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Approximate optimality conditions for approximate efficiency in semi-infinite multiobjective fractional programming problem*

by

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Abstract: In this paper, we obtain approximate necessary and sufficient optimality conditions, characterizing an approximately efficient solution of a semi-infinite multiobjective fractional problem under the closedness qualification condition. As a consequence, we derive approximate necessary and sufficient optimality conditions characterizing an approximately efficient solution for a constrained multiobjective fractional programming problem. Furthermore, we present examples illustrating our main results.

Keywords: fractional semi-infinite multiobjective optimization; multiobjective fractional programming problem; max-function approach; approximate efficient solution; approximate optimality conditions

1. Introduction

A semi-infinite multiobjective fractional programming problem is a mathematical programming problem involving two or more fractional objective functions over a feasible set, described by an infinite (possibly finite) number of constraints. These problems have garnered significant attention from researchers, due to their theoretical and practical importance. Various approaches and perspectives have been proposed to analyze and solve such problems (see Chuong, 2016; Sun, Feng and Teo, 2022; Singh et al., 2021; Pham, 2023; Guo and Jiao, 2021; Zeng, Xu and Fu, 2019, and references therein).

In recent years, much attention has been paid to approximate efficient solutions of multiobjective optimization problems. These solutions are of great

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theoretical and practical significance, as many optimization problems are solved using iterative algorithms or heuristic methods that yield approximations to the theoretical solutions.

To obtain approximate optimality conditions, characterizing an approximate efficient solution for a semi-infinite multiobjective fractional programming problem, which is not generally convex, we formulate an equivalent multiobjective convex optimization problem by using a parametric approach. This equivalent problem can be transformed into a scalar optimization problem through various methods, such as penalty functions (Liu and Yokoyama, 1999; Liu, 1996; 1991) weighted objective sums (Deng, 1997; Dutta and Vetrivel, 2001), and maximum functions (Gutierrez, Jimenez and Novo, 2005, 2006). However, it is necessary to impose some form of constraint qualifications. It is well-known that regularity conditions often fail to hold for finite-dimensional optimization problems and are even more likely to fail in infinite-dimensional cases, such as those involving generalized Slater constraints (Jeyakumar, 2003; Jeyakumar, Lee and Dinh, 2003). For this reason, it is crucial to investigate the approximate optimality conditions for approximate efficient solutions of multiobjective optimization problems under weak constraint qualifications (Liu, Long and Huang, 2023: Lee and Lee, 2018; Chuong and Kim, 2014; Li, Wang and Lin, 2018; Moustaid, Lakhdir and Dali, 2022a,b; Kim and Lee, 2011; Sun, Tan and Teo, 2023; Sun, Huang and Teo, 2024; Kuroiwa and Lee, 2012; Sun, Teo nd Long, 2024; Kerdkaew and Wangkeeree, 2020).

In this paper, we use a max-function approach and combine conjugate analysis with the concept of approximate subdifferential to obtain approximate necessary and sufficient optimality conditions characterizing an approximate efficient solution of a semi-infinite multiobjective fractional problem under the closedness qualification condition. Furthermore, we present examples illustrating the main results of this study. As an application, we derive approximate necessary and sufficient optimality conditions, characterizing an approximate efficient solution of a multiobjective fractional programming problem under a conic and a geometric constraint set.

This paper is organized as follows. In Section 2, we present basic notations and key results from convex analysis. In Section 3, we obtain some approximate necessary and sufficient optimality conditions, characterizing an approximate efficient solution of a semi-infinite multiobjective fractional problem. In Section 4, we provide approximate necessary and sufficient optimality conditions, characterizing an approximate efficient solution for constrained multiobjective fractional programming problem.

2. Preliminaries

In this section, some definitions and results are presented, which will be needed in the sequel. Let $\mathbb{R}^m$ denote the m-dimensional Euclidean space, equipped with the usual Euclidean norm $\|\cdot\|$. We use the notation $\langle \cdot, \cdot \rangle$ for the inner product on $\mathbb{R}^m$ and let $\mathbb{R}^m_+ := \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \; x_i \geq 0, \; i = 1, \ldots, m\}$ be the nonnegative orthant of $\mathbb{R}^m$. For $y = (y_1, \ldots, y_m)$, $z = (z_1, \ldots, z_m)$, the following order notation will be used: $y \leq_{\mathbb{R}^m_+} z$ means $y_i \leq z_i$, for all $i = 1, \ldots, m$, and we denote by $+\infty_{\mathbb{R}^m}$ the greatest element of $\mathbb{R}^m$, i.e. $y \leq_{\mathbb{R}^m_+} +\infty_{\mathbb{R}^m}$, for any $y \in \mathbb{R}^m$ verifying $y + (+\infty_{\mathbb{R}^m}) = (+\infty_{\mathbb{R}^m}) + y = +\infty_{\mathbb{R}^m}$ and $\alpha \cdot (+\infty_{\mathbb{R}^m}) = +\infty_{\mathbb{R}^m}$, for every $\alpha \geq 0$. Let $\mathbb{R}^{(T)}$ be the linear space given below, which has been used for semi-infinite programmming in Goberna and López Cerda (1998).

$$\mathbb{R}^{(T)} := \left\{ \lambda_T = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \;\text{ for all } t \in T \text{ but only finitely many } \; \lambda_t \neq 0 \right\}.$$

Let $\mathbb{R}^{(T)}_+$ be the positive cone in $\mathbb{R}^{(T)}$, defined by

$$\mathbb{R}^{(T)}_+ := \left\{ \lambda_T = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0 \text{ for all } t \in T \right\}.$$

The supporting set of $\lambda_T = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)}$, which is a finite subset of T, is defined by

$$\operatorname{supp} \lambda_T := \{ t \in T \mid \lambda_t \neq 0 \}.$$

For given $\lambda_T = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)}$ and $x_T = (x_t)_{t \in T} \in \mathbb{R}^{(T)}$, we have

$$\sum_{t \in T} \lambda_t x_t = \sum_{t \in \operatorname{supp} \lambda_T} \lambda_t x_t.$$

For a set $D \subset \mathbb{R}^m$ we use the notations cl$D$, co$D$ and cone$D$ to denote the closure, the convex hull of D and the conical hull generated by D, respectively. For a given extended real valued function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, the effective domain and the epigraph of f are defined by $\operatorname{dom} f := \{x \in \mathbb{R}^n : \; f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : \; f(x) \leq r\}$, respectively. The function f is said to be proper if its effective domain is a nonempty subset of $\mathbb{R}^n$. It is said to be lower semicontinuous if and only if its epigraph is closed. Moreover, f is called convex if for every $\lambda \in [0, 1]$ and $x_1, \, x_2 \in \mathbb{R}^n$, we have $f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2)$, or, equivalently, if its epigraph is convex. Furthermore, f is said to be concave if $-f$ is convex.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a given function and $\epsilon \geq 0$. The subdifferential and the ϵ-subdifferential of f at a point $\overline{x} \in \operatorname{dom} f$ are defined, respectively, as follows

$$\partial f(\overline{x}) := \{ x^* \in \mathbb{R}^n, \; \langle x^*, x - \overline{x} \rangle + f(\overline{x}) \leq f(x), \quad \forall x \in \mathbb{R}^n \}$$

and

$$\partial_\epsilon f(\overline{x}) := \{x^* \in \mathbb{R}^n,\ \langle x^*, x-\overline{x}\rangle + f(\overline{x}) - \epsilon \le f(x), \quad \forall x \in \mathbb{R}^n\}.$$

For any proper, convex function f on $\mathbb{R}^n$, the conjugate function of f is defined by

$$\begin{array}{rccl} f^*: & \mathbb{R}^n & \longrightarrow & \mathbb{R}\cup\{-\infty,+\infty\} \\ & x^* & \longmapsto & f^*(x^*) := \sup\limits_{x\in\mathbb{R}^n}\{\langle x^*, x\rangle - f(x)\}. \end{array}$$

Moreover, the Young-Fenchel equality holds

$$f^*(x^*) + f(\overline{x}) = \langle x^*, \overline{x}\rangle \Longleftrightarrow x^* \in \partial f(\overline{x}).$$

As a consequence of that

$$(x^*, \langle x^*, \overline{x}\rangle - f(\overline{x})) \in \mathrm{epi} f^*,\ \ \forall x^* \in \partial f(\overline{x}). \tag{1}$$

The indicator function of a nonempty subset $C \subset \mathbb{R}^n$, δ_C is defined as

$$\delta_C(x) := \begin{cases} 0, \text{ if } x \in C, \\ +\infty, \text{ otherwise.} \end{cases}$$

Recall that δ_C is convex (respectively lower semicontinuous) if the subset C is convex (respectively closed). Let $\epsilon \ge 0$ and $C \subset \mathbb{R}^n$ be a convex subset, the ϵ-normal set of C at $\overline{x}$ is defined by

$$N_\epsilon(\overline{x}, C) := \partial_\epsilon \delta_C(\overline{x}) = \{x^* \in \mathbb{R}^n :\ \langle x^*, x-\overline{x}\rangle \le \epsilon,\ \forall x \in C\}.$$

When $\epsilon = 0$, $N_0(\overline{x}, C)$ collapse to the classical normal cone of C at $\overline{x}$ denoted by $N(\overline{x}, C)$.

Now, let us recall the following results, which will be used in the next section.

Theorem 2.1 (Bot, Grad and Wanka, 2009) *Let $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ be p proper, convex and lower semicontinuous functions such that $\bigcap_{i=1}^{p} \mathrm{dom} f_i \neq \emptyset$. Then*

$$\mathrm{epi}(\sum_{i=1}^{p} f_i)^* = \mathrm{cl}(\sum_{i=1}^{p} \mathrm{epi} f_i^*).$$

Theorem 2.2 (Jeyakumar, Lee ans Dinh, 2003) *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper convex lower semicontinuous function and let $\overline{x} \in \mathrm{dom} f$. Then*

$$\mathrm{epi} f^* = \bigcup_{\epsilon\ge 0}\Big\{(x^*, \langle x^*, \overline{x}\rangle + \epsilon - f(\overline{x}))\ :\ x^* \in \partial_\epsilon f(\overline{x})\Big\}.$$

Lemma 2.1 (Jeyakumar, Lee ans Dinh, 2003) *Let* $h_t : \mathbb{R}^n \to \mathbb{R}, t \in T$, *be convex functions. Then*

$$\bigcup_{\lambda_T \in \mathbb{R}_+^{(T)}} \bigcup_{\theta \geq 0} \left\{ (u^*, \langle u^*, \overline{x} \rangle + \theta - \sum_{t \in T} \lambda_t h_t(\overline{x})),\ u^* \in \partial_\theta \left(\sum_{t \in T} \lambda_t h_t \right) (\overline{x}) \right\}$$

$$= \text{cocone} \left(\bigcup_{t \in T, \epsilon \geq 0} \{(u_t^*, \langle u_t^*, \bar{x} \rangle + \epsilon - h_t(\bar{x})) \mid u_t^* \in \partial_\epsilon h_t(\bar{x})\} \right).$$

Proposition 2.1 (Jeyakumar et al., 1996) *Let* I *be an arbitrary index set and let* f_i , $i \in I$, *be proper lower semicontinuous convex functions on* $\mathbb{R}^n$. *Suppose that there exists* $x_0 \in \mathbb{R}^n$ *such that* $\sup_{i \in I} f_i(x_0) < \infty$. *Then,*

$$\text{epi}(\sup_{i \in I} f_i)^* = \text{cl} \left(\text{co} \bigcup_{i \in I} epi f_i^* \right)$$

where $\sup_{i \in I} f_i : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is defined by* $(\sup_{i \in I} f_i)(x) = \sup_{i \in I} f_i(x)$ *for all* $x \in \mathbb{R}^n$.

3. Approximate necessary and sufficient optimality conditions

In this section, we present approximate necessary and sufficient optimality conditions, characterizing an approximate efficient solution for the following semi-infinite fractional multiobjective optimization problem:

$$(P)\ \inf_{x \in \mathcal{F}} \left\{ \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)} \right) \right\},$$

where the feasible set of (P) is defined by $\mathcal{F} := \{x \in C : h_t(x) \leq 0,\ t \in T\}$, f_i, $-g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(i = 1, \ldots, m)$, $h_t : \mathbb{R}^n \longrightarrow \mathbb{R}$, $t \in T$, are convex, lower semicontinuous functions, T is an arbitrary (possible infinite) index set and C is a nonempty convex closed subset of $\mathbb{R}^n$. Moreover, we assume that for any $x \in \mathcal{F}$, $f_i(x) \geq 0$ and $g_i(x) > 0$ $(i = 1, \ldots, m)$.

Let us recall the following definition of the approximate efficient solution for the problem (P). The first concepts of approximate solution or solution in multiobjective optimization appeared in the works of Kutateladze (1979), Loridan (1984), and White (1986).

Definition 3.1 *Let $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}^m_+$ and $\overline{x}$ be a feasible point of (P), i.e. $\overline{x} \in \mathcal{F}$. The point $\overline{x}$ is called an ϵ-efficient solution of (P) if there is no $x \in \mathcal{F}$ such that*

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i, \text{ for all } i = 1, \dots, m$$
$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(\overline{x})}{g_j(\overline{x})} - \epsilon_j, \text{ for some } j \in \{1, \dots, m\}.$$

Remark 3.1 *When $\epsilon = 0$, then the ϵ-efficient solution becomes the efficient solution.*

Likewise, via max-function approach, we associate to (P) the scalar minimax semi-infinite optimization problem

$$(P_0) \quad \inf_{x \in \mathcal{F}} \Psi(x)$$

where $\Psi(x) := \max\limits_{1 \leq i \leq m} \{f_i(x) - v_i g_i(x)\}$ and $\nu_i = \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i$ $(i = 1, \dots, m)$.

The relationship, linking (P) and (P_0), which will be useful for our purposes, is stated in the following lemma

Lemma 3.1 *Let $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}^m_+$ and suppose that $\overline{x} \in \mathcal{F}$ and $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i \geq 0$ $(i = 1, \dots, m)$, $\gamma_0 := \max\limits_{1 \leq i \leq m} \{\epsilon_i g_i(\overline{x})\}$. Then, the following hold:*
i) If $\overline{x}$ is an ϵ-efficient solution for problem (P), then $\overline{x}$ is a γ-solution of the problem (P_0), $\forall \gamma \geq \gamma_0$.
ii) If $\epsilon \neq 0$, $0 \leq \gamma < \gamma_0$ and $\overline{x}$ is a γ-solution of the problem (P_0), then $\overline{x}$ is an ϵ-efficient solution for problem (P).

Proof *i)* Suppose that there exists a $\gamma_1 \geq \gamma_0$ where $\overline{x}$ is not a γ_1-solution of (P_0). Then, there exists $x_1 \in \mathcal{F}$ such that

$$\Psi(x_1) < \Psi(\overline{x}) - \gamma_1.$$

As $\Psi(\overline{x}) = \max\limits_{1 \leq i \leq m} \{\epsilon_i g_i(\overline{x})\} = \gamma_0$ and $\gamma_1 > \gamma_0$, we have

$$\Psi(x_1) = \max_{1 \leq i \leq m} \{f_i(x_1) - v_i g_i(x_1)\} < 0,$$

i.e.

$$f_i(x_1) - v_i g_i(x_1) < 0, \quad \text{for all } i = 1, \dots, m.$$

Since $g_i(x_1) > 0$ for any $i = 1, \dots, m$, it follows that there exist $x_1 \in \mathcal{F}$ such that for any $i = 1, \dots, m$,

$$\frac{f_i(x_1)}{g_i(x_1)} < \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i.$$

This contradicts the fact that $\overline{x}$ is an ϵ-efficient solution of (P).

ii) Assume that $\overline{x}$ is not an ϵ-efficient solution of (P), then there exists $x \in \mathcal{F}$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i, \text{ for all } i = 1, \dots, m \tag{2}$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(\overline{x})}{g_j(\overline{x})} - \epsilon_j, \text{ for some } j \in \{1, \dots, m\}. \tag{3}$$

As $g_i(x) > 0$ and $\nu_i = \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i$, it follows by (2) and (3) that

$$f_i(x) - \nu_i g_i(x) \leq 0, \text{ for all } i = 1, \dots, m$$
$$f_j(x) - \nu_j g_j(x) < 0, \text{ for some } j \in \{1, \dots, m\}$$

i.e.

$$\Psi(x) = \max_{1 \leq i \leq m} \{f_i(x) - v_i g_i(x)\} \leq 0.$$

Then, for $0 \leq \gamma < \gamma_0$, we have

$$\Psi(x) = \max_{1 \leq i \leq m} \{f_i(x) - v_i g_i(x)\} < \gamma_0 - \gamma = \Psi(\overline{x}) - \gamma.$$

This leads to a contradiction. ■

To obtain the approximate optimality conditions for an approximate solution of problem (P), the following constraint qualification is required

Definition 3.2 *Let* $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}^m_+$, $\overline{x} \in \mathcal{F}$ *and* $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i \geq 0$ $(i = 1, \dots, m)$. *We say that closedness qualification condition* (CQC) *is satisfied at* $\overline{x}$ *if the following set*

$$\text{co}\left(\bigcup_{1 \leq i \leq m} \text{epi} f_i^* + \nu_i \text{epi}(-g_i)^*\right) + \text{cocone}\left(\bigcup_{t \in T} \text{epi} h_t^*\right) + \text{epi}\delta_C^*$$

is closed in the space $\mathbb{R}^{n+1}$.

Now, we give an example intended to show the rationality of the closedness condition (CQC)

Example 3.1 *Let* $\epsilon = \frac{1}{2}$, $\bar{x} = -\frac{1}{2}$, $C = [-1, 1]$, $f(x) = x + 1$, $g(x) = 1$ *and* $h_t(x) = t\max\{x, 0\}$ *for any* $t \in T = [0; +\infty[$. *Then, it is easy to see that* $\nu = \frac{f(\overline{x})}{g(\overline{x})} - \epsilon = 0$, *and* $\overline{x} \in \mathcal{F} = \{x \in C : h_t(x) \leq 0,\ t \in T\}$. *By a simple computation, we have* $\text{epi} f^* = \{1\} \times [-1, +\infty[$, $\text{epi}(-g)^* = \{0\} \times [1, +\infty[$, $\bigcup_{t\in T} \text{epi} h_t^* = [0, +\infty[\times[0, +\infty[$ *and* $\text{epi}\delta_C^* = \{(x^*, r) \in \mathbb{R}^2 :\mid x^* \mid\leq r\} = \text{epi} \mid . \mid$.
Obviously

$$\begin{aligned}&\text{co}\left(\text{epi} f^* + \nu \text{epi}(-g)^*\right) + \text{cocone}\left(\bigcup_{t\in T} \text{epi} h_t^*\right) + \text{epi}\delta_C^* \\ &= \{1\} \times [1, +\infty[+[0, +\infty[\times[0, +\infty[+\text{epi} \mid . \mid\end{aligned}$$

is a closed set. Which yields that closedness qualification condition (CQC) *is satisfied at* $\overline{x}$.

Remark 3.2 *i) In the case where* $m = 1$, *our regularity condition coincides with the closure regularity condition, proposed by Burachik and Jejakumar (2005). This condition has been used by several authors to derive the KKT- optimality conditions.*
ii) Note that when the set $\text{co}\left(\bigcup_{1\leq i\leq m} \text{epi} f_i^* + \nu_i \text{epi}(-g_i)^*\right)$ *is closed, the regularity condition (CQC) can be replaced by the closure of the set*

$$\text{cocone}\left(\bigcup_{t\in T} \text{epi} h_t^*\right) + \text{epi}\delta_C^* = \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \text{epi}\left(\sum_{t\in T} \lambda_t h_t\right)^* + \text{epi}\delta_C^*,$$

which has been utilized to derive the necessary and sufficient optimality conditions for robust optimization problem (see Sun, Feng and Teo, 2022; Zeng, Xu and Fu, 2019; Jiao, Lee and Zhou, 2020, and the references therein).

In the following proposition, we establish a characterization of approximate solutions for the scalar minimax problem (P_0)

Proposition 3.1 *Let* $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}_+^m$, $\gamma \geq 0$, $\overline{x} \in \mathcal{F}$, $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i \geq 0$ $(i = 1, \dots, m)$ *and* $\gamma_0 := \max_{1\leq i\leq m} \{\epsilon_i g_i(\overline{x})\}$. *Then,* $\overline{x}$ *is a* γ*-solution for problem* (P_0) *iff*

$$(0, \gamma - \gamma_0) \in \text{cl}\left\{\text{co}\left(\bigcup_{1\leq i\leq m} \text{epi} f_i^* + \nu_i \text{epi}(-g_i)^*\right) + \text{cocone}\left(\bigcup_{t\in T} \text{epi} h_t^*\right) + \text{epi}\delta_C^*\right\}.$$

Proof Let $\gamma \geq 0$ and suppose that $\overline{x}$ is a γ-solution of the scalar convex problem (P_0). Then, we have

$$0 \in \partial_\gamma \left(\Psi + \delta_{\mathcal{F}}\right)(\overline{x}),$$

where $\Psi(x) := \max\limits_{1\leq i\leq m} \{f_i(x) - v_i g_i(x)\}$. From (1), it results that

$$(0, \gamma - \gamma_0) \in \operatorname{epi}\left(\Psi + \delta_{\mathcal{F}}\right)^*. \tag{4}$$

As f_i, $-g_i$ $h_t : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(i = 1, \ldots, m)$, $t \in T$ are proper, convex, lower semicontinuous functions and $\nu_i \geq 0$, it follows that the function Ψ is also a proper, convex and lower semicontinuous function. Moreover, the indicator function δ_C is proper, convex and lower semicontinuous, since C is a nonempty, convex and closed set. Thus, the functions Ψ and $\delta_{\mathcal{F}}$ satisfy together all the assumptions of Theorem 2.1 and hence it follows from (4) that

$$(0, \gamma - \gamma_0) \in \operatorname{cl}\left(\operatorname{epi}\Psi^* + \operatorname{epi}\delta_{\mathcal{F}}^*\right). \tag{5}$$

On the other hand, by Boţ (2009, Remark 16.12)

$$\operatorname{epi}\delta_{\mathcal{F}}^* = \operatorname{cl}\left(\operatorname{cocone}\bigcup_{t\in T}\operatorname{epi}h_t^* + \operatorname{epi}\delta_C^*\right);$$

further, by Proposition 2.1 and Theorem 2.1 it is easy to see that

$$\operatorname{epi}\Psi^* = \operatorname{clco}\left(\bigcup_{1\leq i\leq m}\operatorname{epi}f_i^* + \nu_i\operatorname{epi}(-g_i)^*\right).$$

Thus, (5) holds if and only if

$$(0, \gamma - \gamma_0) \in \operatorname{cl}\left\{\operatorname{co}\left(\bigcup_{1\leq i\leq m}\operatorname{epi}f_i^* + \nu_i\operatorname{epi}(-g_i)^*\right) + \operatorname{cocone}\left(\bigcup_{t\in T}\operatorname{epi}h_t^*\right) + \operatorname{epi}\delta_C^*\right\}.$$

The proof is complete. ■

Theorem 3.1 *Let* $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}^m_+$, $\overline{x} \in \mathcal{F}$, $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i \geq 0$ $(i = 1, \ldots, m)$ *and* $\gamma_0 := \max\limits_{1\leq i\leq m} \{\epsilon_i g_i(\overline{x})\}$. *Then, we have:*

i) Suppose that the constraint qualification (CQC) is satisfied. If $\overline{x}$ *is an* ϵ-*efficient solution for problem* (P), *then there exist* $\rho_i,\ \alpha_i,\ \beta_i,\ \eta,\ \theta \in \mathbb{R}_+$,

$(i=1,\ldots,m)$, $(\lambda_t)_{t\in T}\in\mathbb{R}_+^{(T)}$, *such that*

$$\sum_{i=1}^{m}\rho_i=1 \tag{6}$$

$$0\in\sum_{i=1}^{m}\rho_i\partial_{\alpha_i}f_i(\overline{x})+\sum_{i=1}^{m}\rho_i\nu_i\partial_{\beta_i}(-g_i)(\overline{x})+\partial_\theta\left(\sum_{t\in T}\lambda_t h_t\right)(\overline{x})+N_\eta(C,\overline{x}) \tag{7}$$

$$\sum_{i=1}^{m}\rho_i\alpha_i+\sum_{i=1}^{m}\rho_i\nu_i\beta_i+\theta+\eta-\sum_{t\in T}\lambda_t h_t(\overline{x})=\sum_{i=1}^{m}\rho_i\epsilon_i g_i(\overline{x}). \tag{8}$$

ii) Let $\epsilon=(\epsilon_1,\ldots,\epsilon_m)\in\mathbb{R}_+^m,\ \epsilon\neq 0$. *If there exist* $\rho_i,\ \alpha_i,\ \beta_i,\ \eta,\ \theta\in\mathbb{R}_+$, $(i=1,\ldots,m)$, $(\lambda_t)_{t\in T}\in\mathbb{R}_+^{(T)}$, *for which we have (6), (7) and*

$$\sum_{i=1}^{m}\rho_i\alpha_i+\sum_{i=1}^{m}\rho_i\nu_i\beta_i+\theta+\eta-\sum_{t\in T}\lambda_t h_t(\overline{x})-\sum_{i=1}^{m}\rho_i\epsilon_i g_i(\overline{x})<0, \tag{9}$$

then $\overline{x}$ *is an* ϵ*-efficient solution for problem* (P).

Proof *i)* Let $\epsilon=(\epsilon_1,\ldots,\epsilon_m)\in\mathbb{R}_+^m$ and $\overline{x}\in\mathcal{F}$. According to Lemma 3.1-(i), if $\overline{x}$ is an ϵ-efficient solution for problem (P) then $\overline{x}$ is a γ_0-solution of the problem (P_0) (with $\gamma=\gamma_0$), and so Proposition 3.1 implies that

$$(0,0)\in\operatorname{cl}\left\{\operatorname{co}\left(\bigcup_{1\le i\le m}\operatorname{epi}f_i^*+\nu_i\operatorname{epi}(-g_i)^*\right)+\operatorname{cocone}\left(\bigcup_{t\in T}\operatorname{epi}h_t^*\right)+\operatorname{epi}\delta_C^*\right\}.$$

Since the constraint qualification (CQC) holds, it follows that

$$(0,0)\in\operatorname{co}\left(\bigcup_{1\le i\le m}\operatorname{epi}f_i^*+\nu_i\operatorname{epi}(-g_i)^*\right)+\operatorname{cocone}\left(\bigcup_{t\in T}\operatorname{epi}h_t^*\right)+\operatorname{epi}\delta_C^*.$$

Thus, there exists $\rho_i\ge 0$ such that $\sum_{i=1}^{m}\rho_i=1$ and

$$(0,0)\in\sum_{i=1}^{m}\rho_i\operatorname{epi}f_i^*+\sum_{i=1}^{m}\rho_i\nu_i\operatorname{epi}(-g_i)^*+\operatorname{cocone}\left(\bigcup_{t\in T}\operatorname{epi}h_t^*\right)+\operatorname{epi}\delta_C^*. \tag{10}$$

By Theorem 2.2 we have

$$\begin{aligned}
\text{epi} f_i^* &= \bigcup_{\alpha_i \geq 0} \Big\{ (x_i^*, \langle x_i^*, \overline{x}\rangle + \alpha_i - f_i(\overline{x})) \ : \ x^* \in \partial_{\alpha_i} f_i(\overline{x}) \Big\} \\
\text{epi}(-g_i)^* &= \bigcup_{\beta_i \geq 0} \Big\{ (y_i^*, \langle y_i^*, \overline{x}\rangle + \beta_i - (-g_i)(\overline{x})) \ : \ y_i^* \in \partial_{\beta_i} (-g_i)(\overline{x}) \Big\} \\
\text{epi}\delta_C^* &= \bigcup_{\eta \geq 0} \Big\{ (c^*, \langle c^*, \overline{x}\rangle + \eta - \delta_C(\overline{x})) \ : \ c^* \in \partial_{\eta} \delta_C(\overline{x}) \Big\} \\
\text{epi} h_t^* &= \bigcup_{\epsilon \geq 0} \left\{ (u_t^*, \langle u_t^*, \bar{x}\rangle + \epsilon - h_t(\bar{x})) : v_i \in \partial_{\epsilon} h_t(\bar{x}) \right\}.
\end{aligned}$$

Now, combining the latter equality of epih_t^* with Lemma 2.1, we have

$$\begin{aligned}
&\text{cocone}\left(\bigcup_{t \in T} \text{epi} h_t^* \right) \\
&= \bigcup_{\lambda_T \in \mathbb{R}_+^{(T)}} \bigcup_{\theta \geq 0} \{ (u^*, \langle u^*, \overline{x}\rangle + \theta - \sum_{t \in T} \lambda_t h_t(\overline{x})) : \ u^* \in \partial_\theta \left(\sum_{t \in T} \lambda_t h_t \right)(\overline{x}) \}.
\end{aligned}$$

Thus, it follows from (10) that

$$\begin{aligned}
(0,0) \in & \sum_{i=1}^m \rho_i \bigcup_{\alpha_i \geq 0} \Big\{ (x_i^*, \langle x_i^*, \overline{x}\rangle + \alpha_i - f_i(\overline{x})) \ : \ x^* \in \partial_{\alpha_i} f_i(\overline{x}) \Big\} \\
& + \sum_{i=1}^m \rho_i \nu_i \bigcup_{\beta_i \geq 0} \Big\{ (y_i^*, \langle y_i^*, \overline{x}\rangle + \beta_i - (-g_i)(\overline{x})) \ : \ y_i^* \in \partial_{\beta_i} (-g_i)(\overline{x}) \Big\} \\
& + \bigcup_{\lambda_T \in \mathbb{R}_+^{(T)}} \bigcup_{\theta \geq 0} \{ (u^*, \langle u^*, \overline{x}\rangle + \theta - \sum_{t \in T} \lambda_t h_t(\overline{x})) : \ u^* \in \partial_\theta \left(\sum_{t \in T} \lambda_t h_t \right)(\overline{x}) \} \\
& + \bigcup_{\eta \geq 0} \Big\{ (c^*, \langle c^*, \overline{x}\rangle + \eta - \delta_C(\overline{x})) \ : \ c^* \in \partial_{\eta} \delta_C(\overline{x}) \Big\}.
\end{aligned}$$

Then, there exist $\rho_i,\ \alpha_i,\ \beta_i,\ \eta,\ \theta \in \mathbb{R}_+$, $(i = 1, \ldots, m)$, $\lambda_T := (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, $x_i^*,\ y_i^*,\ u^*$ and $c^* \in \mathbb{R}^n$ $(i = 1, \ldots, m)$ such that

$$\begin{aligned}
&\sum_{i=1}^m \rho_i = 1 \\
&x_i^* \in \partial_{\alpha_i} f_i(\overline{x}),\ y_i^* \in \partial_{\beta_i}(-g_i)(\overline{x}),\ u^* \in \partial_\theta \left(\sum_{t \in T} \lambda_t h_t \right)(\overline{x}),\ c^* \in N_\eta(C, \overline{x})
\end{aligned} \tag{11}$$

and

$$(0,0)=\sum_{i=1}^m \rho_i(x_i^*,\langle x_i^*,\overline{x}\rangle+\alpha_i-f_i(\overline{x}))+\sum_{i=1}^m \rho_i\nu_i(y_i^*,\langle y_i^*,\overline{x}\rangle+\beta_i-(-g_i)(\overline{x}))$$
$$+(u^*,\langle u^*,\overline{x}\rangle+\theta-\sum_{t\in T}\lambda_t h_t(\overline{x}))+(c^*,\langle c^*,\overline{x}\rangle+\eta-\delta_C(\overline{x})),$$

i.e.

$$0 = \sum_{i=1}^m \rho_i x_i^* + \sum_{i=1}^m \rho_i\nu_i y_i^* + u^* + c^* \tag{12}$$

$$\begin{aligned} 0 &= \sum_{i=1}^m \rho_i(\langle x_i^*,\overline{x}\rangle+\alpha_i-f_i(\overline{x}))+\sum_{i=1}^m \rho_i\nu_i(\langle y_i^*,\overline{x}\rangle+\beta_i-(-g_i)(\overline{x})) \\ &\quad +\langle u^*,\overline{x}\rangle+\theta-\sum_{t\in T}\lambda_t h_t(\overline{x})+\langle c^*,\overline{x}\rangle+\eta-\delta_C(\overline{x}) \\ &= \langle\sum_{i=1}^m \rho_i x_i^*+\sum_{i=1}^m \rho_i\nu_i y_i^*+u^*+c^*,\bar{x}\rangle-\sum_{i=1}^m \rho_i(f_i(\overline{x})-\nu_i g_i(\overline{x})) \\ &\quad -\sum_{t\in T}\lambda_t h_t(\overline{x})-\delta_C(\overline{x})+\sum_{i=1}^m \rho_i\alpha_i+\sum_{i=1}^m \rho_i\nu_i\beta_i+\theta+\eta. \end{aligned} \tag{13}$$

From (11) and (12) we get that

$$0\in\sum_{i=1}^m \rho_i\partial_{\alpha_i}f_i(\overline{x})+\sum_{i=1}^m \rho_i\nu_i\partial_{\beta_i}(-g_i)(\overline{x})+\partial_\theta\left(\sum_{t\in T}\lambda_t h_t\right)(\overline{x})+N_\eta(C,\overline{x}).$$

By (12) and using the fact that $f_i(\overline{x})+\nu_i(-g_i)(\overline{x})=\epsilon_i g_i(\overline{x})$, $\bar{x}\in C$, it follows that the condition (13) reduces to

$$\sum_{i=1}^m \rho_i\epsilon_i g_i(\overline{x}) = \sum_{i=1}^m \rho_i\alpha_i+\sum_{i=1}^m \rho_i\nu_i\beta_i+\theta+\eta-\sum_{t\in T}\lambda_t h_t(\overline{x}).$$

Which allows us to obtain the desired result.

ii) Let $\epsilon=(\epsilon_1,\ldots,\epsilon_m)\in\mathbb{R}_+^m,\ \epsilon\neq 0$, and assume that there exist $\rho_i,\ \alpha_i,\ \beta_i,\ \eta,\ \theta\in\mathbb{R}_+\ \ (i=1,\ldots,m)$, $(\lambda_t)_{t\in T}\in\mathbb{R}_+^{(T)}$, such that (6), (7) and (9) are satisfied. Define

$$\gamma:=\sum_{i=1}^m \rho_i\alpha_i+\sum_{i=1}^m \rho_i\nu_i\beta_i+\theta+\eta-\sum_{t\in T}\lambda_t h_t(\overline{x})-\sum_{i=1}^m \rho_i\epsilon_i g_i(\overline{x})+\gamma_0.$$

By (9) it is easy to see that $\gamma < \gamma_0$, since $\sum_{i=1}^{m} \rho_i \epsilon_i g_i(\overline{x}) \leq \gamma_0$, then we have $0 \leq \gamma$. Thus, $0 \leq \gamma < \gamma_0$.

From the definition of the approximate subdifferential of convex analysis, the condition (7) implies that there exist $x_i^*,\ y_i^*,\ u^*$ and $c^* \in \mathbb{R}^n$ $(i = 1, \ldots, m)$, such that for any $x \in \mathcal{F}$,

$$\begin{aligned}
f_i(x) &\geq f_i(\bar{x}) + \langle x_i^*, x - \bar{x}\rangle - \alpha_i, \quad (i = 1, \ldots, m) &(14)\\
\nu_i(-g_i)(x) &\geq \nu_i(-g_i)(\bar{x}) + \langle y_i^*, x - \bar{x}\rangle - \beta_i, \quad (i = 1, \ldots, m) &(15)\\
0 &\geq \langle c^*, x - \bar{x}\rangle - \eta\\
0 &\geq \sum_{t\in T} \lambda_t h_t(x) \geq \sum_{t\in T} \lambda_t h_t(\bar{x}) + \langle u^*, x - \bar{x}\rangle - \theta
\end{aligned}$$

and

$$\sum_{i=1}^{m} \rho_i x_i^* + \sum_{i=1}^{m} \rho_i \nu_i y_i^* + u^* + c^* = 0. \tag{16}$$

As $\rho_i \geq 0$ $(i = 1, \ldots, m)$, multiplying the inequalities (14), (15) by ρ_i and adding them leads for all $x \in \mathcal{F}$ to

$$\begin{aligned}
\sum_{i=1}^{m} \rho_i (f_i + \nu_i(-g_i))(x) \quad \geq \quad & \sum_{i=1}^{m} \rho_i (f_i + \nu_i(-g_i))(\bar{x})\\
& + \langle \sum_{i=1}^{m} \rho_i x_i^* + \sum_{i=1}^{m} \rho_i \nu_i y_i^* + u^* + c^*, x - \overline{x}\rangle\\
& - \sum_{i=1}^{m} \rho_i \alpha_i - \sum_{i=1}^{m} \rho_i \nu_i \beta_i - \theta - \eta + \sum_{t\in T} \lambda_t h_t(\overline{x}).
\end{aligned}$$

Hence, from (16) and using the fact that $f_i(\overline{x}) + \nu_i(-g_i)(\overline{x}) = \epsilon_i g_i(\overline{x})$, $(i = 1, \ldots, m)$, the above inequality reduces, for any x, to

$$\begin{aligned}
\sum_{i=1}^{m} \rho_i (f_i + \nu_i(-g_i))(x) + \delta_{\mathcal{F}}(x) \quad \geq \quad & \sum_{i=1}^{m} \rho_i \epsilon_i g_i(\overline{x}) + \sum_{t\in T} \lambda_t h_t(\overline{x}) - \sum_{i=1}^{m} \rho_i \alpha_i\\
& - \sum_{i=1}^{m} \rho_i \nu_i \beta_i - \theta - \eta \geq \gamma_0 - \gamma.
\end{aligned}$$

Which leads to

$$(0, \gamma - \gamma_0) \in \operatorname{epi} \left(\sum_{i=1}^{m} \rho_i (f_i + \nu_i(-g_i)) + \delta_{\mathcal{F}} \right)^* .$$

By Theorem 2.1 it follows that

$$(0,\gamma-\gamma_0)\in \mathrm{cl}\left(\sum_{i=1}^{m}\rho_i(\mathrm{epi}f_i^*+\nu_i\mathrm{epi}(-g_i)^*)+\mathrm{epi}\delta_{\mathcal{F}}^*\right).$$

By similar arguments as in the proof of Proposition 3.1, we obtain

$$(0,\gamma-\gamma_0)\in \mathrm{cl}\left\{\mathrm{co}\left(\bigcup_{1\le i\le m}\mathrm{epi}f_i^*+\nu_i\mathrm{epi}(-g_i)^*\right)+\mathrm{cocone}\left(\bigcup_{t\in T}\mathrm{epi}h_t^*\right)+\mathrm{epi}\delta_C^*\right\}.$$

So, it follows from Proposition 3.1 that $\overline{x}$ is a γ-solution for problem (P_0) and since $0\le\gamma<\gamma_0$ we have, by Lemma 3.1-(ii), that x is an ϵ-efficient solution for problem (P), and the proof is complete. ∎

Next, we provide an example illustrating Theorem 3.1.

Example 3.2 *Consider the following semi-infinite multiobjective fractional problem*

$$(\mathrm{Q})\begin{cases}\inf\left(\dfrac{x^2}{2},\dfrac{1+x+x^2+y^2}{x+1}\right)\\ 1-tx\le 0,\quad t\in[1,+\infty[\\ (x,y)\in\mathbb{R}\times\mathbb{R}.\end{cases}$$

Let $\epsilon=(\epsilon_1,\epsilon_2)=(\frac{1}{2},\frac{1}{2})$, $f_1(x,y)=x^2$, $f_2(x,y)=1+x+x^2+y^2$, $g_1(x,y)=2$, $g_2(x,y)=x+1$, $h_t(x,y)=1-tx$, $t\in T=[1,+\infty[$ *and* $C=\mathbb{R}\times\mathbb{R}$.
One can easily observe that $(\overline{x},\overline{y})=(1,0)\in\mathcal{F}=\{x\in C: h_t(x)\le 0,\ t\in T\}$,

$$\nu_1=\frac{f_1(\overline{x},\overline{y})}{g_1(\overline{x},\overline{y})}-\epsilon_1=0 \text{ and } \nu_2=\frac{f_2(\overline{x},\overline{y})}{g_2(\overline{x},\overline{y})}-\epsilon_2=1.$$

1) According to the definition of the conjugate function, we have that

$$f_1^*(x^*,y^*)=\begin{cases}\frac{{x^*}^2}{4}, & \text{if } y^*=0,\\ +\infty, & \text{otherwise.}\end{cases}$$

$$f_2^*(x^*,y^*) = \frac{(x^*-1)^2}{4} + \frac{{y^*}^2}{4} - 1$$

$$(-g_1)^*(x^*,y^*) = \begin{cases} 2, & \textit{if } (x^*,y^*) = (0,0), \\ +\infty, & \textit{otherwise.} \end{cases}$$

$$(-g_2)^*(x^*,y^*) = \begin{cases} 1, & \textit{if } (x^*,y^*) = (-1,0), \\ +\infty, & \textit{otherwise.} \end{cases}$$

$$h_t^*(x^*,y^*) = \begin{cases} -1, & \textit{if } (x^*,y^*) = (-t,0), \\ +\infty, & \textit{otherwise.} \end{cases}$$

Hence, by the definition of the epigraph of a function, it follows that

$$\text{co}\left(\bigcup_{1\le i\le m} \text{epi} f_i^* + \nu_i \text{epi}(-g_i)^*\right) = \{((x^*,y^*),r) : \frac{{x^*}^2}{4} + \frac{{y^*}^2}{4} \le r\}$$

$$\text{cocone}\left(\bigcup_{t\in T} \text{epi} h_t^*\right) =]-\infty,-1] \times \{0\} \times [-1,+\infty[.$$

Thus, it is clear that the set

$$\text{co}\left(\bigcup_{1\le i\le m} \text{epi} f_i^* + \nu_i \text{epi}(-g_i)^*\right) + \text{cocone}\left(\bigcup_{t\in T} \text{epi} h_t^*\right) + \text{epi}\delta_C^*,$$

is closed. Which yields that the closedness qualification condition (CQC) *is satisfied at* $(\overline{x},\overline{y})$.

Moreover, the following inequalities, if at least one of them is strict

$$\frac{x^2}{2} \le \frac{1}{2} - \frac{1}{2} = 0 \quad \textit{and} \quad \frac{1+x+x^2+y^2}{x+1} \le \frac{3}{2} - \frac{1}{2} = 1,$$

are not satisfied for any $(x,y) \in \mathcal{F} \setminus \{(\overline{x},\overline{y})\}$. *So, it follows that* $(\overline{x},\overline{y})$ *is an* ϵ*-efficient solution for problem* (Q).

Let $\rho_1 = \rho_2 = \frac{1}{2}$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 1$, $\theta = \eta = 0$, $(\lambda_t)_{t\in T} = (1,0,0,\ldots)$. *Then, by the definition of the approximate subdifferential, we have* $(2,0) \in \partial_{\alpha_1} f_1(1,0) = [0,4] \times \{0\}$,

$(1,0) \in \partial_{\alpha_2} f_2(1,0) = \{(x^*,y^*) \in \mathbb{R}^2 : \frac{(x^*-1)^2}{4} + \frac{{y^*}^2}{4} - 1 \le x^*\}$,

$(0,0) \in \partial_{\beta_1}(-g_1)(1,0) = \{(0,0)\}$,

$(-1,0) \in \partial_{\beta_2}(-g_2)(1,0) = \{(-1,0)\}$,

$(-1,0) \in \partial_\theta \left(\sum_{t\in T} \lambda_t h_t\right)(1,0) = \partial_\theta h_1(1,0) = \{(-1,0)\}$, $N_\eta((1,0),C) = \{(0,0)\}$.

Then, it follows that $\rho_1 + \rho_2 = 1$,

$$(0,0) = \rho_1(2,0) + \rho_1\nu_1(0,0) + \rho_2(1,0) + \rho_2\nu_2(-1,0) + (-1,0) + (0,0)$$
$$\in \sum_{i=1}^{2} \rho_i \partial_{\alpha_i} f_i(1,0) + \sum_{i=1}^{2} \rho_i\nu_i \partial_{\beta_i}(-g_i)(1,0) + \partial_\theta \left(\sum_{t\in T} \lambda_t h_t\right)(1,0) + N_\eta(C,(1,0))$$

and

$$\sum_{i=1}^{2} \rho_i\alpha_i + \sum_{i=1}^{2} \rho_i\nu_i\beta_i + \theta + \eta - \sum_{t\in T} \lambda_t h_t(1,0) = \sum_{i=1}^{2} \rho_i \epsilon_i g_i(1,0).$$

Thus, (i) of Theorem 3.1 holds.

2) By taking $\rho_1 = \rho_2 = \frac{1}{2}$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = 1$, $\theta = \eta = 0$, $(\lambda_t)_{t\in T} = (1,0,0,\ldots)$, *it follows that* $\rho_1 + \rho_2 = 1$

$$(0,0) \in \sum_{i=1}^{2} \rho_i \partial_{\alpha_i} f_i(1,0) + \sum_{i=1}^{2} \rho_i\nu_i \partial_{\beta_i}(-g_i)(1,0)$$
$$+\partial_\theta \left(\sum_{t\in T} \lambda_t h_t\right)(1,0) + N_\eta(C,(1,0),$$

and

$$\sum_{i=1}^{2} \rho_i\alpha_i + \sum_{i=1}^{m} \rho_i\nu_i\beta_i + \theta + \eta - \sum_{t\in T} \lambda_t h_t(1,0) - \sum_{i=1}^{m} \rho_i \epsilon_i g_i(1,0) = -\frac{1}{4} < 0.$$

Thus, the conditions (6), (7), (9) of Theorem 3.1-(ii) are satisfied. Then, it follows that $(1,0)$ *is an* $(\frac{1}{2},\frac{1}{2})$*-efficient solution for problem* (P)*. Consequently, (ii) of Theorem 3.1 is applicable.*

The following example shows that the constraint qualification (CQC) of Theorem 3.1 (i) is essential.

Example 3.3 *Consider the following semi-infinite multiobjective fractional problem*

$$(\text{S})\begin{cases}\inf\left(\dfrac{y+2}{x+1},\dfrac{3x}{2}\right)\\ tx+y\leq 3, \quad t\in]1,2]\\ (x,y)\in\mathbb{R}_+\times\mathbb{R}.\end{cases}$$

Let $\epsilon=(\epsilon_1,\epsilon_2)=(\frac{1}{2},\frac{1}{2}),\ f_1(x,y)=y+2,\ f_2(x,y)=3x,\ g_1(x,y)=x+1,$ $g_2(x,y)=2,\ h_t(x,y)=tx+y-3,\ t\in T=]1,2],\ C=\mathbb{R}_+\times\mathbb{R}.$
Let $(\overline{x},\overline{y})=(1,1)\in\mathcal{F}=\{x\in C: h_t(x)\leq 0,\ t\in T\},\ \nu_1=\frac{f_1(\overline{x},\overline{y})}{g_1(\overline{x},\overline{y})}-\epsilon_1=1$ *and* $\nu_2=\frac{f_2(\overline{x},\overline{y})}{g_2(\overline{x},\overline{y})}-\epsilon_2=1$. *It is clear that (1, 1) is an* ϵ*-efficient solution to problem (S).*

1) By the definition of the conjugate function, we have

$$f_1^*(x^*,y^*)=\begin{cases}-2, & \textit{if}(x^*,y^*)=(0,1),\\ +\infty, & \textit{otherwise.}\end{cases}$$

$$f_2^*(x^*,y^*)=\begin{cases}0, & \textit{if}(x^*,y^*)=(3,0),\\ +\infty, & \textit{otherwise.}\end{cases}$$

$$(-g_1)^*(x^*,y^*)=\begin{cases}2, & \textit{if }(x^*,y^*)=(-1,0),\\ +\infty, & \textit{otherwise.}\end{cases}$$

$$(-g_2)^*(x^*,y^*)=\begin{cases}2, & \textit{if }(x^*,y^*)=(0,0),\\ +\infty, & \textit{otherwise.}\end{cases}$$

$$h_t^*(x^*,y^*)=\begin{cases}3, & \textit{if }(x^*,y^*)=(t,1),\\ +\infty, & \textit{otherwise.}\end{cases}$$

$$\delta_C^*(x^*,y^*)=\begin{cases}0, & \textit{if }(x^*,y^*)\in]-\infty,0]\times\{0\},\\ +\infty, & \textit{otherwise.}\end{cases}$$

Hence, by simple calculation, the epigraphs of the data functions turn out to be

$$\text{co}\left(\bigcup_{1\leq i\leq m}\text{epi}f_i^*+\nu_i\text{epi}(-g_i)^*\right)=\{(2,1)\}\times[1,+\infty[$$

$$\text{cocone}\left(\bigcup_{t\in T}\text{epi}h_t^*\right)=]1,2]\times\{1\}\times[3,+\infty[$$

$$\text{epi}\delta_C^*=\mathbb{R}_-\times\{0\}\times\mathbb{R}_+.$$

Then, it follows that

$$\text{co}\left(\bigcup_{1\le i\le m}\text{epi}f_i^* + \nu_i\text{epi}(-g_i)^*\right) + \text{cocone}\left(\bigcup_{t\in T}\text{epi}h_t^*\right) + \text{epi}\delta_C^*$$

is not a closed set, i.e. the closedness qualification condition (CQC) *is not satisfied at* $(\overline{x}, \overline{y})$.

On the other hand, let $\rho_i,\ \alpha_i,\ \beta_i,\ \eta,\ \theta \ge 0$, $(i = 1, 2)$, $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, *for which we have (6), (7) and (8). By the definition of the approximate subdifferential, we have*

$\partial_{\alpha_1} f_1(1, 1) = \{(0, 1)\},\ \partial_{\alpha_2} f_2(1, 1) = \{(3, 0)\},\ \partial_{\beta_1}(-g_1)(1, 1) = \{(-1, 0)\},$

$\partial_{\beta_2}(-g_2)(1, 1) = \{(0, 0)\},\ \partial_\theta\left(\sum_{t\in T}\lambda_t h_t\right)(1, 1) = \left\{\left(\sum_{t\in T}\lambda_t t, \sum_{t\in T}\lambda_t\right)\right\},$

$N_\eta((1, 1), C) = \{(x^*, y^*) :\ \ x^*(x - 1) \le \eta,\ y^* = 0\}$ *and by using the fact that*

$\nu_i = 1,\ g_i(1, 1) = 2\ (i = 1, 2),\ (x^*, 0) \in N_\eta((1, 1), C)$

and $(\sum_{t\in T}\lambda_t t, \sum_{t\in T}\lambda_t) \in \partial_\theta\left(\sum_{t\in T}\lambda_t h_t\right)(1, 1)$, *we have*

$$\begin{cases}\rho_1 + \rho_2 = 1\\ (0, 0) = \rho_1(0, 1) + \rho_1\nu_1(-1, 0) + \rho_2(3, 0) + \rho_2\nu_2(0, 0) + (x^*, 0) + (\sum_{t\in T}\lambda_t t, \sum_{t\in T}\lambda_t)\\ \sum_{i=1}^{2}\rho_i\alpha_i + \sum_{i=1}^{2}\rho_i\nu_i\beta_i + \theta + \eta - \sum_{t\in T}\lambda_t h_t(1, 1) = \sum_{i=1}^{2}\rho_i\epsilon_i g_i(1, 1)\end{cases}$$

$$\rho_1 + \rho_2 = 1 \tag{17}$$

$$3\rho_2 - \rho_1 + x^* + \sum_{t\in T}\lambda_t t = 0 \tag{18}$$

$$\rho_1 + \sum_{t\in T}\lambda_t = 0 \tag{19}$$

$$\rho_1\alpha_1 + \rho_2\alpha_2 + \rho_1\beta_1 + \rho_2\beta_2 + \theta + \eta - \sum_{t\in T}\lambda_t(t - 2) = \sum_{i=1}^{2}\rho_i = 1. \tag{20}$$

Since $\rho_1 \in \mathbb{R}_+$ *and* $(\lambda_t)_{t\in T} \in \mathbb{R}_+^{(T)}$, *it follows from (19) that* $\rho_1 = 0, \lambda_t = 0,\ t \in T$, *and from (17) that* $\rho_2 = 0$, *and from (18) that* $x^* = -3$. *From (20)* *we get that*

$$\alpha_2 + \beta_2 + \theta + \eta = 1. \tag{21}$$

Since $x^* = -3 \in N_\eta(C,(1,1))$, it follows that for any $x \in \mathbb{R}_+$, the inequality $-3(x-1) \leq \eta$ holds. Upon substituting $x = 0$, we obtain $3 \leq \eta$, which contradicts the condition (21). Thus, (i) cannot hold.

The next example shows that conditions (6) and (7) of Theorem 3.1 (ii) are essential.

EXAMPLE 3.4 *Consider the following multiobjective fractional problem*

$$(\mathrm{Q}_1)\begin{cases}\inf\left(\dfrac{x^2+1}{-y+1}, \dfrac{3}{2}y+1\right)\\ \max\{0,x\}+y^2 \leq 0\\ (x,y) \in \mathbb{R}^2.\end{cases}$$

Let $\epsilon = (\epsilon_1, \epsilon_2) = (\frac{1}{2}, 0)$, $f_1(x,y) = x^2+1$, $f_2(x,y) = \frac{3}{2}y+1$, $g_1(x,y) = -y+1$,

$g_2(x,y) = 1$, $h_t(x,y) = \max\{0,x\}+y^2$, $t \in T = \{1\}$, $C = \mathbb{R}\times\mathbb{R}$

and $(\overline{x},\overline{y}) = (-1,0)$. It is clear that $(\overline{x},\overline{y}) \in \mathcal{F} = \{x \in C : h_1(x) \leq 0\}$, $\nu_1 = \frac{f_1(\overline{x},\overline{y})}{g_1(\overline{x},\overline{y})} - \epsilon_1 = \frac{3}{2}$

and $\nu_2 = \frac{f_2(\overline{x},\overline{y})}{g_2(\overline{x},\overline{y})} - \epsilon_2 = 1$. It is easy to see that $(\overline{x},\overline{y})$ is not an approximate efficient solution.

On the other hand, by the definition of the approximate subdifferential, we have $\partial_\epsilon f_1(-1,0) = [-\sqrt{\epsilon}-2, \sqrt{\epsilon}-2]\times\{0\}$, $\partial_\epsilon f_2(-1,0) = \{(0,\frac{3}{2})\}$,

$\partial_\epsilon(-g_1)(-1,0) = \{(0,1)\}$, $\partial_\epsilon(-g_2)(-1,0) = \{(0,0)\}$,

$\partial_\epsilon(\lambda_1 h_1)(-1,0) = \{(x^*,y^*) : x^* \in [0,\lambda_1], \frac{y^{*2}}{4\lambda_1} \leq -x^*+\theta\}, \quad \lambda_1 \neq 0.$

Suppose that there exist $\rho_i,\ \alpha_i,\ \beta_i,\ \eta,\ \theta \geq 0,\ (i=1,2)$, $\lambda_1 \in \mathbb{R}_+$,

$(x_1^*,0) \in \partial_{\alpha_1} f_1(-1,0)$ and $(x_2^*,y_2^*) \in \partial_\theta(\lambda_1 h_1)(-1,0)$ such that $\rho_1+\rho_2 = 1$,

$$\begin{aligned}(0,0) &= \rho_1(x_1^*,0)+\rho_2(0,\frac{3}{2})+\rho_1\nu_1(0,1)+\rho_2\nu_2(0,0)+(x_2^*,y_2^*)\\ &\in \partial_{\alpha_1} f_1(-1,0)+\partial_{\alpha_2} f_2(-1,0)+\partial_{\beta_1}(-g_1)(-1,0)+\\ &\partial_{\beta_2}(-g_2)(-1,0)+\partial_\theta(\lambda_1 h_1)(-1,0)\end{aligned}$$

and

$$\sum_{i=1}^{2}\rho_i\alpha_i+\sum_{i=1}^{2}\rho_i\nu_i\beta_i+\theta+\eta-\lambda_1 h_1(-1,0)-\sum_{i=1}^{2}\rho_i\epsilon_i g_i(-1,0)<0.$$

Since $\nu_1=\frac{3}{2},\nu_2=1,\ \epsilon_1 g_1(-1,0)=\frac{1}{2},\ \epsilon_2 g_2(-1,0)=0$ *and* $h_1(-1,0)=0$. *Then, it follows that*

$$\rho_1+\rho_2=1 \tag{22}$$

$$\rho_1 x_1^*+x_2^*=0 \tag{23}$$

$$\frac{3}{2}\rho_1+\frac{3}{2}\rho_2+y_2^*=0 \tag{24}$$

$$\sum_{i=1}^{2}\rho_i\alpha_i+\rho_1\frac{3}{2}\beta_1+\rho_2\beta_2+\theta+\eta-\frac{1}{2}\rho_1<0 \tag{25}$$

and as $(x_1^*,0)\in\partial_{\alpha_1}f_1(-1,0)$, $(x_2^*,y_2^*)\in\partial_\theta\left(\lambda_1 h_1\right)(-1,0)$, *then, by the definition of approximate subdifferential, we have for any* $x_1^*,\ y_2^*\in\mathbb{R}$, $x_2^*\in[0,\lambda_1]$ *and* $\lambda_1\neq 0$ *that*

$$\frac{x_1^{*2}}{4}+1\leq -x_1^*+\alpha_1 \tag{26}$$

$$\frac{y_2^{*2}}{4\lambda_1}\leq -x_2^*+\theta. \tag{27}$$

By multiplying inequality (26) by ρ_1 *and adding the terms of the resulting inequality to the terms of inequality (27), we obtain the following inequality*

$$\rho_1\frac{x_1^{*2}}{4}+\rho_1+\frac{y_2^{*2}}{4\lambda_1}\leq -\rho_1 x_1^*-x_2^*+\rho_1\alpha_1+\theta$$

and by (23) it follows that

$$\rho_1\frac{x_1^{*2}}{4}+\frac{y_2^{*2}}{4\lambda_1}+\rho_1\leq\rho_1\alpha_1+\theta,$$

thus

$$\rho_1\frac{x_1^{*2}}{4}+\frac{y_2^{*2}}{4\lambda_1}+\frac{1}{2}\rho_1\leq\rho_1\alpha_1+\theta-\frac{1}{2}\rho_1,$$

which means that

$$0\leq\rho_1\alpha_1+\theta-\frac{1}{2}\rho_1.$$

On the other hand, since $\rho_2, \beta_1, \alpha_2$ are positive numbers, it follows that

$$\sum_{i=1}^{2} \rho_i \alpha_i + \rho_1 \frac{3}{2}\beta_1 + \rho_2\beta_2 + \theta + \eta - \frac{1}{2}\rho_1 \geq 0$$

which contradicts inequality (25).

If $\lambda_1 = 0$, we have $\partial_\theta (\lambda_1 h_1)(-1,0) = \{(0,0)\}$ then, according to (24), it follows that $\frac{3}{2}\rho_1 + \frac{3}{2}\rho_2 = 0$, which contradicts (22).

Thus, the conditions (6), (7), and (9) of Theorem 3.1-(ii) cannot be satisfied, which implies that these conditions are essential.

REMARK 3.3 *i) In the scalar case, Theorem 3.1-(i) coincides with Theorem 3.1 of Sun et al. (2022) when the uncertainty sets V_t are singletons for all $t \in T$. Furthermore, if $C = \mathbb{R}$, it coincides with Corollary 3.1 of Zeng et al. (2019).*
ii) If the functions $g_i \equiv 1$, then Theorem 3.1 and Theorem 4.1 of Gutiérrez et al. (2006) are equivalent.

REMARK 3.4 *i) In the special case, when T is a finite index set, such as $T := \{1, 2, \ldots, p\}$, the problem known as (P) can be reduced to the following standard fractional multiobjective problem*

$$(S)\ \inf_{x \in \mathcal{A}} \left\{ \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)} \right) \right\},$$

where the feasible set of (S) is defined by $\mathcal{A} := \{x \in C : h_j(x) \leq 0,\ j = 1, ..., p\}$, and then we can derive directly the approximate optimality conditions with the only change being the replacement of the index set T with $\{1, 2, \ldots, p\}$, in Theorem 3.1.
ii) In the case when $\epsilon = 0$, we get the necessary optimality conditions characterizing an efficient solution for problem (P).

As a corollary of Theorem 3.1, we derive a characterization of the approximate optimality condition of approximate efficient solution for the following multiobjective semi-infinite convex optimization problem:

$$(P_1)\ \inf_{x \in \mathcal{F}} \{(f_1(x), \ldots, f_m(x))\}.$$

Under the following constraint qualification (CQC_1) :

$$\text{co}\left(\bigcup_{1 \leq i \leq m} \text{epi} f_i^* \right) + \text{cocone}\left(\bigcup_{t \in T} \text{epi} h_t^* \right) + \text{epi}\delta_C^*$$

is closed in the space $\mathbb{R}^{n+1}$.

Corollary 3.1 *For problem (P_1), let $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}^m_+$, $\overline{x} \in \mathcal{F}$, and $\gamma_0 := \max\limits_{1 \leq i \leq m} \{\epsilon_i\}$. Then, we have:*

i) Suppose that the constraint qualification (CQC_1) is satisfied. If $\overline{x}$ is an ϵ-solution for problem (P_1), then there exist $\rho_i,\ \alpha_i,\ \eta,\ \theta \in \mathbb{R}_+$, $(i = 1, \dots, m)$ and $(\lambda_t)_{t \in T} \in \mathbb{R}^{(T)}_+$, such that

$$\sum_{i=1}^{m} \rho_i = 1 \tag{28}$$

$$0 \in \sum_{i=1}^{m} \rho_i \partial_{\alpha_i} f_i(\overline{x}) + \partial_\theta \left(\sum_{t \in T} \lambda_t h_t \right)(\overline{x}) + N_\eta(C, \overline{x}) \tag{29}$$

$$\sum_{i=1}^{m} \rho_i \alpha_i + \theta + \eta - \sum_{t \in T} \lambda_t h_t(\overline{x}) = \sum_{i=1}^{m} \rho_i \epsilon_i g_i(\overline{x}).$$

ii) Let $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}^m_+$, $\epsilon \neq 0$. If there exist $\rho_i,\ \alpha_i,\ \eta,\ \theta \in \mathbb{R}_+$, $(i = 1, \dots, m)$, $(\lambda_t)_{t \in T} \in \mathbb{R}^{(T)}_+$, for which we have (28) and (29)

$$\sum_{i=1}^{m} \rho_i \alpha_i + \theta + \eta - \sum_{t \in T} \lambda_t h_t(\overline{x}) - \sum_{i=1}^{m} \rho_i \epsilon_i g_i(\overline{x}) < 0,$$

then $\overline{x}$ is an ϵ-efficient solution for problem (P_1).

4. Application

We end this paper with an application of Theorem 3.1 to deduce the approximate necessary and sufficient optimality conditions for the vector optimization problem with geometric and cone constraints. Let Y be a real locally convex Hausdorff topological vector space and its continuous dual space Y^* with duality pairing, denoted by $\langle \cdot, \cdot \rangle$. Let $C \subseteq \mathbb{R}^n$ be a nonempty convex set. Let $K \subseteq Y$ be a nonempty closed convex cone and its polar cone be defined as follow

$$K^* := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0,\ \forall y \in K\}.$$

On Y we define the partial order, namely:

$$y_1 \leq_K y_2 \Longleftrightarrow y_2 - y_1 \in -K,$$

for any $y_1, y_2 \in Y$ and we attach to Y an abstract maximal element, denoted by $+\infty_Y$, with respect to " $\leq_K$ " and we denote $\overline{Y} = Y \cup \{+\infty_Y\}$. Then, for every $y \in Y$ one has $y \leq_K +\infty_Y$. The algebraic operations of Y are extended as follows

$$y + (+\infty_Y) = (+\infty_Y) + y = +\infty_Y, \quad \forall y \in Y \cup \{+\infty_Y\},$$

$$\alpha \cdot (+\infty_Y) = +\infty_Y, \quad \forall \alpha \geq 0.$$

Consider the following multiobjective fractional programming problem under a conic and a geometric constraint set:

$$\text{(Q)} \quad \inf_{\substack{x \in C \\ h(x) \in -K}} \left\{ \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)} \right) \right\}$$

where the functions f_i, -$g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $(i = 1, \ldots, m)$ are convex, lower semicontinuous and $h : X \longrightarrow \overline{Y}$ is proper, K-convex, i.e., for any $x, y \in X$, $\lambda \in [0,1]$, $f(\lambda x + (1-\lambda)y) \leq_K \lambda f(x) + (1-\lambda) f(y)$. Moreover, h is star K-lower semicontinuous if and only if, for each $\lambda \in K^*$, the function $(\lambda h) : X \to \mathbb{R} \cup \{+\infty\}$, defined by

$$(\lambda h)(x) = \begin{cases} \langle \lambda, h(x) \rangle, & \text{if } x \in \operatorname{dom} h, \\ +\infty, & \text{otherwise} \end{cases}$$

is a lower semicontinuous function, where $\operatorname{dom} h := \{x \in X : h(x) \in Y\}$ is the effective domain of the mapping h.

Remark 4.1 *i) It is easy to see that h is K-convex if and only if (λh) is a convex function for each $\lambda \in K^*$.*
ii) The mapping h is called star K-lower semicontinuous if and only if the function (λh) is a lower semicontinuous function for any $\lambda \in K^*$.

The problem (Q) has been studied extensively under various angles, see Zeng, Xu and Fu (2019), Moustaid, Laghdir and Dali (2022), and Moustaid et al. (2022), and the references therein. In addition, the problem (Q) can be viewed as an example of (P) by setting

$$T := K^* \text{ and } h_\lambda := \lambda h \text{ for any } \lambda \in T.$$

As before, we use $\mathcal{F}$ to denote the solution set

$$\mathcal{F} := \{x \in C : h(x) \in -K\} = \{x \in C : (\lambda h)(x) \leq 0, \forall \lambda \in T\}.$$

Moreover, since $\bigcup_{\lambda \in K^*}$ epi $(\lambda h)^*$ is a convex cone (see Jeyakumar et al., 1996), it follows that

$$\begin{aligned}
&\operatorname{co}\left(\bigcup_{1\le i\le m}\operatorname{epi}f_i^* + \nu_i\operatorname{epi}(-g_i)^*\right) + \operatorname{cocone}\left(\bigcup_{\lambda\in T}\operatorname{epi}h_\lambda^*\right) + \operatorname{epi}\delta_C^*\\
&= \operatorname{co}\left(\bigcup_{1\le i\le m}\operatorname{epi}f_i^* + \nu_i\operatorname{epi}(-g_i)^*\right) + \operatorname{cocone}\left(\bigcup_{\lambda\in K^*}\operatorname{epi}(\lambda h^*)\right) + \operatorname{epi}\delta_C^*\\
&= \operatorname{co}\left(\bigcup_{1\le i\le m}\operatorname{epi}f_i^* + \nu_i\operatorname{epi}(-g_i)^*\right) + \bigcup_{\lambda\in K^*}\operatorname{epi}(\lambda h)^* + \operatorname{epi}\delta_C^*.
\end{aligned}$$

Thus, we can immediately see that the qualification condition (CQC), introduced in Section 3, becomes the follows qualification condition (CQC_2):

$$\operatorname{co}\left(\bigcup_{1\le i\le m}\operatorname{epi}f_i^* + \nu_i\operatorname{epi}(-g_i)^*\right) + \bigcup_{\lambda\in K^*}\operatorname{epi}(\lambda h)^* + \operatorname{epi}\delta_C^*$$

is closed in the space $\mathbb{R}^{n+1}$.

Now, utilizing Theorem 3.1, we can derive the approximate necessary and sufficient approximate optimality conditions for problem (Q)

THEOREM 4.1 *Let $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}_+^m$, let f_i, $-g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ $(i = 1, \ldots, m)$ be m convex, lower semicontinuous functions and $h : X \longrightarrow \overline{Y}$ be a proper, K-convex and star K-lower semicontinuous mapping. Let $\overline{x} \in \mathcal{F}$, $\nu_i := \frac{f_i(\overline{x})}{g_i(\overline{x})} - \epsilon_i \geq 0$ $(i = 1, \ldots, m)$ and $\gamma_0 := \max_{1\le i\le m}\{\epsilon_i g_i(\overline{x})\}$. Then, we have:*

i) Suppose that the constraint qualification (CQC_2) is satisfied. If $\overline{x}$ is an ϵ-efficient solution for problem (Q), then there exist $\rho_i,\ \alpha_i,\ \beta_i,\ \eta,\ \theta \in \mathbb{R}_+$ $(i = 1, \ldots, m)$, $\lambda \in K^*$, such that

$$\sum_{i=1}^m \rho_i = 1 \tag{30}$$

$$0 \in \sum_{i=1}^m \rho_i\partial_{\alpha_i} f_i(\overline{x}) + \sum_{i=1}^m \rho_i\nu_i\partial_{\beta_i}(-g_i)(\overline{x}) + \partial_\theta\left(\lambda h\right)(\overline{x}) + N_\eta(C, \overline{x}) \tag{31}$$

$$\sum_{i=1}^m \rho_i\alpha_i + \sum_{i=1}^m \rho_i\nu_i\beta_i + \theta + \eta - (\lambda h)(\overline{x}) = \sum_{i=1}^m \rho_i\epsilon_i g_i(\overline{x}).$$

ii) Let $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}_+^m,\ \epsilon \neq 0$. If there exist $\rho_i,\ \alpha_i,\ \beta_i,\ \eta,\ \theta \in \mathbb{R}_+$, $(i = 1, \ldots, m)$, $\lambda \in K^*$ for which we have (30), (31) and

$$\sum_{i=1}^{m}\rho_i\alpha_i+\sum_{i=1}^{m}\rho_i\nu_i\beta_i+\theta+\eta-(\lambda h)(\overline{x})-\sum_{i=1}^{m}\rho_i\epsilon_i g_i(\overline{x})<0,$$

then $\overline{x}$ is an ϵ-efficient solution for problem (Q).

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References

Bot, R. I. (2009) *Conjugate Duality in Convex Optimization.* **637**. Springer Science & Business Media.

Bot, R. I., Grad, S.-M. and Wanka, G. (2009) *Duality in Vector Optimization.* Springer Science & Business Media.

Burachik, R. and Jeyakumar, V. (2005) A dual condition for the convex subdifferential sum formula with applications. *J. Convex Anal.*, 15: 540–554.

Chuong, T. D. (2016) Nondifferentiable fractional semi-infinite multiobjective optimization problems. *Operations Research Letters*, **44**(2): 260–266.

Chuong, T. D. and Kim, D. S. (2014) Nonsmooth semi-infinite multiobjective optimization problems. *J. Optim. Theory Appl.*, **160**(3): 748–762.

Deng, S. (1997) On approximate solutions in convex vector optimization. *SIAM Journal on Control and Optimization*, **35**(6): 2128–2136.

Dutta, J. and Vetrivel, V. (2001) On approximate minima in vector optimization. *Numer. Funct. Anal. Optim.*, **22**(7-8): 845–859.

Goberna, M. A. and López Cerda, M. A. (1998) *Linear Semi-Infinite Optimization.* Wiley. Chichester.

Guo, F. and Jiao, L. (2021) On solving a class of fractional semi-infinite polynomial programming problems. *Comput. Optim. Appl.*, **80**(2): 439–481.

Gutiérrez, C., Jiménez, B. and Novo, V. (2005) Multiplier rules and saddlepoint theorems for Helbig's approximate solutions in convex Pareto problems. *J. Glob. Optim.*, **32**(3): 367–383.

Gutiérrez, C., Jiménez, B., and Novo, V. (2006) ϵ-Pareto optimality conditions for convex multiobjective programming via Max function. *Numer. Funct. Anal. Optim.*, **27**(1): 57–70.

Jeyakumar, V. (2003) Characterizing set containments involving infinite convex constraints and reverse-convex constraints. *SIAM J. Optim.*, **13**(4): 947–959.

Jeyakumar, V., Lee, G. M. and Dinh, N. (2003) New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs. *SIAM J. Optim.*, **14**(2): 534–547.

Jeyakumar, V., Rubinov, A. M., Glover, B. M. and Ishizuka, Y. (1996) Inequality systems and global optimization. *J. Math. Anal. Appl.*, **202**(3): 900–919.

Jiao, L., Lee, J. H. and Zhou, Y. (2020) A hybrid approach for finding efficient solutions in vector optimization with sos-convex polynomials. *Oper. Res. Lett.*, 48: 188–194.

Kerdkaew, J. and Wangkeeree, R. (2020) Characterizing robust weak sharp solution sets of convex optimization problems with uncertainty. *J. Ind. Manag. Optim.*, 16: 2651–2673.

Kim, M. H., Kim, G. S. and Lee, G. M. (2011) On ϵ-optimality conditions for multiobjective fractional optimization problems. *Fixed Point Theory Appl.*, 2011: 13.

Kuroiwa, D. and Lee, G. M. (2012) On robust multiobjective optimization. *Vietnam J. Math.*, 40: 305–317.

Kutateladze, S. S. (1979) Convex ϵ-programming. *Sov. Math., Dokl.*, 20: 391–393.

Lee, J. H. and Lee, G. M. (2018) On optimality conditions and duality theorems for robust semi-infinite multiobjective optimization problems. *Ann. Oper. Res.*, **269**(1-2): 419–438.

Li, X.-B., Wang, Q.-L. and Lin, Z. (2019) Optimality conditions and duality for minimax fractional programming problems with data uncertainty. *J. Ind. Manag. Optim.*, **15**(3): 1133–1151.

Liu, J., Long, X.-J. and Huang, N.-J. (2023) Approximate optimality conditions and mixed type duality for semi-infinite multiobjective programming problems involving tangential subdifferentials. *J. Ind. Manag. Optim.*, **19**(9): 6500–6519.

Liu, J. C. (1992) ϵ-duality theorem of nondifferentiable nonconvex multiobjective programming. *J. Optim. Theory Appl.*, **74**(3): 567–568.

Liu, J. C. (1996) ϵ-Pareto optimality for nondifferentiable multiobjective programming via penalty function. *J. Math. Anal. Appl.*, **198**(1): 248—-61.

Liu, J.-C. and Yokoyama, K. (1999) ϵ-optimality and duality for multiobjective fractional programming. *Comput. Math. Appl.*, **37**(8): 119–128.

Loridan, P. (1984) ϵ-solutions in vector minimization problems. *J. Optim. Theory Appl.*, 43: 265–276.

Moustaid, M., Laghdir, M. and Dali, I. (2022a) Sequential optimality conditions of approximate proper efficiency for a multiobjective fractional programming problem. *SeMA Journal*, **80**(4), 611–627.

MOUSTAID, M. B., RIKOUANE, A. DALI, I. AND LAGHDIR, M. (2022b) Sequential approximate weak optimality conditions for multiobjective fractional programming problems via sequential calculus rules for the Brondsted-Rockafellar approximate subdifferential. *Rend. Circ. Mat. Palermo (2)*, **71**(2): 737–754.

PHAM, T.-H. (2023) On isolated/properly efficient solutions in nonsmooth robust semi-infinite multiobjective optimization. *Bull. Malays. Math. Sci. Soc. (2)*, **46**(2): 31.

SINGH, V., JAYSWAL, A., STANCU-MINASIAN, I. AND RUSU-STANCU, A. M. (2021) Isolated and proper efficiencies for semi-infinite multiobjective fractional problems. *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, **83**(3): 111–124.

SUN, X., FENG, X. AND TEO, K. L. (2022) Robust optimality, duality and saddle points for multiobjective fractional semi-infinite optimization with uncertain data. *Optim. Lett.*, **16**(5): 1457–1476.

SUN, X., HUANG, J. AND TEO, K. L. (2024) On semidefinite programming relaxations for a class of robust sos-convex polynomial optimization problems. *J. Glob. Optim.*, 88: 755–776.

SUN, X., TAN, W. AND TEO, K. L. (2023) Characterizing a class of robust vector polynomial optimization via sum of squares conditions. *J. Optim. Theory Appl.*, 197: 737–764.

SUN, X., TEO, K. L. AND LONG, X.-J. (2021) Some characterizations of approximate solutions for robust semi-infinite optimization problems. *J. Glob. Optim.*, 191: 281–310.

WHITE, D. J. (1986) Epsilon efficiency. *J. Optim. Theory Appl.*, 49: 319–337.

ZENG, J., XU, P. AND FU, H. (2019) On robust approximate optimal solutions for fractional semi-infinite optimization with uncertainty data. *J. Inequal. Appl.*, 2019: 16.