Z, \end{cases}$$

for all $(a,b) \in \mathbb{B}_{Y\times Z}(0,\delta)$. Then, $(p,q) + \mathbb{B}_Y(0,\delta) \subset \Xi$. This shows that Ξ is open.

C - Considering the aforementioned properties of the set Ξ and applying a separation theorem, we can find that $(0_{Y^*}, 0_{Z^*}) \neq (y^*, z^*) \in Y^* \times Z^*$ such that

$$\langle (y^*, z^*), (p, q) \rangle \geq 0 \text{ for all } (p, q) \in \Xi.$$

Let $x \in U \cap C,\ r \in intK_Y,\ s \in intK_Z$ and $\tau, \sigma > 0$. Taking

$$p = G(x) - G(\overline{x}) - T(x - \overline{x}) + \tau r + 2\varepsilon \|x - \overline{x}\|$$

and

$$q = G_1(x) - H_1(\overline{x}) - T_1(x - \overline{x}) + \sigma s,$$

one has $(p, q) \in \Xi$; then

$$\begin{aligned} &\langle y^*, G(x) - G(\overline{x}) - T(x - \overline{x}) + \tau r + 2\varepsilon \|x - \overline{x}\| \rangle \\ &+ \langle z^*, G_1(x) - H_1(\overline{x}) - T_1(x - \overline{x}) + \sigma s \rangle \geq 0. \end{aligned} \tag{7}$$

As $\tau > 0$ and $\sigma > 0$ are arbitrary, (7) yields

$$\begin{aligned} &\langle y^*, G(x) - G(\overline{x}) - T(x - \overline{x}) + 2\varepsilon \|x - \overline{x}\| \rangle \\ &+ \langle z^*, G_1(x) - H_1(\overline{x}) - T_1(x - \overline{x}) \rangle \geq 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \langle y^*, G(x) \rangle + \langle z^*, G_1(x) \rangle - \langle y^* \circ T + z^* \circ T_1, x - \overline{x} \rangle + \langle y^*, 2\varepsilon \rangle \|x - \overline{x}\| \\ \geq \langle y^*, G(\overline{x}) \rangle + \langle z^*, H_1(\overline{x}) \rangle. \end{aligned} \tag{8}$$

Considering again (7), for $x = \overline{x}$, we have

$$\tau \langle y^*, r \rangle + \sigma \langle z^*, s \rangle + \langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle \geq 0. \tag{9}$$

Since $\tau > 0$ and $\sigma > 0$ are arbitrary, it follows that

$$\sigma \langle z^*, s \rangle + \langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle \geq 0$$

and

$$\tau \langle y^*, r \rangle + \langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle \geq 0.$$

Hence,

$$\langle z^*, s \rangle \geq -\frac{1}{\sigma} \left(\langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle \right) \text{ and } \langle y^*, r \rangle \geq -\frac{1}{\tau} \langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle,$$

for all $\tau > 0$ and $\sigma > 0$. Letting $\sigma \to +\infty$ and $\tau \to +\infty$, we get

$$\langle z^*, s \rangle \geq 0 \text{ and } \langle y^*, r \rangle \geq 0.$$

As $r \in intK_Y$ and $s \in intK_Z$ are arbitrary, we conclude that $y^* \in K_Y^*$ and $z^* \in K_Z^*$.

- Since $G_1(\overline{x}) - H_1(\overline{x}) \in -K_Z$ and $\varepsilon \in K_Y$, we have

$$\rho = \langle y^*, 2\varepsilon \rangle \geq 0 \text{ and } \langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle \leq 0. \tag{10}$$

With $\sigma > 0$ and $\tau > 0$ being arbitrary, (9) implies

$$\langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle \geq 0. \tag{11}$$

Upon combining (10) and (11), we conclude that

$$\langle z^*, H_1(\overline{x}) \rangle = \langle z^*, G_1(\overline{x}) \rangle. \tag{12}$$

- Now, by combining (8) and (12), we get

$$\begin{aligned} &\langle y^*, G(x) \rangle + \langle z^*, G_1(x) \rangle - \langle y^* \circ T + z^* \circ T_1, x - \overline{x} \rangle + \rho \|x - \overline{x}\| \\ &\geq \langle y^*, G(\overline{x}) \rangle + \langle z^*, G_1(\overline{x}) \rangle, \ \forall x \in C \cap U. \end{aligned}$$

By considering the indicator functions, we obtain

$$\begin{aligned} &\langle y^*, G(x) \rangle + \delta_C(x) + \langle z^*, G_1(x) \rangle - \langle y^* \circ T + z^* \circ T_1, x - \overline{x} \rangle + \rho \|x - \overline{x}\| \\ &\quad \geq \langle y^*, G(\overline{x}) \rangle + \langle z^*, G_1(\overline{x}) \rangle, \ \forall x \in U. \end{aligned}$$

Although this inequality is established only in a neighborhood of $\overline{x}$, the convexity of the scalar function

$$x \longmapsto \langle y^*, G(x) \rangle + \delta_C(x) + \langle z^*, G_1(x) \rangle - \langle y^* \circ T + z^* \circ T_1, x - \overline{x} \rangle + \rho \|x - \overline{x}\|$$

guarantees that

$$y^* \circ T + z^* \circ T_1 \in \partial_\rho^r (y^* \circ G + z^* \circ G_1 + \delta_C)(\overline{x}).$$

This concludes the proof. ∎

The following example illustrates Theorem 1.

Example 4 *Let* $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = [-1, 1]$, $K_Y = \mathbb{R}^2_+$, $K_Z = \mathbb{R}_+$, $\varepsilon = \left(\frac{1}{4}, \frac{1}{4}\right) \in K_Y$ *and* $\overline{x} = 0$. *Consider the functions* $G,\ H,\ G_1,\ H_1 : \mathbb{R} \to \mathbb{R}^2$, *defined by*

$$G(x) = \left(x^4, |x|\right), \ H(x) = \left(x^2, e^x\right), \ G_1(x) = \tfrac{1}{4}x \text{ and } H_1(x) = x^2.$$

It is worth noting that $G,\ H$ *are* K_Y*-convex and* $G_1,\ H_1$ *are* K_Z*-convex. Moreover, we have* $2\varepsilon = \left(\frac{1}{2}, \frac{1}{2}\right)$, $K_Y^* = \mathbb{R}^2_+$, $K_Z^* = \mathbb{R}_+$, $\partial_\varepsilon^r H(\overline{x}) = \left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[\frac{3}{4}, \frac{5}{4}\right]$, $\partial H_1(\overline{x}) = \{0\}$ *and* $\Omega = [-1, 0] \cup \left[\frac{1}{4}, 1\right]$.

- *First and foremost, it is important to highlight that* $\overline{x}$ *is not a weak local efficient solution of* (P). *Indeed,*

$$(G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) = \left(x^4 - x^2, |x| - e^x + 1\right), \ \forall x \in \Omega.$$

For any neighborhood U *of* $\overline{x}$, *there exists a sufficiently large* n *such that* $\frac{1}{n} \in U \cap \Omega$, *with*

$$\tfrac{1}{n^4} - \tfrac{1}{n^2} < 0 \ \textit{and} \ \left|\tfrac{1}{n}\right| - e^{\frac{1}{n}} + 1 < 0.$$

As a result:

$$\left(G\left(\tfrac{1}{n}\right) - H\left(\tfrac{1}{n}\right)\right) - (G(\overline{x}) - H(\overline{x})) = \left(\tfrac{1}{n^4} - \tfrac{1}{n^2}, \left|\tfrac{1}{n}\right| - e^{\frac{1}{n}} + 1\right) \in -intK_Y.$$

- $\overline{x}$ *is an* ε*-weak local quasi-efficient solution of* (P). *Indeed,*

$$(G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) + \varepsilon|x - \overline{x}| = \left(x^4 - x^2 + \tfrac{1}{4}|x|, \tfrac{5}{4}|x| - e^x + 1\right), \ \forall x \in \mathbb{R}.$$

As $x^4 - x^2 + \frac{1}{4}|x| \geq 0$, *for all* $x \in U := \left]-\frac{1}{8}, \frac{1}{8}\right[$, *we deduce that*

$$(G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) + \varepsilon|x - \overline{x}| \notin -int\mathbb{R}^2_+, \ \forall x \in U \cap \Omega = \left]-\tfrac{1}{8}, 0\right].$$

- *Let* $T \in \partial^r_\varepsilon H(\overline{x})$ *and* $T_1 \in \partial H_1(\overline{x})$ *be arbitrarily chosen. For* $y^* = (1,1)$ *and* $z^* = 1$, *we have*

$$\begin{cases} y^* \circ T + z^* \circ T_1 \in \partial^r_{\langle y^*, 2\varepsilon\rangle}(y^* \circ G + z^* \circ G_1 + \delta_C)(\overline{x}), \\ \langle z^*, G_1(\overline{x}) - H_1(\overline{x})\rangle = 0. \end{cases}$$

Indeed, as $\varepsilon = \left(\frac{1}{4}, \frac{1}{4}\right)$, $y^* = (1,1)$, $z^* = 1$, $T_1 = 0$ *and* $T \in \left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[\frac{3}{4}, \frac{5}{4}\right]$, *we have*

$$\rho = \langle y^*, 2\varepsilon\rangle = \tfrac{1}{2} + \tfrac{1}{2} = 1, \ y^* \circ T + z^* \circ T_1 \in \left[\tfrac{1}{2}, \tfrac{3}{2}\right] \ \textit{and} \ \langle z^*, G_1(\overline{x}) - H_1(\overline{x})\rangle = 0.$$

Consequently,

$$y^* \circ G(x) + z^* \circ G_1(x) + \delta_C(x) + \rho\|x - \overline{x}\| = x^4 + \tfrac{1}{4}x + 2|x| + \delta_C(x), \ x \in \mathbb{R}, \tag{13}$$

and

$$(y^* \circ T + z^* \circ T_1)(x - \overline{x}) \leq \tfrac{3}{2}|x|, \ x \in \mathbb{R}. \tag{14}$$

- *Upon combining* (13) *and* (14), *we obtain*

$$(y^* \circ T + z^* \circ T_1)(x - \overline{x}) \leq y^* \circ G(x) + z^* \circ G_1(x) + \delta_C(x) + \rho \|x - \overline{x}\|, \; x \in \mathbb{R},$$

and thus

$$y^* \circ T + z^* \circ T_1 \in \partial_\rho^r (y^* \circ G + z^* \circ G_1 + \delta_C)(\overline{x}).$$

The following result offers necessary optimality conditions for a weak local efficient solution to the vector optimization problem (P), derived from Theorem 1, by setting $\varepsilon = 0_Y$.

COROLLARY 1 *Suppose that $\overline{x} \in \Omega$ is a weak local efficient solution of problem (P). Then, for all $T \in \partial H(\overline{x})$ and $T_1 \in \partial H_1(\overline{x})$, there exist $y^* \in K_Y^*$ and $z^* \in K_Z^*$ such that $(y^*, z^*) \neq (0_{Y^*}, 0_{Z^*})$, and*

$$\begin{cases} y^* \circ T + z^* \circ T_1 \in \partial (y^* \circ G + z^* \circ G_1 + \delta_C)(\overline{x}), \\ \langle z^*, G_1(\overline{x}) - H_1(\overline{x}) \rangle = 0. \end{cases}$$

If $G_1 \equiv 0_Z$ and $H_1 \equiv 0_Z$, then problem (P) reduces to a D.C. vector optimization problem

$$(P_1): \quad \begin{cases} K_Y - \text{Min} & G(x) - H(x) \\ \text{subject to}: & x \in C. \end{cases}$$

This leads to the following corollary, derived from Theorem 1. Notably, in Corollary 2, if $\varepsilon = 0_Y$, we recover a result from Gadhi and Metrane (2005, Theorem 3.2). Furthermore, if we also have $Y = Z = \mathbb{R}$, $K_Y = K_Z = \mathbb{R}_+$ and $C = X$, then we recover a well-known result from Hiriart-Urruty (1989 Proposition 3.1).

COROLLARY 2 *Let $\overline{x} \in C$ and $\varepsilon \in K_Y$. Suppose that $\overline{x}$ is an ε-weak local quasi-efficient solution of problem (P_1). Then, for all $T \in \partial_\varepsilon^r G_1(\overline{x})$, there exist $y^* \in K_Y^* \backslash \{0_{Y^*}\}$ such that*

$$y^* \circ T \in \partial_\rho^r (y^* \circ G + \delta_C)(\overline{x}),$$

where $\rho := \langle y^*, 2\varepsilon \rangle$.

If $H \equiv 0_Y$ and $H_1 \equiv 0_Z$, then problem (P) simplifies to the convex vector optimization problem

$$(P_2): \quad \begin{cases} K_Y - \text{Min} & G(x) \\ \text{subject to}: & x \in C \text{ and } G_1(x) \in -K_Z, \end{cases}$$

leading to the following corollary, based on Theorem 1.

COROLLARY 3 *Let $\overline{x} \in \Omega$ and $\varepsilon \in K_Y$. Suppose that $\overline{x}$ is an ε-weak local quasi-efficient solution of problem (P_2). Then, there exist $y^* \in K_Y^*$ and $z^* \in K_Z^*$ such that $(y^*, z^*) \neq (0_{Y^*}, 0_{Z^*})$, and*

$$\begin{cases} 0 \in \partial_\rho^r \left(y^* \circ G + z^* \circ G_1 + \delta_C\right)(\overline{x}), \\ \qquad \langle z^*, G_1(\overline{x})\rangle = 0, \end{cases}$$

where $\rho := \langle y^*, 2\varepsilon\rangle$.

4. Sufficient optimality conditions

In this section, we present sufficient optimality conditions for the ε-weak local quasi-efficient solution of problem (P).

THEOREM 2 *Let $\overline{x} \in \Omega$, $\varepsilon \in K_Y$. Suppose that there exist a scalar $\gamma > 0$ and vectors $y^* \in K_Y^* \backslash \{0\}$ and $z^* \in K_Z^*$, such that*

$$\partial_\alpha \left(y^* \circ H + z^* \circ H_1\right)(\overline{x}) \subseteq \partial_\alpha \left(y^* \circ G + z^* \circ G_1 + \langle y^*, \varepsilon\rangle \|. - \overline{x}\|\right)(\overline{x}),\ \forall \alpha \in [0, \gamma], \tag{15}$$

and

$$\langle z^*, G_1(\overline{x}) - H_1(\overline{x})\rangle = 0. \tag{16}$$

Then, $\overline{x}$ is an ε-weak local quasi-efficient solution of problem (P).

PROOF By hypothesis, there exist a scalar $\gamma > 0$ and vectors $y^* \in K_Y^* \backslash \{0\}$ and $z^* \in K_Z^*$, such that (15) and (16) are satisfied. According to Dür (2003, Theorem 2.1), $\overline{x}$ is a local minimum of the D.C. function

$$y^* \circ G + z^* \circ G_1 + \langle y^*, \varepsilon\rangle \|. - \overline{x}\| - y^* \circ H - z^* \circ H_1.$$

Consequently, there exists a neighborhood U of $\overline{x}$ such that, for all $x \in U$, the following inequality holds:

$$\begin{aligned} &y^* \circ G(\overline{x}) + z^* \circ G_1(\overline{x}) - y^* \circ H(\overline{x}) - z^* \circ H_1(\overline{x}) \\ &\leq y^* \circ G(x) + z^* \circ G_1(x) - y^* \circ H(x) - z^* \circ H_1(x) + \langle y^*, \varepsilon\rangle \|x - \overline{x}\|. \end{aligned}$$

Let $x \in \Omega \cap U$ be arbitrarily chosen. Since $G_1(x) - H_1(x) \in -K_Z$ and $z^* \in K_Z^*$, it follows that

$$z^* \circ (G_1(x) - H_1(x)) \leq 0.$$

Thus, we obtain

$$y^* \circ G(\overline{x}) + z^* \circ G_1(\overline{x}) - y^* \circ H(\overline{x}) - z^* \circ H_1(\overline{x}) \leq y^* \circ G(x) - y^* \circ H(x) + \langle y^*, \varepsilon\rangle \|x - \overline{x}\|.$$

Since $\langle z^*, H_1(\overline{x}) - G_1(\overline{x})\rangle = 0$, we deduce

$$y^* \circ G(\overline{x}) - y^* \circ H(\overline{x}) \leq y^* \circ G(x) - y^* \circ H(x) + \langle y^*, \varepsilon\rangle \|x - \overline{x}\|.$$

Consequently,

$$0 \leq \langle y^*, (G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) + \langle y^*, \varepsilon\rangle \|x - \overline{x}\|\rangle.$$

Finally, we obtain

$$(G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) + \varepsilon \|x - \overline{x}\| \notin -int\ K_Y.$$

As $x \in \Omega \cap U$ was chosen arbitrarily, it follows that

$$(G(x) - H(x)) - (G(\overline{x}) - H(\overline{x})) + \varepsilon \|x - \overline{x}\| \notin -int\ K_Y, \text{ for all } x \in \Omega \cap U,$$

which implies that $\overline{x}$ is an ε-weak local quasi-efficient solution of (P). ■

Below, we give an example that illustrates our findings.

EXAMPLE 5 *Let* $\varepsilon = (1, 0)$, $\overline{x} = 0$, $C = \mathbb{R}$, $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $K_Y = \mathbb{R}^2_+$, $K_Z = \mathbb{R}_+$ *and let* $G(x) = (|x|, x^4)$, $H(x) = (x^2, x^2)$, $G_1(x) = |x|$ *and* $H_1(x) = x$. *Here,* $\Omega = \mathbb{R}_+$ *and*

$$intK_Y =]0, +\infty[\times]0, +\infty[,\ K^*_Y \setminus \{(0,0)\} = \mathbb{R}^2_+ \setminus \{(0,0)\} \text{ and } K^*_Z = [0, +\infty[.$$

Notice that

$$G(x) - H(x) = \{(|x| - x^2, x^4 - x^2)\}, \text{ for all } x \in X.$$

- *It is straightforward to verify that conditions* (15) *and* (16) *are met for* $\gamma = \frac{1}{3} > 0$, $y^* = \left(\frac{1}{3}, 0\right) \in K^*_Y \setminus \{(0,0)\}$ *and* $z^* = \frac{1}{3} \in K^*_Z$. *Specifically, we have*

$$\langle z^*, G_1(\overline{x}) - H_1(\overline{x})\rangle = 0$$

and

$$y^* \circ H(x) + z^* \circ H_1(x) = \frac{1}{3}x^2 + \frac{1}{3}x \text{ and } y^* \circ G(x) + z^* \circ G_1(x) + \langle y^*, \varepsilon\rangle \|x - \overline{x}\| = |x|.$$

Given that for all $\alpha \in [0, \gamma]$ *we have*

$$\partial_\alpha (y^* \circ H + z^* \circ H_1)(\overline{x}) = \left[\frac{1 - 2\sqrt{3\alpha}}{3}, \frac{1 + 2\sqrt{3\alpha}}{3}\right]$$

and

$$\partial_\alpha (y^* \circ G + z^* \circ G_1 + \langle y^*, \varepsilon\rangle \|. - \overline{x}\|)(\overline{x}) = [-1, 1],$$

we deduce that (15) *and* (16) *are both satisfied.*

- *Applying Theorem 2, we can conclude that $\overline{x}$ is an ε-weak local quasi-efficient solution of* (P).

Remark 1 *Condition* (15) *extends the sufficient optimality condition, proposed by Dür (2003, Theorem 2.1), to the vector setting. Notably, when* $Y = \mathbb{R}$, $K_Y = \mathbb{R}^+$, $G_1 \equiv 0_Z$ *and* $H_1 \equiv 0_Z$, *it reduces precisely to the scalar condition, given in Dür (2003, Theorem 2.1). This condition may be satisfied if the scalar functions* $y^* \circ G$ *et* $z^* \circ G_1$ *dominate* $y^* \circ H$ *et* $z^* \circ H_1$ *in terms of convexity or growth.*

If $G_1 \equiv 0_Z$ and $H_1 \equiv 0_Z$, then, considering Theorem 2, we derive Corollary 4, which offers sufficient optimality conditions for the nonconvex vector optimization problem (P_1). It is worth mentioning that, in Corollary 4, if $Y = \mathbb{R}$ and $\varepsilon = 0$, we recover the result from Bomze and Lemaréchal (2010, Theorem 1.2).

Corollary 4 *Let* $\overline{x} \in C$ *and* $\varepsilon \in K_Y$. *Suppose that there exist a scalar* $\gamma > 0$ *and vectors* $y^* \in K_Y^* \backslash \{0\}$ *and* $z^* \in K_Z^*$ *such that*

$$\partial_\alpha \left(y^* \circ H\right)(\overline{x}) \subseteq \partial_\alpha \left(y^* \circ G + \langle y^*, \varepsilon\rangle \left\|. - \overline{x}\right\|\right)(\overline{x}), \ \forall \alpha \in [0, \gamma].$$

Then, $\overline{x}$ *is an* ε*-weak local quasi-efficient solution of problem* (P_1).

If $H \equiv 0_Y$ and $H_1 \equiv 0_Z$, then, by leveraging Theorem 2, we obtain Corollary 5, which provides sufficient optimality conditions for ensuring solutions to the convex vector optimization problem (P_2).

Corollary 5 *Let* $\overline{x} \in \Omega$ *and* $\varepsilon \in K_Y$. *Suppose that there exist a scalar* $\gamma > 0$ *and vectors* $y^* \in K_Y^* \backslash \{0\}$ *and* $z^* \in K_Z^*$ *such that*

$$0 \in \partial_\alpha \left(y^* \circ G + z^* \circ G_1 + \langle y^*, \varepsilon\rangle \left\|. - \overline{x}\right\|\right)(\overline{x}), \ \forall \alpha \in [0, \gamma],$$

and

$$\langle z^*, G_1(\overline{x})\rangle = 0.$$

Then, $\overline{x}$ *is an* ε*-weak local quasi-efficient solution of problem* (P_2).

5. Conclusion

In recent years, vector optimization problems have attracted significant attention from mathematicians, with efficient solutions in the Pareto sense being widely recognized as a crucial concept. In this paper, we employ a separation argument along with ε-regularized subdifferentials to establish the necessary optimality conditions for a point to be an ε-weak local quasi-efficient solution to a D.C. vector problem. Next, we present sufficient optimality conditions for ε-weak local quasi-efficiency and illustrate our findings through suitable examples.

Our results open several promising avenues for future research, including the development of numerical methods for identifying ε-weak local quasi-efficient points. Another promising direction for future work could be to explore approximate solutions for bilevel optimization problems, particularly when data functions are fully convex (i.e., convex with respect to all their variables), using a similar approach to the one applied in this research.

Acknowledgements

Our sincere acknowledgements to the anonymous referees for their insightful remarks and valuable suggestions, which have helped to improve the paper.

6. References

Attouch, H. and Riahi, H. (1993) Stability results for Ekeland's ε-variational principle and cone extremal solutions. *Math. Oper. Res.* 18: 173-201. https://doi.org/10.1287/moor.18.1.173

Benslimane, O., Gadhi, N. A. and Zerrifi, A. I. (2024) ε-weak Pareto minimality in DC vector optimization. *NACO.* https://doi.org/10.3934/naco.2024001

Bomze, I. M. and Lemaréchal, C. (2010) Necessary conditions for local optimality in difference-of-convex programming. *Journal of Convex Analysis* 17: 673-680.

Craven, B. D. (1995) *Control and Optimization.* Chapman & Hall, London.

Dür, M. (2003) A parametric characterization of local optimality. *Math. Methods Oper. Res.* 57: 101-109. https://doi.org/10.1007/s001860200232

Dutta, J. (2005) Necessary optimality conditions and saddle points for approximate optimization in Banach spaces. *TOP* 13: 127-143. https://link.springer.com/article/10.1007/bf02578991

Ekeland, I. (1974) On the variational principle. *J. Math. Anal. Appl.* 47: 324-353. https://doi.org/10.1016/0022-247X(74)90025-0

Gadhi, N. A. (2024) Focus on sufficient optimality conditions in D.C. vector optimization. *Optimization* 73: 1611-1623. https://doi.org/10.1080/02331934.2023.2170700

Gadhi, N. and Metrane, A. (2004) Sufficient optimality condition for vector optimization problems under D.C. date. *J. Glob. Optim.* 28: 55-66. https://doi.org/10.1023/B:JOGO.0000006715.69153.8b

Gadhi, N., Laghdir, M. and Metrane, A. (2005) Optimality conditions for D.C. vector optimization problems under reverse convex constraints. *J. Glob. Optim.* 33: 527-540. https://doi.org/10.1007/s10898-004-8318-4

Ghaznavi-Ghosoni, B.A. and Khorram, E. (2011) On approximating weakly/properly efficient solutions in multi-objective programming. *Math.*

Comput. Modelling 54: 3172-3181. https://doi.org/10.1016/j.mcm.2011.08.013

Guo, X. L. and Li, S. J. (2014) Optimality conditions for vector optimization problems with difference of convex maps. *J. Optim. Theory Appl.* 162: 821-844. https://doi.org/10.1007/s10957-013-0327-3

Hamel, A. (2001) An ε-Lagrange multiplier rule for a mathematical programming problem on Banach spaces. *Optimization* 49: 137-149. https://doi.org/10.1080/02331930108844524

Hiriart-Urruty, J. B. (1989) From Convex Optimization to Nonconvex Optimization. Necessary and Sufficient Conditions for Global Optimality. In: F. H. Clarke, V. F. Dem'yanov and F. Giannessi, eds., *Nonsmooth Optimization and Related Topics. Ettore Majorana International Science Series*, **43**. Springer, Boston, MA. https://doi.org/10.1007/978-1-4757-6019-4_13

Li, Z. and Wang, S. (1998) ε-approximation solutions in multiobjective optimization. *Optimization* 44: 161-174. https://doi.org/10.1080/02331939808844406

Liu, J. C. (1999) ε-properly efficient solutions to nondifferentiable multiobjective programming problems. *Applied Mathematics Letters* 12: 109-113. https://doi.org/10.1016/S0893-9659(99)00087-7

Khazayel, B. and Farajzadeh, A. (2022) On the optimality conditions for D.C. vector optimization problems. *Optimization* 71, 2033-2045. https://doi.org/10.1080/02331934.2020.1847109

Loridan, P. (1982) Necessary conditions for ϵ-optimality. *Math. Progr. Stud.* 19: 140-152. https://doi.org/10.1007/BFb0120986

Loridan, P. (1984) ε-solutions in vector minimization problems. *J. Optim. Theory Appl.* 43: 265-276. https://doi.org/10.1007/BF00936165

Mordukhovich, B. S. and Wang, B. (2002) Necessary suboptimality and optimality conditions via variational principles. *SIAM J. Control Optim.* 41: 623-640. https://doi.org/10.1137/S036301290037481

Penot, J. P. (2011) The directional subdifferential of the difference of two convex functions. *J. Glob. Optim.* **49**, 505-519. https://doi.org/10.1007/s10898-010-9615-8

Shafie, A. and Bozorgnia, F. (2019) A note on the paper "Optimality Conditions for Vector Optimization Problems with Difference of Convex Maps". *Journal of Optimization Theory and Applications* 182: 837-849. https://doi.org/10.1007/s10957-019-01530-x

Su, T. V. and Hang, D. D. (2022) Optimality and duality in nonsmooth multiobjective fractional programming problem with constraints. *4OR* 20:105-137. https://doi.org/10.1007/s10288-020-00470-x

Zălinescu, C. (2008) Hahn-Banach extension theorems for multifunctions revisited. *Math. Meth. Oper. Res.* 68: 493-508. https://doi.org/10.1007/s00186-007-0193-6

Zowe, J. (1974) Subdifferentiability of convex functions with values in an ordered vector space. *Math. Scand.* 34: 69-83. https://doi.org/10.7146/math.scand.a-11507