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Set-valued fractional-type optimization problems with κ-cone arcwise connectedness of higher-order*

by

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Abstract: The aim of this study is to establish the sufficient higher-order KKT (Karush Kuhn Tucker) criteria of optimality for a set-valued fractional-type optimization problem (SFP) (FP). Under the presumptions of higher-order contingent epi-derivative and higher-order κ-arcwisely connectedness, these requirements are derived. We also look into the effects of these constraints on the higher-order duality of Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds.

Keywords: convex cone; set-valued map; contingent epiderivative; arcwise connectedness; duality

1. Introduction

A recent development in optimization theory is related to the set-valued optimization problems, or SOPs, on which researchers have been working for the past few years. Set-valued functions, or, briefly, SVFs, are used as objective functions and constraints in SOPs. Applications include the theory of viability, image processing, engineering and mathematical economics. The notion of convexity of SVFs was put forth by Borwein (1977). It is crucial for obtaining the optimal conditions of the SOPs. There have been several presentations of differentiabilities of set-valued functions (SVFs). The notion of contingent epiderivative of SVFs was developed by Jahn and Rauh (1997). The Lagrangian results of duality of SFPs were established by Bhatia and Mehra (1997). Bhatia and Mehra (1998) also formulated the duality results for Geoffrion efficiency of

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the SFPs. Gadhi and Jawhar (2013) presented the necessary requirements of optimality of SFPs. Kaul and Lyall (1989), Bhatia and Garg (1998), Suneja and Gupta (1990), Suneja and Lalitha (1993), and Lee and Ho (2008) studied the requirements of optimality and provided the results of duality in terms of generalized convexity. Das (2023, 2024) suggested the notion of κ-convex SVFs. As a result, he proposed the KKT sufficiency criteria and derived the dualities of various SOP types under a set of presumptions, concerning contingent epiderivative and κ-convexity. Das (2022) presented the proposal of second-order κ-convex SVFs and determined the requirement of optimality for SFPs. Li et al. (2008a,b) utilized higher-order contingent derivative to create both the sufficient and necessary requirements of optimality. The Mond-Weir dual of higher-order for SOP was also introduced, and the results of duality under the assumptions of convexity were examined.

Avriel (1976) developed the notion of arcwisely connectedness. Lalitha et al. (2003) and Wang (2003) presented the notion of SVFs with arcwise connections, which is a protracted version of convex SVFs. Lalitha et al. (2003) developed the sufficient requirement of optimality for SOPs under the set of assumptions, the cone arcwisely connectedness hypothesiss and contingent epiderivative. Qiu and Yang (2012) presented the connectedness of Henig weak efficiency. Yu (2013) provided both the sufficient and necessary requirements of optimality for proper global efficient solutions in vector optimization problem (VP). Later, Yihong and Min (2016) presented the notion of SVFs of near arcwisely connectedness of higher order. Additionally, they provided both the sufficient and necessary conditions for SOPs. Yu (2016) presented both the sufficient and necessary criteria for the proper global efficient solutions of VPs employing SVFs with arcwise connections. Peng and Xu (2018) developed the notion of subarcwise connected SVFs. As a result, they provided the necessary conditions of the local proper global efficient second-order requirements of optimality of the SOPs.

The Fréchet differentiability concept is essentially extended from the single-valued to the set-valued case by the concept of contingent derivative. In the set-valued optimization theory, this idea has also been applied extensively. However, under conventional assumptions, the necessary and sufficient optimality conditions do not coincide. Consequently, contingent derivatives are not the ideal tool for establishing optimality requirements in set-valued optimization problems. One potential extension of directional derivatives in the single-valued convex case is the concept of contingent epiderivative. The contingent derivative and the contingent epiderivative differ primarily in that the derivative is now single-valued and the graph is substituted with an epigraph. If contingent epiderivatives exist in the cone-convex situation, one of their unique characteristics is that they are sublinear. Therefore, contingent epiderivatives are of greater relevance to us when studying the set-valued optimization problems.

Arcwise connectedness is a generalization of convexity, in which a continuous arc is used in place of the line segment, connecting two places. As a generalization of cone arcwise connected set-valued mappings, we provide the concept of κ-cone arcwise connectedness of set-valued maps. The standard concept of cone arcwise connected set-valued mappings is obtained for $\kappa = 0$. Additionally, we provide an example of a set-valued map that is κ-cone arcwise connected but not cone arcwise connected. Using contingent epiderivative and κ-cone arcwise connectivity assumptions on the objective functions and constraints, we are primarily concerned with determining the sufficient optimality requirements of set-valued fractional-type optimization problems in a more generalized context. Additionally, we establish the weak, strong, and converse duality theorems of the Mond-Weir, Wolfe, and mixed types under the conditions of contingent epiderivative and κ-cone arcwise connectedness. Our findings generalize the ones that are already accessible in the literature for $\kappa = 0$.

In this paper, we introduce the idea of higher-order κ-cone arcwisely connectedness of SVFs as a generalization of higher-order SVFs with arcwise connections. The concept of SVFs with arcwise connections of higher-order is what we typically think of when $\kappa = 0$. Additionally, we develop an example of a higher-order κ-cone arcwisely connected SVF that is not higher-order cone arcwisely connected. With the assistance of generalized contingent epiderivatives and κ-cone arcwisely connectedness of higher-order, we are mainly interested in finding out the sufficient condition of optimality of higher-order of the SFP (FP) in a more generalized case. Assuming κ-cone arcwisely connectedness and generalized contingent epiderivative of higher-order, we also investigate higher-order strong, converse, and weak results of duality for Mond-Weir (MWD), mixed (MD), and Wolfe (WD) types. Our results are more accurate for $\kappa = 0$ than those previously published.

This article is structured as follows. In the second section, certain terminology is defined and several fundamental concepts are presented. The higher-order sufficient criteria of optimality for the weak efficient solutions of the SFPs under the higher-order generalized cone arcwisely connected hypothesis are determined in Section 3. In Sections 4 through 6, we also formulate the results of higher-order duality for the Wolfe (WD), Mond-Weir (MWD), and mixed (MD) types. Section 7 ends with the concluding remarks of the work.

2. Definitions and preliminary information

Suppose that Θ is a real normed space (in brief, NS) and $\emptyset \neq \Gamma \subseteq \Theta$. Then, Γ is termed a cone if $\tau\theta \in \Gamma$, $\forall\ \theta \in \Gamma$ and $\tau \geq 0$. Additionally, the cone Γ is termed pointed if $\Gamma \cap (-\Gamma) = \{\mathbf{0}_\Theta\}$, closed if $\overline{\Gamma} = \Gamma$, solid if $\operatorname{int}(\Gamma) \neq \emptyset$, and convex if

$$\tau\Gamma + (1-\tau)\Gamma \subseteq \Gamma, \forall \tau \in [0,1],$$

where $\overline{\Gamma}$ and $\mathrm{int}(\Gamma)$ stand for the closure and interior of Γ, correspondingly, and $\mathbf{0}_\Theta$ remains the zero element of Θ.

Let Γ be a solid convex pointed cone in Θ. Cone orderings can be of two different types in Θ in relation to Γ. For $\theta_1, \theta_2 \in \Theta$, we have

$$\theta_1 \leq \theta_2 \text{ if } \theta_2 - \theta_1 \in \Gamma$$

and

$$\theta_1 < \theta_2 \text{ if } \theta_2 - \theta_1 \in \mathrm{int}(\Gamma).$$

The most often utilized minimality concepts in relation to a solid convex pointed cone Γ in an NS Θ include the following.

Definition 2.1 *Let $\widetilde{\Theta}$ be a non-empty subset of an NS Θ.*

(i) *$\theta' \in \widetilde{\Theta}$ is termed a minimal element of $\widetilde{\Theta}$ if there is no $\theta \in \widetilde{\Theta} \setminus \{\theta'\}$, such that $\theta \leq \theta'$.*
(ii) *$\theta' \in \widetilde{\Theta}$ is termed a weakly minimal element of $\widetilde{\Theta}$ if there is no $\theta \in \widetilde{\Theta}$, such that $V\ \theta < \theta'$.*
(iii) *$\theta' \in \widetilde{\Theta}$ is termed an ideal minimal element of $\widetilde{\Theta}$ if $\theta' \leq \theta$, $\forall\ \theta \in \widetilde{\Theta}$.*

The sets of minimal elements, weakly minimal elements, and ideal minimal elements of $\widetilde{\Theta}$ are denoted by $\min(\widetilde{\Theta})$, w-$\min(\widetilde{\Theta})$, and I-$\min(\widetilde{\Theta})$, correspondingly.

Definition 2.2 (Aubin, 1981; Aubin and Frankowska, 1990) *Suppose that Θ is NS and $\emptyset \neq \Gamma \subseteq \Theta$, $\theta' \in \overline{\Gamma}$, $k \in \mathbb{N}$, with $k \geq 2$, and $\theta_1, ..., \theta_{k-1} \in \Theta$. The contingent set $T^{(k)}(\Gamma, \theta', \theta_1, ..., \theta_{k-1})$ of k-th order to Γ at $(\theta', \theta_1, ..., \theta_{k-1})$ is defined as:*

$\theta \in T^{(k)}(\Gamma, \theta', \theta_1, ..., \theta_{k-1})$ if there appear $\{\tau_n\}$ in $\mathbb{R}$ and $\{\theta_n\}$ in Θ, in addition to $\tau_n \to 0^+$ and $\theta_n \to \theta$, so that

$$\theta' + \tau_n\theta_1 + ... + {\tau_n}^{k-1}\theta_{k-1} + {\tau_n}^k\theta_n \in \Gamma, \quad \forall n \in \mathbb{N},$$

equivalently, $\theta \in T^{(k)}(\Gamma, \theta', \theta_1, ..., \theta_{k-1})$ if there appear $\{\tau_n\}$ in $\mathbb{R}$ and $\{\theta'_n\}$ in Θ, in addition to $\tau_n \to 0^+$, $\theta'_n \in \Gamma$, and $\theta'_n \to \theta'$, so that

$$\frac{\theta'_n - \theta' - \tau_n\theta_1 - ... - {\tau_n}^{k-1}\theta_{k-1}}{{\tau_n}^k} \to \theta, \text{ as } n \to \infty.$$

Let Δ, Θ be NSs and Γ be a solid convex pointed cone in Θ. Let $\Phi : \Delta \to 2^\Theta$ be a SVF from Δ to Θ, i.e., $\Phi(\delta) \subseteq \Theta$, $\forall\ \delta \in \Delta$. The effective domain, graph, as well as epigraph of Φ are comprehended by

$$\mathrm{DM}(\Phi) = \{\delta \in \Delta : \Phi(\delta) \neq \emptyset\},$$

$$\mathrm{grp}(\Phi) = \{(\delta, \theta) \in \Delta \times \Theta : \theta \in \Phi(\delta)\},$$

and

$$\mathrm{epigrp}(\Phi) = \{(\delta, \theta) \in \Delta \times \Theta : \theta \in \Phi(\delta) + \Gamma\}.$$

Aubin and Frankowska (1990) developed SVFs with contingent derivative of higher-order.

Definition 2.3 (Aubin and Frankowska, 1990) *Let us consider that* $k \in \mathbb{N}$*, with* $k \geq 2$*,* Δ*,* Θ *are NSs,* $\Phi : \Delta \to 2^{\Theta}$ *is an SVF in addition to* $\mathrm{DM}(\Phi) = \Delta$*,* $(\delta', \theta') \in \mathrm{grp}(\Phi)$*, and* $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1}) \in \Delta \times \Theta$*. Then, the contingent derivative of k-th order* $\Delta^{(k)}\Phi(\delta', \theta', \delta_1, \theta_1, ..., \delta_{k-1}, \theta_{k-1})$ *of* Φ *at* (δ', θ') *for* $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1})$ *is the SVF from* Δ *to* Θ*, as stated by*

$$\begin{aligned}&\mathrm{grp}(\Delta^{(k)}\Phi(\delta', \theta', \delta_1, \theta_1, ..., \delta_{k-1}, \theta_{k-1}))\\ &= T^{(k)}(\mathrm{grp}(\Phi), (\delta', \theta'), (\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1})).\end{aligned}$$

Jahn, Khan and Zeilinger (2005) introduced the contingent epi-derivatives of second-order of SVFs.

Definition 2.4 (Jahn, Khan and Zeilinger, 2005) *Let us consider that* Δ*,* Θ *are NSs,* $\Phi : \Delta \to 2^{\Theta}$ *is an SVF in addition to* $\mathrm{DM}(\Phi) = \Delta$*,* $(\delta', \theta') \in \mathrm{grp}(\Phi)$*, and* $(\delta_1, \theta_1) \in \Delta \times \Theta$*. Then, the contingent epi-derivative of second-order* $\overrightarrow{\Delta}^2\Phi(\delta', \theta', \delta_1, \theta_1)$ *of* Φ *at* (δ', θ') *in the direction* (δ_1, θ_1) *is the function from* Δ *to* Θ*, described by*

$$\mathrm{epigrp}(\overrightarrow{\Delta}^2\Phi(\delta', \theta', \delta_1, \theta_1)) = T^{(2)}(\mathrm{epigrp}(\Phi), (\delta', \theta'), (\delta_1, \theta_1)).$$

Let Δ, Θ be NSs, Γ be a solid convex pointed cone in Θ, and $\Phi : \Delta \to 2^{\Theta}$ be a SVF. Let us define an SVF $\Phi + \Gamma : \Delta \to 2^{\Theta}$ by

$$(\Phi + \Gamma)(\delta) = \Phi(\delta) + \Gamma, \forall \delta \in \mathrm{dom}(\Phi).$$

Li and Chen (2006) developed the concept of generalized contingent epi-derivative of higher-order of SVFs.

Definition 2.5 (Li and Chen, 2006) *Let us consider that* $k \in \mathbb{N}$*, with* $k \geq 2$*,* Δ*,* Θ *are NSs,* Γ *is a solid convex pointed cone in* Θ*,* $\Phi : \Delta \to 2^{\Theta}$ *is an SVF in addition to* $\mathrm{DM}(\Phi) = \Delta$*,* $(\delta', \theta') \in \mathrm{grp}(\Phi)$*, and* $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1}) \in \Delta \times \Theta$*. Then, the generalized contingent epi-derivative of k-th order of* Φ *at* (δ', θ') *for* $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1})$*, denoted by* $\overrightarrow{\Delta}_g^{(k)}\Phi(\delta', \theta', \delta_1, \theta_1, ..., \delta_{k-1}, \theta_{k-1})$*, is the SVF from* Δ *to* Θ*, described by*

$$\begin{aligned}&\overrightarrow{\Delta}_g^{(k)}\Phi(\delta', \theta', \delta_1, \theta_1, ..., \delta_{k-1}, \theta_{k-1})(\delta)\\ &= \min\{\theta \in \Theta : (\delta, \theta) \in T^{(k)}(\mathrm{epigrp}(\Phi), (\delta', \theta'), (\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1}))\},\\ &\delta \in \mathrm{DM}(\Delta^{(k)}(\Phi + \Gamma)(\delta', \theta', \delta_1, \theta_1, ..., \delta_{k-1}, \theta_{k-1})).\end{aligned}$$

Borwein (1977) introduced the idea of convexity of SVFs.

Definition 2.6 (Borwein, 1977) *Let Λ be a non-empty convex subset of an NS Δ. An SVF $\Phi : \Delta \to 2^{\Theta}$, in addition to $\Lambda \subseteq \mathrm{DM}(\Phi)$, is termed Γ-convex on Λ if $\forall \delta_1, \delta_2 \in \Lambda$ and $\tau \in [0,1]$,*

$$\tau\Phi(\delta_1) + (1-\tau)\Phi(\delta_2) \subseteq \Phi(\tau\delta_1 + (1-\tau)\delta_2) + \Gamma.$$

Li, Theo and Yang (2008) formulated the following result for contingent derivative of higher-order of SVFs.

Proposition 2.1 (Li, Theo and Yang, 2008) *Assume that Δ, Θ are NSs and Φ is Γ-convex on Λ of Δ, then $\forall \delta, \delta' \in \Lambda$ and $\forall \theta' \in \Phi(\delta')$,*

$$\Phi(\delta) - \theta' \subseteq \Delta^{(k)}\Phi(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')(\delta - \delta'),$$

where $\delta_1, ..., \delta_{k-1} \in \Lambda$, $\theta_1 \in \Phi(\delta_1) + \Gamma, ..., \theta_{k-1} \in \Phi(\delta_{k-1}) + \Gamma$.

Let Δ be an NS and Λ be a non-empty subset of Δ. Let $\Phi : \Delta \to 2^{\mathbb{R}^j}$, $\Psi : \Delta \to 2^{\mathbb{R}^j}$, and $\Omega : \Delta \to 2^{\mathbb{R}^l}$ be SVFs, along with

$$\Lambda \subseteq \mathrm{DM}(\Phi) \cap \mathrm{DM}(\Psi) \cap \mathrm{DM}(\Omega).$$

We take the followings notation throughout the entire work.

$$(0, ..., 0) = \mathbf{0}_{\mathbb{R}^j} \in \mathbb{R}^j$$

and

$$(1, ..., 1) = \mathbf{1}_{\mathbb{R}^j} \in \mathbb{R}^j.$$

Let $\Phi = (\Phi_1, \Phi_2, ..., \Phi_j)$, $\Psi = (\Psi_1, \Psi_2, ..., \Psi_j)$, and $\Omega = (\Omega_1, \Omega_2, ..., \Omega_l)$, where the SVFs $\Phi_i : \Delta \to 2^{\mathbb{R}}$, $\Psi_i : \Delta \to 2^{\mathbb{R}}$, $i = 1, , ..., j$, and $\Omega_n : \Delta \to 2^{\mathbb{R}}$, $n = 1, , ..., l$, are comprehended by:

$$\mathrm{DM}(\Phi_i) = \mathrm{DM}(\Phi), \mathrm{DM}(\Psi_i) = \mathrm{DM}(\Psi) \text{ and } \mathrm{DM}(\Omega_n) = \mathrm{DM}(\Omega),$$

$$\delta \in \Lambda, \theta = (\theta_1, \theta_2, ..., \theta_j) \in \Phi(\delta) \Longleftrightarrow \theta_i \in \Phi_i(\delta), \forall i = 1, , ..., j,$$

$$\psi = (\psi_1, \psi_2, ..., \psi_j) \in \Psi(\delta) \Longleftrightarrow \psi_i \in \Psi_i(\delta), \forall i = 1, , ..., j,$$

and

$$\omega = (\omega_1, \omega_2, ..., \omega_l) \in \Omega(\delta) \Longleftrightarrow \omega_n \in \Omega_n(\delta), \forall n = 1, , ..., l.$$

Take note that $\Phi_i(\delta) \subseteq \mathbb{R}_+$ and $\Psi_i(\delta) \subseteq \mathrm{int}(\mathbb{R}_+), \forall i = 1, , ..., j$ and $\delta \in \Lambda$. Let $\tau' = (\tau'_1, \tau'_2, ..., \tau'_j) \in \mathbb{R}^j_+$. Define $\frac{\theta}{\psi} \in \mathbb{R}^j$ and $\tau'\psi \in \mathbb{R}^j$ by:

$$\frac{\theta}{\psi} = \Big(\frac{\theta_1}{\psi_1}, \frac{\theta_2}{\psi_2}, ..., \frac{\theta_j}{\psi_j}\Big)$$

and

$$\tau'\psi = (\tau'_1\psi_1, \tau'_1\psi_2, ..., \tau'_j\psi_j).$$

For $\delta \in \Lambda$, define the subset $\frac{\Phi(\delta)}{\Psi(\delta)}$ of $\mathbb{R}^j$ by:

$$\frac{\Phi(\delta)}{\Psi(\delta)} = \left\{ \frac{\theta}{\psi} = \Big(\frac{\theta_1}{\psi_1}, \frac{\theta_2}{\psi_2}, ..., \frac{\theta_j}{\psi_j}\Big) : \theta_i \in \Phi_i(\delta), \psi_i \in \Psi_i(\delta), i = 1, , ..., j \right\}.$$

For now, let us assume the SFP (FP).

$$\begin{aligned} &\underset{\delta \in \Lambda}{\text{minimize}} \quad \frac{\Phi(\delta)}{\Psi(\delta)} = \left(\frac{\Phi_1(\delta)}{\Psi_1(\delta)}, \frac{\Phi_2(\delta)}{\Psi_2(\delta)}, ..., \frac{\Phi_j(\delta)}{\Psi_j(\delta)}\right) \\ &\text{s. t.} \qquad \Omega(\delta) \cap (-\mathbb{R}^l_+) \neq \emptyset. \end{aligned} \tag{FP}$$

The set of feasible solutions of the problem (FP) is classified as

$$S = \Big\{\delta \in \Lambda : \Omega(\delta) \cap (-\mathbb{R}^l_+) \neq \emptyset\Big\}.$$

Definition 2.7 *A point* $(\delta', \frac{\theta'}{\psi'}) \in \Delta \times \mathbb{R}^j$*, in addition to* $\delta' \in S$*,* $\theta' \in \Phi(\delta')$*, and* $\psi' \in \Psi(\delta')$*, is said to minimize* (FP) *if there are no* $\delta \in S$*,* $\theta \in \Phi(\delta)$*, and* $\psi \in \Psi(\delta)$ *such that*

$$\tfrac{\theta}{\psi} - \tfrac{\theta'}{\psi'} \in -\mathbb{R}^j_+ \setminus \{\mathbf{0}_{\mathbb{R}^j}\}.$$

Definition 2.8 *A point* $(\delta', \frac{\theta'}{\psi'}) \in \Delta \times \mathbb{R}^j$*, in addition to* $\delta' \in S$*,* $\theta' \in \Phi(\delta')$*, and* $\psi' \in \Psi(\delta')$*, is said to weakly minimize* (FP) *if there are no* $\delta \in S$*,* $\theta \in \Phi(\delta)$*, and* $\psi \in \Psi(\delta)$ *such that*

$$\frac{\theta}{\psi} - \frac{\theta'}{\psi'} \in -\mathrm{int}(\mathbb{R}^j_+).$$

For now, let us assume the parametric problem $(FP_{\tau'})$, connected to the SFP (FP):

$$\begin{aligned} &\underset{\delta \in \Lambda}{\text{minimize}} \quad \Phi(\delta) - \tau'\Psi(\delta) \\ &\text{s. t.} \qquad \Omega(\delta) \cap (-\mathbb{R}^l_+) \neq \emptyset. \end{aligned} \tag{$FP_{\tau'}$}$$

DEFINITION 2.9 *A point* $(\delta', \theta' - \tau'\psi') \in \Delta \times \mathbb{R}^j$*, in addition to* $\delta' \in S$*,* $\theta' \in \Phi(\delta')$*, and* $\psi' \in \Psi(\delta')$*, is said to minimize* $(FP_{\tau'})$*, if there are no* $\delta \in S$*,* $\theta \in \Phi(\delta)$*, and* $\psi \in \Psi(\delta)$ *such that*

$$(\theta - \tau'\psi) - (\theta' - \tau'\psi') \in -\mathbb{R}^j_+ \setminus \{\mathbf{0}_{\mathbb{R}^j}\}.$$

DEFINITION 2.10 *A point* $(\delta', \theta' - \tau'\psi') \in \Delta \times \mathbb{R}^j$*, in addition to* $\delta' \in S$*,* $\theta' \in \Phi(\delta')$*, and* $\psi' \in \Psi(\delta')$*, is said to weakly minimize* $(FP_{\tau'})$*, if there are no* $\delta \in S$*,* $\theta \in \Phi(\delta)$*, and* $\psi \in \Psi(\delta)$ *such that*

$$(\theta - \tau'\psi) - (\theta' - \tau'\psi') \in -\mathrm{int}(\mathbb{R}^j_+).$$

Gadhi and Jawhar (2013) demonstrated the connections between the solutions to (FP) and $(FP_{\tau'})$.

LEMMA 2.1 (GADHI AND JAWHAR, 2013) *A point* $(\delta', \frac{\theta'}{\psi'}) \in \Delta \times \mathbb{R}^j$ *weakly minimizes* (FP) *if and only if* $(\delta', \mathbf{0}_{\mathbb{R}^j})$ *weakly minimizes the problem* $(FP_{\tau'})$*, where* $\tau' = \frac{\theta'}{\psi'}$.

Avriel (1976) presented the concept of arcwisely connectedness. Convexity is primarily generalized in the following statement.

DEFINITION 2.11 *A subset* Λ *of a NS* Δ *is assumed to be an arcwisely connected set if* $\forall\ \delta_1, \delta_2 \in \Lambda$ *there is a continuous arc* $\chi_{\delta_1,\delta_2} : [0,1] \to \Lambda$*, for which* $\chi_{\delta_1,\delta_2}(0) = \delta_1$ *and* $\chi_{\delta_1,\delta_2}(1) = \delta_2$.

Lalitha, Dutta and Govil (2003) and Fu and Wang (2003) presented the notion of SVFs with arcwise connections.

DEFINITION 2.12 (FU AND WANG, 2003; LALITHA, DUTTA AND GOVIL, 2003) *Let* Λ *be an arcwisely connected subset of an NS* Δ *and* $\Phi : \Delta \to 2^\Theta$ *be a SVF, in addition to* $\Lambda \subseteq \mathrm{DM}(\Phi)$*. Then,* Φ *is said to be* Γ*-arcwisely connected on* Λ *if*

$$(1-\tau)\Phi(\delta_1) + \tau\Phi(\delta_2) \subseteq \Phi(\chi_{\delta_1,\delta_2}(\tau)) + \Gamma, \quad \forall \delta_1, \delta_2 \in \Lambda \text{ and } \forall \tau \in [0,1].$$

Peng and Xu (2018) presented the notion of subarcwise cone connected SVFs.

DEFINITION 2.13 (PENG AND XU, 2018) *Let* Λ *be an arcwisely connected subset of an NS* Δ*,* $e \in \mathrm{int}(\Gamma)$*, and* $\Phi : \Delta \to 2^\Theta$ *be an SVF, in addition to* $\Lambda \subseteq \mathrm{DM}(\Phi)$*. Then* Φ *is called* Γ*-subarcwise connected on* Λ *if*

$$(1-\tau)\Phi(\delta_1) + \tau\Phi(\delta_2) + \epsilon e \subseteq \Phi(\chi_{\delta_1,\delta_2}(\tau)) + \Gamma,$$
$$\forall \delta_1, \delta_2 \in \Lambda, \forall \epsilon > 0, \text{ and } \forall \tau \in [0,1].$$

Khanh and Tung (2014) formulated the concept of η-arcwisely connectedness.

Definition 2.14 (Khanh and Tung, 2014) *A subset Λ of a NS Δ is said to be an η-arcwisely connected set, where $\eta : \Lambda \times \Lambda \times [0,1] \to \Delta$, if for any $\delta_1, \delta_2 \in \Lambda$ and $\tau \in [0,1]$,*

$$\delta_1 + \tau\eta(\delta_1, \delta_2, \tau) \in \Lambda.$$

For $\eta : \Lambda \times \Lambda \times [0,1] \to \Delta$, an SVF $\Phi : \Delta \to 2^\Theta$ is said to be Γ-η-arcwisely connected on an η-arcwisely connected set Λ if for any $\delta_1, \delta_2 \in \Lambda$ and $\tau \in [0,1]$,

$$\lim_{\tau \to 0^+} \tau\eta(\delta_1, \delta_2, \tau) = 0$$

and

$$(1-\tau)\Phi(\delta_1) + \tau\Phi(\delta_2) \subseteq \Phi(\delta_1 + \tau\eta(\delta_1, \delta_2, \tau)) + \Gamma.$$

Das, Ahmad and Treanta (2024) and Das, Treanta and Botmart (2013) introduced higher-order κ-convexity of SVFs.

Definition 2.15 (Das, Ahmad and Treanta, 2014; Das and Treanta, 2021; Das Treanta and Botmart, 2023) *Consider that Δ, Θ are NSs, Λ is a non-empty subset of Δ, Γ is a convex pointed solid cone of Θ, $\kappa \in \mathbb{R}$, $e \in \mathrm{int}(\Gamma)$, and $\Phi : \Delta \to 2^\Theta$ is an SVF, in addition to $\Lambda \subseteq \mathrm{DM}(\Phi)$. Suppose that $(\delta', \theta'), (\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1}) \in \Delta \times \Theta$ in addition to $\delta', \delta_1, ..., \delta_{k-1} \in \Lambda$, $\theta' \in \Phi(\delta')$, $\theta_1 \in \Phi(\delta_1) + \Gamma, ..., \theta_{k-1} \in \Phi(\delta_{k-1}) + \Gamma$. Make the assumption that Φ is the generalized contingent epiderivable of k-th order at (δ', θ') for $(\delta_1 - \delta', \theta_1 - \theta'), ..., (\delta_{k-1} - \delta', \theta_{k-1} - \theta')$. Then, Φ is considered to be κ-Γ-convex of k-th order in relation to e at (δ', θ') for $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1})$ on Λ if*

$$\begin{aligned}\Phi(\delta) - \theta' \subseteq \overrightarrow{\Delta}_g^{(k)}\Phi(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')\\ (\delta - \delta') + \kappa\|\delta - \delta'\|^2 e + \Gamma, \quad \forall \delta \in \Lambda.\end{aligned}$$

As a generalization of SVFs with arcwise connections, Das, Treanta and Khan (2023) and Das (2022, 2024) offered the concept of κ-cone arcwisely connectedness of SVFs and subsequently developed sufficient condition of optimality.

Definition 2.16 (Das, 2023, 2024; Das, Treanta and Khan, 2023) *Let Λ be an arcwisely connected subset of a NS Δ, $e \in \mathrm{int}(\Gamma)$, and $\Phi : \Delta \to 2^\Theta$ be an SVF, with $\Lambda \subseteq \mathrm{DM}(\Phi)$. Then, Φ is said to be κ-Γ-arcwisely connected in relation to e on Λ if there is $\kappa \in \mathbb{R}$, such that*

$$\begin{aligned}(1-\tau)\Phi(\delta_1) + \tau\Phi(\delta_2) \subseteq \Phi(\chi_{\delta_1,\delta_2}(\tau)) + \kappa\tau(1-\tau)\|\delta_1 - \delta_2\|^2 e + \Gamma,\\ \forall \delta_1, \delta_2 \in \Lambda \text{ and } \forall \tau \in [0,1].\end{aligned}$$

Let Λ be an arcwisely connected subset of an NS Δ. In this study, we assume that $\chi'_{\delta',\delta}(0+)$ exists $\forall\ \delta, \delta' \in \Lambda$, where

$$\chi'_{\delta',\delta}(0+) = \lim_{\tau \to 0^+} \frac{\chi_{\delta',\delta}(\tau) - \chi_{\delta',\delta}(0)}{\tau}.$$

3. Main results

3.1. The preliminaries

We construct sufficient optimality criteria for SOPs by introducing the idea of κ-cone arcwisely connectedness of higher-order of SVFs as a generalization of higher-order SVFs with arcwise connections.

Let $\Phi : \Delta \to 2^{\Theta}$ be an SVF, $\Lambda \subseteq \mathrm{DM}(\Phi)$, $(\delta', \theta'), (\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1}) \in \Delta \times \Theta$ in addition to $\delta', \delta_1, ..., \delta_{k-1} \in \Lambda$, $\theta' \in \Phi(\delta')$, $\theta_1 \in \Phi(\delta_1) + \Gamma, ..., \theta_{k-1} \in \Phi(\delta_{k-1}) + \Gamma$.

Definition 3.1 (Das, Ahmad and Treanta, 2024; Das and Treanta, 2021; Das, Treanta and Botmart, 2023) *Let us consider that Δ, Θ are NSs, Λ is an arcwisely connected subset of Δ, Γ is a convex pointed solid cone of Θ, $\kappa \in \mathbb{R}$, $e \in \mathrm{int}(\Gamma)$, and $\Phi : \Delta \to 2^{\Theta}$ is an SVF, in addition to $\Lambda \subseteq \mathrm{DM}(\Phi)$. Make the assumption that Φ is generalized contingent epiderivable of k-th order at (δ', θ') for $(\delta_1 - \delta', \theta_1 - \theta'), ..., (\delta_{k-1} - \delta', \theta_{k-1} - \theta')$. Then, Φ is considered to be κ-Γ-arcwisely connected of k-th order in relation to e at (δ', θ') for $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1})$ on Λ if*

$$\begin{aligned}\Phi(\delta) - \theta' \subseteq \overrightarrow{\Delta}_g^{(k)} \Phi(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')(\chi'_{\delta',\delta}(0+)) \\ +\kappa \|\delta - \delta'\|^2 e + \Gamma, \quad \forall \delta \in \Lambda.\end{aligned}$$

Remark 3.1 *For $\kappa = 0$ and $\chi_{\delta_1,\delta_2}(\tau) = \delta_1 + \tau\eta(\delta_1, \delta_2, \tau)$, with $\delta_1 + \eta(\delta_1, \delta_2, 1) = \delta_2$, we get the notion of η-arcwisely connectedness of k-th order. When $\chi_{\delta_1,\delta_2}(\tau) = \delta_1 + \tau\eta(\delta_1, \delta_2, \tau)$, with $\delta_1 + \eta(\delta_1, \delta_2, 1) = \delta_2$, κ-Γ-arcwisely connectedness of k-th order $\Rightarrow$ η-arcwisely connectedness of k-th order.*

Remark 3.2 *When the SVF $\Phi : \Delta \to 2^{\Theta}$ reduces to the single-valued function $f : \Delta \to \Theta$, i.e., $\Phi(\delta) = \{f(\delta)\}$, $\forall\ \delta \in dom(\Phi)$, we can draw the following conclusions.*

Let $\delta' \in \Lambda, \theta' = f(\delta'), (\delta_1, \theta_1) \in \Delta \times \Theta$, and $(\delta_2, \theta_2) \in \Delta \times \Theta$, with $\theta_1 \in f(\delta_1) + \Gamma$ and $\theta_2 \in f(\delta_2) + \Gamma$. Let $f : \Delta \to \Theta$ be continuously differentiable of 3-rd order at δ'. f is referred to as κ-Γ-arcwisely connected of 3-rd order in *relation to e at (δ', θ') for the vectors (δ_1, θ_1) and (δ_2, θ_2) if*

$$f(\delta) - f(\delta') \in f'(\delta')(\chi'_{\delta',\delta}(0+)) + f''(\delta')(\delta_1 - \delta', \delta_2 - \delta')$$
$$+ \frac{1}{3!} f'''(\delta')(\delta_1 - \delta', \delta_1 - \delta', \delta_1 - \delta') + \kappa\|\delta - \delta'\|^2 e + \Gamma, \quad \forall \delta \in \Delta,$$

where

$$\theta_1 - \theta' = f'(\delta')(\delta_1 - \delta') \text{ and } \theta_2 - \theta' = f'(\delta')(\delta_2 - \delta') + \tfrac{1}{2} f''(\delta')(\delta_1 - \delta', \delta_1 - \delta').$$

- *When $\Theta = \mathbb{R}^m$, $\Gamma = \mathbb{R}^m_+$, $f = (f_1, f_2, ..., f_m)$, and $e = (1, 1, ..., 1)$,*
$$f_i(\delta) - f_i(\delta') \geq f'_i(\delta')(\chi'_{\delta',\delta}(0+)) + f''_i(\delta')(\delta_1 - \delta', \delta_2 - \delta')$$
$$+ \tfrac{1}{3!} f'''_i(\delta')(\delta_1 - \delta', \delta_1 - \delta', \delta_1 - \delta') + \kappa\|\delta - \delta'\|^2, \ \forall \delta \in \Delta \text{ and } i = 1, , ..., m.$$

- *When $\Theta = \mathbb{R}$, $\Gamma = \mathbb{R}_+$, and $e = 1$,*
$$f(\delta) - f(\delta') \geq f'(\delta')(\chi'_{\delta',\delta}(0+)) + f''(\delta')(\delta_1 - \delta', \delta_2 - \delta') +$$
$$\tfrac{1}{3!} f'''(\delta')(\delta_1 - \delta', \delta_1 - \delta', \delta_1 - \delta') + \kappa\|\delta - \delta'\|^2, \ \forall \delta \in \Delta.$$

- *When $\Delta = \mathbb{R}^n$, $\Theta = \mathbb{R}$, $\Gamma = \mathbb{R}_+$, and $e = 1$,*
$$f(\delta) - f(\delta') \geq (\chi'_{\delta',\delta}(0+))^T \nabla f(\delta') + (\delta_1 - \delta')^T H(\delta')(\delta_2 - \delta') +$$
$$\tfrac{1}{3!} f_{xxx}((\delta_1 - \delta') \otimes (\delta_1 - \delta') \otimes (\delta_1 - \delta')) + \kappa\|\delta - \delta'\|^2, \ \forall \delta \in \mathbb{R}^n,$$

where $\nabla f(\delta')$ and $H(\delta')$ are the gradient and Hessian matrix of Φ at δ', correspondingly, and f_{xxx} is tensor of dimension $1 \times 3 \times 3 \times 3$.

3.2. Higher-order requirements of optimality

Let $\delta' \in \Lambda$, $\theta' \in \Phi(\delta')$, $\psi' \in \Psi(\delta')$, $\tau' = \frac{\theta'}{\psi'}$, and $\nu' \in \Omega(\delta')$. Let us consider $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1}) \in \Delta \times \mathbb{R}^j$, $(\delta_1, \psi_1), ..., (\delta_{k-1}, \psi_{k-1}) \in \Delta \times \mathbb{R}^j$, and $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1}) \in \Delta \times \mathbb{R}^l$ in addition to $\delta_1, ..., \delta_{k-1} \in \Lambda$, $\theta_1 \in \Phi(\delta_1) + \mathbb{R}^j_+, ..., \theta_{k-1} \in \Phi(\delta_{k-1}) + \mathbb{R}^j_+$, $\psi_1 \in \Psi(\delta_1) + \mathbb{R}^j_+, ..., \psi_{k-1} \in \Psi(\delta_{k-1}) + \mathbb{R}^j_+$, and $\nu_1 \in \Omega(\delta_1) + \mathbb{R}^l_+, ..., \nu_{k-1} \in \Omega(\delta_{k-1}) + \mathbb{R}^l_+$.

We assume that Φ be contingent epidifferentiable of k-th order at (δ', θ') for $(\delta_1 - \delta', \theta_1 - \theta'), ..., (\delta_{k-1} - \delta', \theta_{k-1} - \theta')$, $-\tau'\Psi$ be contingent epidifferentiable of k-th order at $(\delta', -\tau'\psi')$ for $(\delta_1 - \delta', -\tau'\psi_1 + \tau'\psi'), ..., (\delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi')$, and Ω be contingent epidifferentiable of k-th order at (δ', ν'), for $(\delta_1 - \delta', \nu_1 - \nu'), ..., (\delta_{k-1} - \delta', \nu_{k-1} - \nu')$.

Under higher-order κ-arcwisely connectedness and higher-order contingent epi-derivative hypothesis, we identify the higher-order sufficient requirements of optimality for the problem (FP).

Theorem 3.1 (Higher-order sufficient requirements of optimality) *Let Λ be an arcwisely connected subset of Δ, δ' be an element of the set of feasible solutions S of* (FP), *$\theta' \in \Phi(\delta')$, $\psi' \in \Psi(\delta')$, $\tau' = \frac{\theta'}{\psi'}$, and $\nu' \in \Omega(\delta') \cap (-\mathbb{R}^l_+)$. We suppose that the SVF Φ is κ_1-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at (δ', θ') for $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1})$, the SVF $-\tau'\Psi$ is κ_2-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', -\tau'\psi')$ for $(\delta_1, -\tau'\psi_1), ..., (\delta_{k-1}, -\tau'\psi_{k-1})$, and the SVF Ω is κ_3-$\mathbb{R}^l_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^l}$ at (δ', ν') for $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, on Λ. Make the assumption that there is $(\theta^*, \psi^*) \in \mathbb{R}^j_+ \times \mathbb{R}^l_+$, in addition to $\theta^* \neq \mathbf{0}_{\mathbb{R}^j}$, and*

$$(\kappa_1 + \kappa_2)\langle \theta^*, \mathbf{1}_{\mathbb{R}^j} \rangle + \kappa_3 \langle \psi^*, \mathbf{1}_{\mathbb{R}^l} \rangle \geq 0, \tag{3.1}$$

such that

$$\begin{aligned} &\Big\langle \theta^*, \overrightarrow{\Delta}^{(k)}_g \Phi(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')(\chi'_{\delta',\delta}(0+)) \\ &+ \overrightarrow{\Delta}^{(k)}_g (-\tau'\Psi)(\delta', -\tau'\psi', \delta_1 - \delta', -\tau'\psi_1 + \tau'\psi', ..., \delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi') \\ &(\chi'_{\delta',\delta}(0+))\Big\rangle \\ &+ \Big\langle \psi^*, \overrightarrow{\Delta}^{(k)}_g \Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+))\Big\rangle \\ &\geq 0, \quad \forall \delta \in \Lambda, \end{aligned} \tag{3.2}$$

$$\theta' - \tau'\psi' = 0, \tag{3.3}$$

and

$$\langle \psi^*, \nu' \rangle = 0. \tag{3.4}$$

Then, $(\delta', \frac{\theta'}{\psi'})$ weakly minimizes the problem (FP).

Proof Make the assumption that $(\delta', \frac{\theta'}{\psi'})$ does not minimize the problem (FP). Then, there are $\delta \in S$, $\theta \in \Phi(\delta)$ and $\psi \in \Psi(\delta)$ such that

$$\frac{\theta}{\psi} < \frac{\theta'}{\psi'}.$$

As $\theta' - \tau'\psi' = 0$, we have

$$\frac{\theta}{\psi} < \tau'.$$

So,

$$\theta - \tau'\psi < 0.$$

Hence,

$$\langle \theta^*, \theta - \tau'\psi \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^j} \neq \theta^* \in \mathbb{R}^j_+.$$

Again, as $\theta' - \tau'\psi' = 0$, we have

$$\langle \theta^*, \theta' - \tau'\psi' \rangle = 0.$$

As $\delta \in S$, there is $\psi \in \Omega(\delta) \cap (-\mathbb{R}^l_+)$.

Therefore,

$$\langle \psi^*, \psi \rangle \leq 0.$$

So,

$$\langle \psi^*, \nu - \nu' \rangle \leq 0, \text{ as } \langle \psi^*, \nu' \rangle = 0.$$

Hence,

$$\langle \theta^*, \theta - \tau'\psi - (\theta' - \tau'\psi') \rangle + \langle \psi^*, \nu - \nu' \rangle < 0. \tag{3.5}$$

As the SVF Φ is κ_1-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at (δ', θ') for $(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1})$, $-\tau'\Psi$ is κ_2-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', -\tau'\psi')$ for $(\delta_1, -\tau'\psi_1), ..., (\delta_{k-1}, -\tau'\psi_{k-1})$, and Ω is κ_3-$\mathbb{R}^l_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^l}$ at (δ', ν') for $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, on Λ,

$$\begin{aligned}\Phi(\delta) - \theta' \subseteq \overrightarrow{\Delta}^{(k)}_g \Phi(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')(\chi'_{\delta',\delta}(0+)) \\ + \kappa_1 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,\end{aligned}$$

$$\begin{aligned}&(-\tau'\Psi)(\delta) + \tau'\psi \subseteq \\ &\overrightarrow{\Delta}^{(k)}_g (-\tau'\Psi)(\delta', -\tau'\psi', \delta_1 - \delta', -\tau'\psi_1 + \tau'\psi', ..., \delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi') \\ &(\chi'_{\delta',\delta}(0+)) + \kappa_2 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,\end{aligned}$$

and

$$\begin{aligned}\Omega(\delta) - \nu' \subseteq \overrightarrow{\Delta}^{(k)}_g \Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+)) \\ + \kappa_3 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}^l_+.\end{aligned}$$

Hence,

$$\begin{aligned}\theta - \theta' \in \overrightarrow{\Delta}^{(k)}_g \Phi(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')(\chi'_{\delta',\delta}(0+)) \\ + \kappa_1 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,\end{aligned}$$

$$-\tau'\psi + \tau'\psi' \in$$
$$\overrightarrow{\Delta}_g^{(k)}(-\tau'\Psi)(\delta', -\tau'\psi', \delta_1 - \delta', -\tau'\psi_1 + \tau'\psi', ..., \delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi')$$
$$(\chi'_{\delta',\delta}(0+)) + \kappa_2\|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,$$

and

$$\nu - \nu' \in \overrightarrow{\Delta}_g^{(k)}\Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+)) + \kappa_3\|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}^l_+.$$

So, (3.1) and (3.2) imply that

$$\langle \theta^*, \theta - \tau'\psi - (\theta' - \tau'\psi') \rangle + \langle \psi^*, \nu - \nu' \rangle \geq 0,$$

which is a contradiction with respect to (3.5). In consequence, $(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')$ weakly minimizes the problem (FP). ■

In consequence, using the same methodology, we may demonstrate the subsequent theorem.

Theorem 3.2 (Higher-order sufficient requirements of optimality) *Let Λ be an arcwisely connected subset of Δ, δ' be an element of the set of feasible solutions S of* (FP), *$\theta' \in \Phi(\delta')$, $\psi' \in \Psi(\delta')$, and $\nu' \in \Omega(\delta') \cap (-\mathbb{R}^l_+)$. We suppose that the SVF $\psi'\Phi$ is κ_1-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', \psi'\theta')$ for $(\delta_1, \psi'\theta_1), ..., (\delta_{k-1}, \psi'\theta_{k-1})$, the SVF $-\theta'\Psi$ is κ_2-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', -\theta'\psi')$ for $(\delta_1, -\theta'\psi_1), ..., (\delta_{k-1}, -\theta'\psi_{k-1})$, and the SVF Ω is κ_3-$\mathbb{R}^l_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^l}$ at (δ', ν') for $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, on Λ. Make the assumption that there is $(\theta^*, \psi^*) \in \mathbb{R}^j_+ \times \mathbb{R}^l_+$, in addition to $\theta^* \neq \mathbf{0}_{\mathbb{R}^j}$, and* (3.1) *and* (3.4) *are satisfied, in addition to*

$$\Big\langle \theta^*, \overrightarrow{\Delta}_g^{(k)}(\psi'\Phi)(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta')(\chi'_{\delta',\delta}(0+)) + \overrightarrow{\Delta}_g^{(k)}(-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \theta'\psi', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi')(\chi'_{\delta',\delta}(0+)) \Big\rangle + \Big\langle \psi^*, \overrightarrow{\Delta}_g^{(k)}\Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+)) \Big\rangle \geq 0, \quad \forall \delta \in \Lambda. \tag{3.6}$$

Then, $(\delta', \frac{\theta'}{\psi'})$ weakly minimizes (FP).

We consider the higher-order duals of the Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds, linked to (FP). As a result, we investigate the associated higher-order results of duality.

Let us assume that

$(\delta_1, \theta_1), ..., (\delta_{k-1}, \theta_{k-1}) \in \Delta \times \mathbb{R}^j$, $(\delta_1, \psi_1), ..., (\delta_{k-1}, \psi_{k-1}) \in \Delta \times \mathbb{R}^j$,

and

$(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1}) \in \Delta \times \mathbb{R}^l$

in addition to

$\delta_1, ..., \delta_{k-1} \in \Lambda$, $\theta_1 \in \Phi(\delta_1) + \mathbb{R}^j_+, ..., \theta_{k-1} \in \Phi(\delta_{k-1}) + \mathbb{R}^j_+$,

$\psi_1 \in \Psi(\delta_1) + \mathbb{R}^j_+, ..., \psi_{k-1} \in \Psi(\delta_{k-1}) + \mathbb{R}^j_+$,

and

$\nu_1 \in \Omega(\delta_1) + \mathbb{R}^l_+, ..., \nu_{k-1} \in \Omega(\delta_{k-1}) + \mathbb{R}^l_+$.

4. Higher-order Mond-Weir kind of dual

For now, let us assume the higher-order Mond-Weir kind of dual (MWD), linked to (FP).

$$\text{maximize } \frac{\theta'}{\psi'},$$

s. t.,

$$\begin{aligned}
&\Big\langle \theta^*, \overrightarrow{\Delta}_g^{(k)}(\psi'\Phi)(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta') \\
&\qquad (\chi'_{\delta',\delta}(0+)) \\
&+ \overrightarrow{\Delta}_g^{(k)}(-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \theta'\psi', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi') \\
&\qquad (\chi'_{\delta',\delta}(0+))\Big\rangle \\
&+ \Big\langle \psi^*, \overrightarrow{\Delta}_g^{(k)}\Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+))\Big\rangle \\
&\qquad \geq 0, \forall \delta \in \Lambda, \\
&\qquad \langle \psi^*, \nu' \rangle \geq 0, \\
&\delta' \in \Lambda, \theta' \in \Phi(\delta'), \psi' \in \Psi(\delta'), \nu' \in \Omega(\delta'), \theta^* \in \mathbb{R}^j_+, \psi^* \in \mathbb{R}^l_+ \text{ and } \langle \theta^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1.
\end{aligned} \tag{MWD}$$

A point $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$, which satisfies all the constraints of (MWD) is termed feasible with respect to (MWD).

Definition 4.1 *A point* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *in the set of feasible solutions of the problem* (MWD) *is termed a weak maximizer of* (MWD) *if there is no point* $(\delta, \theta, \psi, \nu, \theta_1^*, \psi_1^*)$ *in the set of feasible solutions of* (MWD) *such that*

$$\frac{\theta}{\psi} - \frac{\theta'}{\psi'} \in \text{int}(\mathbb{R}^j_+).$$

Theorem 4.1 (Higher-order weak duality) *Let Λ be an arcwisely connected subset of Δ, $\overline{\delta}$ be an element of the set of feasible solutions S of the problem* (FP)*, and $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ be feasible with respect to the problem* (MWD)*. Take note that $\psi'\Phi$ is κ_1-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', \psi'\theta')$ for $(\delta_1, \psi'\theta_1), ..., (\delta_{k-1}, \psi'\theta_{k-1})$, $-\theta'\Psi$ is κ_2-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', -\theta'\psi')$ for $(\delta_1, -\theta'\psi_1), ..., (\delta_{k-1}, -\theta'\psi_{k-1})$, and Ω is κ_3-$\mathbb{R}^l_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^l}$ at (δ', ν') for $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, on Λ, so that*

$$(\kappa_1 + \kappa_2) + \kappa_3 \langle \psi^*, \mathbf{1}_{\mathbb{R}^l} \rangle \geq 0. \tag{4.7}$$

Then,

$$\frac{\Phi(\overline{\delta})}{\Psi(\overline{\delta})} - \frac{\theta'}{\psi'} \subseteq \mathbb{R}^j \setminus -\mathrm{int}(\mathbb{R}^j_+).$$

Proof Make the assumption that for some $\overline{\theta} \in \Phi(\overline{\delta})$ and $\overline{\psi} \in \Psi(\overline{\delta})$,

$$\frac{\overline{\theta}}{\overline{\psi}} - \frac{\theta'}{\psi'} \in -\mathrm{int}(\mathbb{R}^j_+).$$

Therefore,

$$\frac{\overline{\theta}}{\overline{\psi}} < \frac{\theta'}{\psi'}.$$

So,

$$\psi'\overline{\theta} - \theta'\overline{\psi} < 0.$$

Hence,

$$\langle \theta^*, \psi'\overline{\theta} - \theta'\overline{\psi} \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^j} \neq \theta^* \in \mathbb{R}^j_+.$$

As $\overline{\delta} \in S$, we have

$$\Omega(\overline{\delta}) \cap (-\mathbb{R}^l_+) \neq \emptyset.$$

We select $\overline{\nu} \in \Omega(\overline{\delta}) \cap (-\mathbb{R}^l_+)$.
So,

$$\langle \psi^*, \overline{\nu} \rangle \leq 0.$$

From the prerequisites of (MWD), we have

$$\langle \psi^*, \nu' \rangle \geq 0.$$

So,

$$\langle \psi^*, \overline{\nu} - \nu' \rangle = \langle \psi^*, \overline{\nu} \rangle - \langle \psi^*, \nu' \rangle \leq 0.$$

Hence,

$$\langle \theta^*, \psi' \overline{\theta} - \theta' \overline{\psi} \rangle + \langle \psi^*, \overline{\nu} - \nu' \rangle < 0. \tag{4.8}$$

As $\psi'\Phi$ is κ_1-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', \psi'\theta')$ for the vectors $(\delta_1, \psi'\theta_1), ..., (\delta_{k-1}, \psi'\theta_{k-1})$, $-\theta'\Psi$ is κ_2-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', -\theta'\psi')$ for $(\delta_1, -\theta'\psi_1), ..., (\delta_{k-1}, -\theta'\psi_{k-1})$, and Ω is κ_3-$\mathbb{R}^l_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^l}$ at (δ', ν') for the vectors $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, on Λ,

$$\begin{aligned}(\psi'\Phi)(\overline{\delta}) - \psi'\theta' \subseteq \overrightarrow{\Delta}_g^{(k)} \Phi(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta') \\ (\chi'_{\delta', \overline{\delta}}(0+)) + \kappa_1 \|\overline{\delta} - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,\end{aligned}$$

$$\begin{aligned}&(-\theta'\Psi)(\overline{\delta}) + \theta'\psi' \subseteq \\ &\overrightarrow{\Delta}_g^{(k)} (-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \theta'\psi', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi') \\ &(\chi'_{\delta', \overline{\delta}}(0+)) + \kappa_2 \|\overline{\delta} - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} +, \mathbb{R}^j_+,\end{aligned}$$

and

$$\begin{aligned}&\Omega(\overline{\delta}) - \nu' \subseteq \\ &\overrightarrow{\Delta}_g^{(k)} \Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta', \overline{\delta}}(0+)) \\ &+ \kappa_3 \|\overline{\delta} - \delta'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}^l_+.\end{aligned}$$

Hence,

$$\begin{aligned}&\psi'\overline{\theta} - \psi'\theta' \in \\ &\overrightarrow{\Delta}_g^{(k)} (\psi'\Phi)(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta')(\chi'_{\delta', \overline{\delta}}(0+)) \\ &+ \kappa_1 \|\overline{\delta} - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,\end{aligned}$$

$$\begin{aligned}&-\theta'\overline{\psi} + \theta'\psi' \in \\ &\overrightarrow{\Delta}_g^{(k)} (-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \psi'\theta', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi') \\ &(\chi'_{\delta', \overline{\delta}}(0+)) + \kappa_2 \|\overline{\delta} - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,\end{aligned}$$

and

$$\begin{aligned}\overline{\nu} - \nu' \in \overrightarrow{\Delta}_g^{(k)} \Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta', \overline{\delta}}(0+)) \\ + \kappa_3 \|\overline{\delta} - \delta'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}^l_+.\end{aligned}$$

From the prerequisites of (MWD) and (4.7),

$$\langle \theta^*, \psi'\overline{\theta} - \theta'\overline{\psi}\rangle + \langle \psi^*, \overline{\nu} - \nu'\rangle \geq 0,$$

which contradicts (4.8).

Therefore,

$$\frac{\overline{\theta}}{\overline{\psi}} - \frac{\theta'}{\psi'} \notin -\mathrm{int}(\mathbb{R}^j_+).$$

Since $\overline{\theta} \in F(\overline{\delta})$ is chosen arbitrarily,

$$\frac{\Phi(\overline{\delta})}{\Psi(\overline{\delta})} - \frac{\theta'}{\psi'} \subseteq \mathbb{R}^j \setminus -\mathrm{int}(\mathbb{R}^j_+).$$

■

Theorem 4.2 *[Higher-order strong duality] Let* $(\delta', \frac{\theta'}{\psi'})$ *weakly minimize the problem* (FP) *and* $\nu' \in \Omega(\delta') \cap (-\mathbb{R}^l_+)$. *Suppose that for some* $(\theta^*, \psi^*) \in \mathbb{R}^j_+ \times \mathbb{R}^l_+$, *in addition to* $\langle \theta^*, \mathbf{1}_{\mathbb{R}^j}\rangle = 1$, *(3.4) and (3.2) are fulfilled at* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$. *Then,* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *is feasible to* (MWD). *Moreover, if Theorem 4.1 between* (FP) *and* (MWD) *is true, then the element* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *weakly maximizes* (MWD).

Proof As (3.4) and (3.2) are fulfilled at $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$, we have

$$\begin{aligned}
&\Big\langle \theta^*, \overrightarrow{\Delta}^{(k)}_g(\psi'\Phi)(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta')\\
&(\chi'_{\delta',\delta}(0+))\\
&+ \overrightarrow{\Delta}^{(k)}_g(-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \theta'\psi', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi')\\
&(\chi'_{\delta',\delta}(0+))\Big\rangle\\
&+ \Big\langle \psi^*, \overrightarrow{\Delta}^{(k)}_g\Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+))\Big\rangle\\
&\geq 0, \quad \forall \delta \in \Lambda,
\end{aligned}$$

and

$$\langle \psi^*, \nu'\rangle = 0.$$

So, $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ is feasible to (MWD). Make the assumption that the Theorem 4.1 between (FP) and (MWD) becomes valid and $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$

does not maximize weakly (MWD). And then there exists a point $(\delta, \theta, \psi, \nu, \theta_1^*, \psi_1^*)$ in the set of feasible solutions of (MWD), for which

$$\frac{\theta'}{\psi'} < \frac{\theta}{\psi},$$

but this contradicts the Theorem 4.1 between (FP) and (MWD). In consequence, $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ weakly maximizes (MWD). ■

THEOREM 4.3 *[Higher-order converse duality] Let Λ be an arcwisely connected subset of Δ and $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ be feasible to the problem* (MWD). *Take note that $\psi'\Phi$ is κ_1-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', \psi'\theta')$ for $(\delta_1, \psi'\theta_1), ..., (\delta_{k-1}, \psi'\theta_{k-1})$, $-\theta'\Psi$ is κ_2-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', -\theta'\psi')$ for $(\delta_1, -\theta'\psi_1), ..., (\delta_{k-1}, -\theta'\psi_{k-1})$, and Ω is κ_3-$\mathbb{R}^l_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^l}$ at (δ', ν') for $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, on Λ, satisfying* (4.7). *If δ' is an element of the set of feasible solutions S of the problem* (FP), *then $(\delta', \frac{\theta'}{\psi'})$ weakly minimizes the problem* (FP).

PROOF Suppose that $(\delta', \frac{\theta'}{\psi'})$ does not minimize the problem (FP). Therefore, there are $\delta \in S$, $\theta \in \Phi(\delta)$ and $\psi \in \Psi(\delta)$ such that

$$\frac{\theta}{\psi} < \frac{\theta'}{\psi'}.$$

So,

$$\theta\psi' - \theta'\psi < 0.$$

Therefore,

$$\langle \theta^*, \psi'\theta - \theta'\psi \rangle < 0, \text{ since } \mathbf{0}_{\mathbb{R}^j} \neq \theta^* \in \mathbb{R}^j_+.$$

As $\delta \in S$,

$$\Omega(\delta) \cap (-\mathbb{R}^l_+) \neq \emptyset.$$

We select $\psi \in \Omega(\delta) \cap (-\mathbb{R}^l_+)$.
So,

$$\langle \psi^*, \psi \rangle \leq 0.$$

From the prerequisites of (WD),

$$\langle \psi^*, \nu' \rangle \geq 0.$$

So,

$$\langle \psi^*, \nu - \nu' \rangle = \langle \psi^*, \psi \rangle - \langle \psi^*, \nu' \rangle \leq 0.$$

Hence,

$$\langle \theta^*, \psi'\theta - \theta'\psi \rangle + \langle \psi^*, \nu - \nu' \rangle < 0. \tag{4.9}$$

As $\psi'\Phi$ is κ_1-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', \psi'\theta')$ for the elements $(\delta_1, \psi'\theta_1), ..., (\delta_{k-1}, \psi'\theta_{k-1})$, $-\theta'\Psi$ is κ_2-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', -\theta'\psi')$ for $(\delta_1, -\theta'\psi_1), ..., (\delta_{k-1}, -\theta'\psi_{k-1})$, and Ω is κ_3-$\mathbb{R}^l_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^l}$ at (δ', ν') for the elements $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, on Λ,

$$(\psi'\Phi)(\delta) - \psi'\theta' \subseteq \overrightarrow{\Delta}^{(k)}_g \Phi(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta')$$
$$(\chi'_{\delta',\delta}(0+)) + \kappa_1 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,$$

$$(-\theta'\Psi)(\delta) + \theta'\psi' \subseteq$$
$$\overrightarrow{\Delta}^{(k)}_g (-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \theta'\psi', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi')$$
$$(\chi'_{\delta',\delta}(0+)) + \kappa_2 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,$$

and

$$\Omega(\delta) - \nu' \subseteq \overrightarrow{\Delta}^{(k)}_g \Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+))$$
$$+ \kappa_3 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}^l_+.$$

Hence,

$$\psi' q - \psi'\theta' \in$$
$$\overrightarrow{\Delta}^{(k)}_g (\psi'\Phi)(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta')(\chi'_{\delta',\delta}(0+))$$
$$+ \kappa_1 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,$$

$$-\theta'\psi + \theta'\psi' \in$$
$$\overrightarrow{\Delta}^{(k)}_g (-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \psi'\theta', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi')$$
$$(\chi'_{\delta',\delta}(0+)) + \kappa_2 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^j} + \mathbb{R}^j_+,$$

and

$$\nu - \nu' \in \overrightarrow{\Delta}^{(k)}_g \Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+))$$
$$+ \kappa_3 \|\delta - \delta'\|^2 \mathbf{1}_{\mathbb{R}^l} + \mathbb{R}^l_+.$$

So, from the requirements of (WD) and (4.7),

$$\langle \theta^*, \psi'\theta - \theta'\psi\rangle + \langle \psi^*, \nu - \nu'\rangle \geq 0,$$

which contradicts (4.9). Therefore, $(\delta', \frac{\theta'}{\psi'})$ weakly minimizes (FP). ∎

5. Higher-order Wolfe kind of dual

For now, let us assume the higher-order Wolfe kind of dual (WD), linked to (FP).

$$\begin{aligned}
&\text{maximize } \frac{\theta' + \langle \psi^*, \nu'\rangle \mathbf{1}_{\mathbb{R}^j}}{\psi'},\\
&\text{s. t.}\\
&\Big\langle \theta^*, \overrightarrow{\Delta}_g^{(k)}(\psi'\Phi)(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta')\\
&(\chi'_{\delta',\delta}(0+))\\
&+\overrightarrow{\Delta}_g^{(k)}(-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \theta'\psi', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi')\\
&(\chi'_{\delta',\delta}(0+))\Big\rangle\\
&+\Big\langle \psi^*, \overrightarrow{\Delta}_g^{(k)}\Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+))\Big\rangle\\
&\geq 0, \quad \forall \delta \in \Lambda,\\
&\delta' \in \Lambda, \theta' \in \Phi(\delta'), \psi' \in \Psi(\delta'), \nu' \in \Omega(\delta'), \theta^* \in \mathbb{R}^j_+, \psi^* \in \mathbb{R}^l_+ \text{ and } \langle \theta^*, \mathbf{1}_{\mathbb{R}^j}\rangle = 1.
\end{aligned} \tag{WD}$$

A point $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$, which satisfies all the constraints of (WD) is termed feasible with respect to (WD).

Definition 5.1 *A point $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ from the set of feasible solutions of the problem* (WD) *is termed a weak maximizer of* (WD) *if there is no point $(\delta, \theta, \psi, \nu, \theta_1^*, \psi_1^*)$ in the set of feasible solutions of* (WD) *such that*

$$\frac{q + \langle \psi_1^*, \nu\rangle \mathbf{1}_{\mathbb{R}^j}}{\psi} - \frac{\theta' + \langle \psi^*, \nu'\rangle \mathbf{1}_{\mathbb{R}^j}}{\psi'} \in \operatorname{int}(\mathbb{R}^j_+).$$

Theorem 5.1 (Higher-order weak duality) *Let Λ be an arcwisely connected subset of Δ, $\overline{\delta}$ be an element of the set of feasible solutions S of the problem* (FP) *and $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ be feasible with respect to* (WD)*. Suppose that the SVF $\psi'\Phi$ is κ_1-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', \psi'\theta')$ for $(\delta_1, \psi'\theta_1), ..., (\delta_{k-1}, \psi'\theta_{k-1})$, the SVF $-\theta'\Psi$ is κ_2-$\mathbb{R}^j_+$-arcwisely connected of k-th order in relation to $\mathbf{1}_{\mathbb{R}^j}$ at $(\delta', -\theta'\psi')$ for $(\delta_1, -\theta'\psi_1), ..., (\delta_{k-1},$*

$-\theta'\psi_{k-1})$*, and the SVF* Ω *is* κ_3*-*$\mathbb{R}^l_+$*-arcwisely connected of k-th order in relation to* $\mathbf{1}_{\mathbb{R}^l}$ *at* (δ',ν') *for* $(\delta_1,\nu_1),...,(\delta_{k-1},\ \nu_{k-1})$*, on* Λ*, satisfying* (4.7)*.*
Then,

$$\frac{\Phi(\bar{\delta})}{\Psi(\bar{\delta})}-\frac{\theta'+\langle\psi^*,\nu'\rangle\mathbf{1}_{\mathbb{R}^j}}{\psi'}\subseteq\mathbb{R}^j\setminus-\mathrm{int}(\mathbb{R}^j_+).$$

Proof The proof bears similarities to that for Theorem 4.1. ■

Theorem 5.2 (Higher-order strong duality) *Let* $(\delta',\frac{\theta'}{\psi'})$ *weakly minimize the problem* (FP) *and* $\nu'\in\Omega(\delta')\cap(-\mathbb{R}^l_+)$*. Suppose that for some* $(\theta^*,\psi^*)\in\mathbb{R}^j_+\times\mathbb{R}^l_+$*, in addition to* $\langle\theta^*,\mathbf{1}_{\mathbb{R}^j}\rangle=1$*, (3.4) and (3.2) are fulfilled at* $(\delta',\theta',\psi',\nu',\theta^*,\psi^*)$*. Then,* $(\delta',\theta',\psi',\nu',\theta^*,\psi^*)$ *is feasible with respect to* (WD)*. Moreover, if Theorem 5.1 between* (FP) *and* (WD) *is true, then* $(\delta',\theta',\psi',\nu',\theta^*,\psi^*)$ *weakly maximizes* (WD)*.*

Proof The proof bears similarities to that for Theorem 4.2. ■

Theorem 5.3 (Higher-order converse duality) *Let* Λ *be an arcwisely connected subset of* Δ*,* $(\delta',\theta',\psi',\nu',\theta^*,\psi^*)$ *be feasible with respect to the problem* (WD) *and* $\langle\psi^*,\nu'\rangle\geq 0$*. Suppose that* $\psi'\Phi$ *is* κ_1*-*$\mathbb{R}^j_+$*-arcwisely connected of k-th order in relation to* $\mathbf{1}_{\mathbb{R}^j}$ *at* $(\delta',\psi'\theta')$ *for* $(\delta_1,\psi'\theta_1),...,(\delta_{k-1},\psi'\theta_{k-1})$*,* $-\theta'\Psi$ *is* κ_2*-*$\mathbb{R}^j_+$*-arcwisely connected of k-th order in relation to* $\mathbf{1}_{\mathbb{R}^j}$ *at* $(\delta',-\theta'\psi')$ *for the vectors* $(\delta_1,-\theta'\psi_1),...,(\delta_{k-1},-\theta'\psi_{k-1})$*, and* Ω *is* κ_3*-*$\mathbb{R}^l_+$*-arcwisely connected of k-th order in relation to* $\mathbf{1}_{\mathbb{R}^l}$ *at* (δ',ν') *for* $(\delta_1,\nu_1),...,(\delta_{k-1},\nu_{k-1})$*, on* Λ*, satisfying* (4.7)*. If* δ' *is an element of the set of feasible solutions* S *of the problem* (FP)*, then* $(\delta',\frac{\theta'}{\psi'})$ *weakly minimizes the problem* (FP)*.*

Proof The proof bears similarities to that for Theorem 4.3. ■

6. Higher-order mixed kind of dual

For now, let us assume the higher-order mixed kind of dual (MD) linked to (FP).

maximize $\frac{\theta'+\langle\psi^*,\nu'\rangle\mathbf{1}_{\mathbb{R}^j}}{\psi'}$,
s. t.

$$\Big\langle \theta^*, \overrightarrow{\Delta}_g^{(k)}(\psi'\Phi)(\delta', \psi'\theta', \delta_1 - \delta', \psi'\theta_1 - \psi'\theta', ..., \delta_{k-1} - \delta', \psi'\theta_{k-1} - \psi'\theta')$$
$$(\chi'_{\delta',\delta}(0+))$$
$$+\overrightarrow{\Delta}_g^{(k)}(-\theta'\Psi)(\delta', -\theta'\psi', \delta_1 - \delta', -\theta'\psi_1 + \theta'\psi', ..., \delta_{k-1} - \delta', -\theta'\psi_{k-1} + \theta'\psi')$$
$$(\chi'_{\delta',\delta}(0+))\Big\rangle$$
$$+\Big\langle \psi^*, \overrightarrow{\Delta}_g^{(k)}\Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+))\Big\rangle$$
$$\geq 0, \forall \delta \in \Lambda,$$
$$\langle \psi^*, \nu' \rangle \geq 0,$$
$$\delta' \in \Lambda, \theta' \in \Phi(\delta'), \psi' \in \Psi(\delta'), \nu' \in \Omega(\delta'), \theta^* \in \mathbb{R}^j_+, \psi^* \in \mathbb{R}^l_+ \text{ and } \langle \theta^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1. \tag{MD}$$

A point $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$, fulfilling every criterion of the problem (MD) is termed feasible with respect to (MD).

Definition 6.1 *A point* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *in the set of feasible solutions of the problem* (MD) *is termed a weak maximizer of* (MD) *if there is no point* $(\delta, \theta, \psi, \nu, \theta_1^*, \psi_1^*)$ *in the set of feasible solutions of* (MD) *such that*

$$\frac{q + \langle \psi_1^*, \nu \rangle \mathbf{1}_{\mathbb{R}^j}}{\psi} - \frac{\theta' + \langle \psi^*, \nu' \rangle \mathbf{1}_{\mathbb{R}^j}}{\psi'} \in \mathrm{int}(\mathbb{R}^j_+).$$

Theorem 6.1 (Higher-order weak duality) *Let* Λ *be an arcwisely connected subset of* Δ, $\bar{\delta}$ *be an element of the set of feasible solutions* S *of the problem* (FP) *and* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *be feasible with respect to* (MD). *Suppose that the SVF* $\psi'\Phi$ *is* κ_1-$\mathbb{R}^j_+$*-arcwisely connected of* k*-th order in relation to* $\mathbf{1}_{\mathbb{R}^j}$ *at* $(\delta', \psi'\theta')$ *for* $(\delta_1, \psi'\theta_1), ..., (\delta_{k-1}, \psi'\theta_{k-1})$, *the SVF* $-\theta'\Psi$ *is* κ_2-$\mathbb{R}^j_+$*-arcwisely connected of* k*-th order in relation to* $\mathbf{1}_{\mathbb{R}^j}$ *at* $(\delta', -\theta'\psi')$ *for* $(\delta_1, -\theta'\psi_1), ..., (\delta_{k-1}, -\theta'\psi_{k-1})$, *and the SVF* Ω *is* κ_3-$\mathbb{R}^l_+$*-arcwisely connected of* k*-th order in relation to* $\mathbf{1}_{\mathbb{R}^l}$ *at* (δ', ν') *for* $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, *on* Λ, *satisfying* (4.7).
Then,

$$\frac{\Phi(\bar{\delta})}{\Psi(\bar{\delta})} - \frac{\theta' + \langle \psi^*, \nu' \rangle \mathbf{1}_{\mathbb{R}^j}}{\psi'} \subseteq \mathbb{R}^j \setminus -\mathrm{int}(\mathbb{R}^j_+).$$

Proof The proof bears similarities to that for Theorem 4.1. ■

Theorem 6.2 (Higher-order strong duality) *Let* $(\delta', \frac{\theta'}{\psi'})$ *weakly minimize the problem* (FP) *and* $\nu' \in \Omega(\delta') \cap (-\mathbb{R}^l_+)$. *Suppose that for some* $(\theta^*, \psi^*) \in \mathbb{R}^j_+ \times \mathbb{R}^l_+$, *in addition to* $\langle \theta^*, \mathbf{1}_{\mathbb{R}^j} \rangle = 1$, *(3.4) and (3.2) are fulfilled at* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$.

Then $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *is feasible with respect to* (MD). *Moreover, if Theorem 6.1 between* (FP) *and* (MD) *is true, then* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *weakly maximizes* (MD).

PROOF The proof bears similarities to that for Theorem 4.2. ■

THEOREM 6.3 (HIGHER-ORDER CONVERSE DUALITY) *Let* Λ *be an arcwisely connected subset of* Δ *and* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *be feasible with respect to the problem* (MD). *Consider that* $\psi'\Phi$ *is* κ_1-$\mathbb{R}^j_+$*-arcwisely connected of* k*-th order in relation to* $\mathbf{1}_{\mathbb{R}^j}$ *at* $(\delta', \psi'\theta')$ *for* $(\delta_1, \psi'\theta_1), ..., (\delta_{k-1}, \psi'\theta_{k-1})$, $-\theta'\Psi$ *is* κ_2-$\mathbb{R}^j_+$*-arcwisely connected of* k*-th order in relation to* $\mathbf{1}_{\mathbb{R}^j}$ *at* $(\delta', -\theta'\psi')$ *for* $(\delta_1, -\theta'\psi_1), ..., (\delta_{k-1}, -\theta'\psi_{k-1})$, *and* Ω *is* κ_3-$\mathbb{R}^l_+$*-arcwisely connected of* k*-th order in relation to* $\mathbf{1}_{\mathbb{R}^l}$ *at* (δ', ν') *for* $(\delta_1, \nu_1), ..., (\delta_{k-1}, \nu_{k-1})$, *on* Λ, *satisfying*(4.7). *If* δ' *is an element of the set of feasible solutions* S *of the problem* (FP), *then* $(\delta', \frac{\theta'}{\psi'})$ *weakly minimizes the problem* (FP).

PROOF The proof bears similarities to that for Theorem 4.3. ■

We have included the following example to help make the conclusions more clear.

EXAMPLE 6.1 *Let* $k \in \mathbb{N}$, *with* $k \geq 2$, $j = 2, l = 1$, $\Delta = \mathbb{R}^2$, *and* $\Lambda = \{(t, 0) : t \in [-1, 1]\} \subseteq \mathbb{R}^2$. *Let* $\delta, \delta' \in \mathbb{R}^2$. *Consider a continuous arc* $\chi_{\delta',\delta} : [0, 1] \to \mathbb{R}^2$ *defined by*

$$\chi_{\delta',\delta}(\tau) = (1 - \tau^2)\delta' + \tau^2 p, \quad \forall \tau \in [0, 1].$$

Evidently, Λ *is an arcwisely connected subset of* Δ. *Now,*

$$\chi'_{\delta',\delta}(0+) = \lim_{\tau \to 0^+} \frac{\chi_{\delta',\delta}(\tau) - \chi_{\delta',\delta}(0)}{\tau} = (0, 0).$$

We consider three SVFs $\Phi : \Lambda \subseteq \mathbb{R}^2 \to 2^{\mathbb{R}^2}$, $\Psi : \Lambda \subseteq \mathbb{R}^2 \to 2^{\mathbb{R}^2}$, *and* $\Omega : \Lambda \subseteq \mathbb{R}^2 \to 2^{\mathbb{R}}$, *with* $\mathrm{DM}(\Phi) = \mathrm{DM}(\Psi) = \mathrm{DM}(\Omega) = M$, *defined by*

$$\Phi(t, 0) = \begin{cases} \{(x - 10t^2, x^2 - 10t^2) : x \geq 0\}, & \textit{if } 0 \leq t \leq 1, \\ \{(x - 10t^2, x - 10t^2) : x < 0\}, & \textit{if } -1 \leq t < 0, \end{cases}$$

$$\Psi(t, 0) = (1, 1), \quad \forall t \in [-1, 1],$$

and

$$\Omega(t, 0) = \begin{cases} \{x^2 + 11t^2 : x \geq 0\}, & \textit{if } 0 \leq t \leq 1, \\ \{x + 11t^2 : x > 0\}, & \textit{if } -1 \leq t < 0. \end{cases}$$

Let us assume that $(\delta_1,\theta_1),...,(\delta_{k-1},\theta_{k-1})\in\Delta\times\mathbb{R}^j$, $(\delta_1,\psi_1),...,(\delta_{k-1},\psi_{k-1})\in\Delta\times\mathbb{R}^j$, *and* $(\delta_1,\nu_1),...,(\delta_{k-1},\nu_{k-1})\in\Delta\times\mathbb{R}^l$, *in addition to* $\delta_1,...,\delta_{k-1}\in\Lambda$, $\theta_1\in\Phi(\delta_1)+\mathbb{R}^j_+,...,\theta_{k-1}\in\Phi(\delta_{k-1})+\mathbb{R}^j_+$, $\psi_1\in\Psi(\delta_1)+\mathbb{R}^j_+,...,\psi_{k-1}\in\Psi(\delta_{k-1})+\mathbb{R}^j_+$, *and* $\nu_1\in\Omega(\delta_1)+\mathbb{R}^l_+,...,\nu_{k-1}\in\Omega(\delta_{k-1})+\mathbb{R}^l_+$.

Let us choose $\delta'=(\overline{\delta'},\overline{\overline{\delta'}})=(t',0)=(0,0)$, *with* $t'\in[-1,1]$, $\theta'=(\overline{\theta'},\overline{\overline{\theta'}})=(0,0)\in\Phi(\delta')$, $\psi'=(1,1)\in\Psi(\delta')$, *and* $\nu'=0\in\Omega(\delta')\cap(-\mathbb{R}_+)$. *Additionally, choose* $\delta_1=(1,0)$, $\delta_2=(\frac{1}{2},0),...,\delta_{k-1}=(\frac{1}{k-1},0)$, $\theta_1=(-10,-10)$, $\theta_2=(-\frac{10}{2^2},-\frac{10}{2^2}),...,$ $\theta_{k-1}=(-\frac{10}{(k-1)^2},-\frac{10}{(k-1)^2})$, $\psi_1=\psi_2=...=\psi_{k-1}=(1,1)$, *and* $\nu_1=11,\nu_2=\frac{11}{2^2},...,\nu_{k-1}=\frac{11}{(k-1)^2}$. *So,* $\tau'=\frac{\theta'}{\psi'}=(0,0)$.

We can show that Φ *is* (-10)-$\mathbb{R}^2_+$-*arcwisely connected of k-th order in relation to* $(1,1)$ *at* (δ',θ') *for* $(\delta_1,\theta_1),...,(\delta_{k-1},\theta_{k-1})$, $-\tau'\Psi$ *is* 0-$\mathbb{R}^2_+$-*arcwisely connected of k-th order in relation to* $(1,1)$ *at* $(\delta',-\tau'\psi')$ *for* $(\delta_1,-\tau'\psi_1),...,(\delta_{k-1},-\tau'\psi_{k-1})$, *and* Ω *is* 11-$\mathbb{R}_+$-*arcwisely connected of k-th order in relation to* 1 *at* (δ',ν') *for* $(\delta_1,\nu_1),...,(\delta_{k-1},\nu_{k-1})$, *on* Λ. *So,* $\kappa_1=-10$, $\kappa_2=0$, *and* $\kappa_3=11$. *Now, we have*

$$\begin{aligned}\operatorname{epigrp}(\Phi)&=\{(\delta,\theta)\in\mathbb{R}^2\times\mathbb{R}^2:\delta\in\Lambda,\theta\in\Phi(\delta)+\mathbb{R}^2_+\}\\&=\{((t,0),(\overline{\theta},\overline{\overline{\theta}})):\delta=(t,0),\theta=(\overline{\theta},\overline{\overline{\theta}}),t\in[-1,1],\theta\in\Phi(\delta)+\mathbb{R}^2_+\}\\&=\{((t,0),(\overline{\theta},\overline{\overline{\theta}})):0\le t\le 1,\overline{\theta}\ge x-10t^2,\overline{\overline{\theta}}\ge x^2-10t^2,x\ge 0\}\\&\cup\{((t,0),(\overline{\theta},\overline{\overline{\theta}})):-1\le t<0,\overline{\theta}\ge x-10t^2,\overline{\overline{\theta}}\ge x-10t^2,x<0\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&T^{(k)}(\operatorname{epigrp}(\Phi),((t',0),(\overline{\theta'},\overline{\overline{\theta'}})),(\delta_1-\delta',\theta_1-\theta'),...,(\delta_{k-1}-\delta',\theta_{k-1}-\theta'))\\&=\begin{cases}\mathbb{R}_+\times\{0\}\times\mathbb{R}_+\times\mathbb{R}_+, & \text{if } t'=0,\overline{\theta'}=\overline{\overline{\theta'}}=0,\\ \mathbb{R}\times\{0\}\times\mathbb{R}\times\mathbb{R}, & \text{otherwise.}\end{cases}\end{aligned}$$

It is obvious that

$$\begin{aligned}&\text{I-min}\{(\overline{\theta},\overline{\overline{\theta}}):((t,0),(\overline{\theta},\overline{\overline{\theta}}))\in\\&T^{(k)}(\operatorname{epigrp}(\Phi),((t',0),(\overline{\theta'},\overline{\overline{\theta'}})),(\delta_1-\delta',\theta_1-\theta'),...,(\delta_{k-1}-\delta',\theta_{k-1}-\theta'))\}\end{aligned}$$

exists for $0\le t\le 1$ *if and only if* $t'=0,\overline{\theta'}=0$, *and* $\overline{\overline{\theta'}}=0$. *Evidently,*

$$\begin{aligned}&\text{I-min}\{(\overline{\theta},\overline{\overline{\theta}}):((t,0),(\overline{\theta},\overline{\overline{\theta}}))\in\\&T^{(k)}(\operatorname{epigrp}(\Phi),((0,0),(0,0)),(\delta_1-\delta',\theta_1-\theta'),...,(\delta_{k-1}-\delta',\theta_{k-1}-\theta'))\}\\&=\{(0,0)\},\ \textit{for } 0\le t\le 1.\end{aligned}$$

Now, we have

$$\overrightarrow{\Delta}^{(k)}_g\Phi((0,0),(0,0),\delta_1-\delta',\theta_1-\theta',...,\delta_{k-1}-\delta',\theta_{k-1}-\theta')(t,0)$$

$$= \text{I-min}\{(\overline{\theta}, \overline{\overline{\theta}}) : ((t,0),(\overline{\theta}, \overline{\overline{\theta}})) \in T^{(k)}(\text{epigrp}(\Phi), ((0,0),(0,0)), (\delta_1 - \delta', \theta_1 - \theta'), ..., (\delta_{k-1} - \delta', \theta_{k-1} - \theta'))\}.$$

So, Φ is contingent epidifferentiable of k-th order at $((0,0),(0,0))$ for $(\delta_1 - \delta', \theta_1 - \theta'), ..., (\delta_{k-1} - \delta', \theta_{k-1} - \theta')$ with

$$\overrightarrow{\Delta}_g^{(k)}\Phi((0,0),(0,0),\delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')(t,0) = \{(0,0)\}, \quad \forall t, \text{ with } 0 \leq t \leq 1.$$

Likewise, we can demonstrate that $-\tau'\Psi$ is contingent epidifferentiable of k-th order everywhere for $(\delta_1 - \delta', -\tau'\psi_1 + \tau'\psi'), ..., (\delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi')$ and Ω is contingent epidifferentiable of k-th order only at $((0,0),0)$ for $(\delta_1 - \delta', \nu_1 - \nu'), ..., (\delta_{k-1} - \delta', \nu_{k-1} - \nu')$ with

$$\overrightarrow{\Delta}_g^{(k)}(-\tau'\Psi)((0,0),(0,0),\delta_1 - \delta', -\tau'\psi_1 + \tau'\psi', ..., \delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi')(t,0) = \{(0,0)\}, \quad \forall t, \text{ with } -1 \leq t \leq 1,$$

and

$$\overrightarrow{\Delta}_g^{(k)}\Omega((0,0),0,\delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(t,0) = \{0\}, \quad \forall t, \text{ with } 0 \leq t \leq 1.$$

Since $\delta' = (0,0)$, $\theta' = (0,0) \in \Phi(\delta')$, $\psi' = (1,1) \in \Psi(\delta')$, $\nu' = 0 \in \Omega(\delta') \cap (-\mathbb{R}_+)$, and $\tau' = \frac{\theta'}{\psi'} = (0,0)$, the feasible set of the problem (FP) is

$$S = \{(t,0) : \Omega(t,0) \cap (-\mathbb{R}_+) \neq \emptyset\} = \{(0,0)\}.$$

Therefore, $(\delta', \frac{\theta'}{\psi'}) = ((0,0),(0,0))$ weakly minimizes the problem (FP).

Now, we have

$$\begin{aligned}&\overrightarrow{\Delta}_g^{(k)}\Phi(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')(\chi'_{\delta',\delta}(0+)) \\ &= \overrightarrow{\Delta}_g^{(k)}\Phi(\delta', \theta', \delta_1 - \delta', \theta_1 - \theta', ..., \delta_{k-1} - \delta', \theta_{k-1} - \theta')(0,0) = (0,0),\end{aligned}$$

$$\begin{aligned}&\overrightarrow{\Delta}_g^{(k)}(-\tau'\Psi)(\delta', -\tau'\psi', \delta_1 - \delta', -\tau'\psi_1 + \tau'\psi', ..., \delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi') \\ &(\chi'_{\delta',\delta}(0+)) = \\ &\overrightarrow{\Delta}_g^{(k)}(-\tau'\Psi)(\delta', -\tau'\psi', \delta_1 - \delta', -\tau'\psi_1 + \tau'\psi', ..., \delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi')(0,0) \\ &= (0,0),\end{aligned}$$

and

$$\begin{aligned}&\overrightarrow{\Delta}_g^{(k)}\Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(\chi'_{\delta',\delta}(0+)) \\ &= \overrightarrow{\Delta}_g^{(k)}\Omega(\delta', \nu', \delta_1 - \delta', \nu_1 - \nu', ..., \delta_{k-1} - \delta', \nu_{k-1} - \nu')(0,0) = 0.\end{aligned}$$

It is clear that for $\theta^* = (\frac{1}{2}, \frac{1}{2})$ *and* $\psi^* = 1$, *the sufficient conditions of optimality (3.1)–(3.4) are satisfied and* $\langle \theta^*, (1,1) \rangle = 1$. *So,* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *is feasible with respect to the (MWD). Again,*

$$(\kappa_1 + \kappa_2)\langle \theta^*, (1,1) \rangle + \kappa_3 \langle \psi^*, 1 \rangle = 1 \geq 0.$$

Hence the weak duality Theorem (4.1) *holds between (FP) and (MWD).*

We prove that $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$ *weakly maximizes (MWD).*
Let any point feasible to the Mond-Weir category of dual (MWD) be $(\delta, \theta, \psi, \nu, \theta_1^*, \psi_1^*)$, *where* $\delta \in \Lambda, \theta \in \Phi(\delta), \psi \in \Omega(\delta), \nu \in \Omega(\delta) \cap (-\mathbb{R}_+), \theta_1^* \in \mathbb{R}_+^2, \psi_1^* \in \mathbb{R}_+$, *and* $\langle \theta_1^*, (1,1) \rangle = 1$. *Since* Φ *is contingent epidifferentiable of k-th order only at* $((0,0),(0,0))$ *for* $(\delta_1 - \delta', \theta_1 - \theta'), ..., (\delta_{k-1} - \delta', \theta_{k-1} - \theta')$, $-\tau'\Psi$ *is contingent epidifferentiable of k-th order everywhere for* $(\delta_1 - \delta', -\tau'\psi_1 + \tau'\psi'), ..., (\delta_{k-1} - \delta', -\tau'\psi_{k-1} + \tau'\psi')$, Ω *is contingent epidifferentiable of k-th order only at* $((0,0),0)$, *for* $(\delta_1 - \delta', \nu_1 - \nu'), ..., (\delta_{k-1} - \delta', \nu_{k-1} - \nu')$, *and the conditions of the Mond-Weir category of dual (MWD) are satisfied at the point* $(\delta, \theta, \psi, \nu, \theta_1^*, \psi_1^*)$, *we have* $\delta = (0,0)$, $\theta = (0,0)$, $\psi = (1,1)$, *and* $\nu = 0$. *Hence,* $(\delta', \theta', \psi', \nu', \theta^*, \psi^*)$, *with* $\delta' = (0,0)$, $\theta' = (0,0)$, $\psi' = (1,1)$, *and* $\nu' = 0$, *weakly maximizes (MWD).*

In the same way, we can confirm the strong, converse, and weak duality results for mixed, and Wolfe kinds for the problem (FP).

7. Concluding remarks

To solve the set-valued fractional-type optimization problems, we deal with κ-arcwisely connectedness of set-valued maps. Compared to κ-cone convexity, the concept of κ-arcwisely connectedness is more general. In our work, this is demonstrated by providing an example. In essence, we solve the set-valued fractional-type optimization problems using κ-arcwisely connected set-valued mappings as objective functions and constraints. There are other areas for expansion, based on the results of the work offered in this article. The study of symmetric duals and optimality requirements for weakly efficient solutions to set-valued fractional-type optimization problems is possible. The approximate duality results of set-valued fractional-type optimization problems can be established using approximation assumptions for cone convexity. These findings can be expanded upon using a variety of generalized concepts of set-valued map differentiability, including the generalized contingent epiderivative and the $\mathbb{T}$-epiderivative. It is also possible to study the complementarity and control problems in the context of set-valued maps.

In this study, using higher-order contingent epi-derivative hypothesis, we find the higher-order KKT requirements of sufficiency for the SFP (FP). So, under the assumption of higher-order cone arcwisely connectedness, we present the higher-order duals of the Mond-Weir (MWD), Wolfe (WD), and mixed (MD) kinds and derive the higher-order strong, weak, and converse results of duality.

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