sciendo
Control and Cybernetics
vol. 53 (2024) No. 3
pages: 409-428
DOI: 10.2478/candc-2024-0018

Stabilization of KdV equation with boundary time-delay feedback and input saturation*

by

Ahmat Mahamat Taboye and Toufik Ennouari

Department of Mathematics, University Chouaib Doukkali,
El Jadidda, 24000, Morocco
as.ahmat.taboye@gmail.com
ennouari.t@ucd.ac.ma

Abstract: In this paper, we study the question of stabilization of nonlinear Korteweg-de Vries equation with boundary time-delay feedback in presence of saturated source term. Thanks to Banach fixed-point theorem the well-posedness is proved. The exponential stability result is demonstrated, using an appropriate Lyapunov functional.

Keywords: Korteweg-de Vries equation, time-delay, exponential stability, input saturated, Lyapunov functional

1. Introduction

In recent decades, significant effort has been devoted to studying infinite-dimensional systems with constraints on the feedback law (see, e.g., Achhab and Laabissi, 2002; Chen, 2018, Zuo and Wang, 2018; Curtain and Zwart, 2016; Jacob, Schwenninger and Vorberg, 2020). The majority of practical systems are subject to constraints, arising from various factors, such as economic conditions and technological restrictions. Therefore, it is more practical and advisable to examine cases, where the controls are subject to constraints. Various types of constraints have been considered, including bounded control and saturation constraints (see Achhab and Laabissi, 2002; Curtain and Zwart, 2016; Marx et al., 2017; Slemrod, 1989; or Tarbouriech et al., 2011).

The saturation constraint belongs to the large class of constraints, referred to as cone-bounded nonlinearity (see Marx, Andrieu and Prieur, 2017; Prieur, Tarbouriech and Da Silva, 2016). Over the past few decades, there has been

*Submitted: May 2024; Accepted: December 2024.

considerable attention given to the saturation constraint (see, e.g., Lasiecka and Seidman, 2001; Marx et al., 2015, 2016; Marx, Chitour and Prieur, 2018; Seidman and Li, 2001; Sene and Ndiaye, 2014; Slemrod, 1989; Tarbouriech et al., 2011). Dealing with the saturation constraint is inevitable for the control systems. In fact, restrictions on the amplitude of signals, whether due to practical or physical constraints, can lead to adverse consequences for the control system. The conventional approach to the analysis of stability with input saturation, follows a two step process. Initially, the study is carried out without saturation constraint. Subsequently, in the second step, an investigation of the closed-loop system is performed when adding the saturation. This approach leads to achieving local stabilization.

The Korteweg-de Vries equation

$$u_t + u_x + u_{xxx} + uu_x = 0,$$

is a mathematical model of waves on shallow water surfaces. Various methods have been employed to investigate the properties of this equation (see, for example, Linares and Ponce, 2014, pp. 151-184; Coron, 2007, pp. 38–50; Cerpa, 2014; Rosier and Zhang, 2006). In particular, extensive research has been conducted on its controllability and stabilizability characteristics (see Coron, 2007; Rosier and Zhang, 2009).

In the literature of this particular subject, there are several papers that study the Korteweg-de Vries equation with input saturation (see, e.g., Marx et al., 2016, 2017; Parada, 2022; Sene, 2022; Taboye and Ennouari, 2024; Taboye and Laabissi, 2022). In Marx et al. (2017), the global stabilization of Korteweg-de Vries equation with saturated distributed feedback has been studied. Thanks to the Banach fixed-point theorem, the well-posedness was proven. To prove the asymptotic stability, the authors have worked on two cases. In the first case, when the control acts on all the domain saturated, they used a sector condition and Lyapunov theory for infinite-dimensional systems. For the second case, where the control acts only on a part of the saturated domain, they prove the asymptotic stability of the closed-loop system, using an argument by contradiction. In Marx et al. (2015), the question of asymptotic stability of a linear Korteweg-de Vries equation with the following saturated distributed control

$$f(x,t) = sat(au(x,t)),$$

where a is a positive constant, has been investigated. Using the nonlinear semigroup theory, the well-posedness was proven. The authors referred to use a sector condition and a Lyapunov function to prove the asymptotic stability of the closed-loop system.

In this paper, we focus on studying the nonlinear Korteweg-de Vries equation with boundary time-delay feedback in the presence of a saturated source term

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + uu_x(x,t) & \\ = -sat(au(x,t)), & t>0, x\in(0,L); \\ u(0,t) = u(L,t) = 0, & t>0 \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-h), & t>0; \\ u(x,0) = u_0(x), & x\in(0,L); \\ u_x(0,t) = z_0(t), & t\in[-h,0]. \end{cases} \tag{1}$$

Here, u represents the state, $L > 0$ is the length of the spatial domain, $h > 0$ is the delay, and α and β are real constants that satisfy certain conditions to be specified later on: $a = a(x) \in L^\infty(0, L)$, is a nonnegative function that satisfies a specific hypothesis that will be provided later.

In the literature, to the best of our knowledge, the study of the Korteweg-de Vries equation with time-delay on the boundary or internal feedback has started with Baudouin, Crépeau and Valein (2018). More recently, we find some other papers that study this problem (Parada, Timimoun and Valein, 2023; Valein, 2022). In Baudouin, Crépeau and Valein (2018), the following nonlinear Korteweg-de Vries equation with time-delay on the boundary feedback

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) = 0, & t>0, x\in(0,L); \\ u(0,t) = u(L,t) = 0, & t>0 \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-h), & t>0; \\ u(x,0) = u_0(x), & x\in(0,L); \\ u_x(0,t) = z_0(t), & t\in[-h,0]. \end{cases} \tag{2}$$

has been studied. Baudouin, Crépeau and Valein (2018) presented two different approaches that prove the exponential stability results for the system (2). The first approach involves using a Lyapunov functional with an estimation of the decay rate, while assuming that the length L of the spatial domain satisfies $L < \pi\sqrt{3}$. Secondly, by employing an observability inequality approach without providing an estimation of the decay rate, and for any length such that

$$L \notin \mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, k, l \in \mathbb{N}^\star \right\},$$

the same authors also demonstrate the exponential stability of the system (2). In Valein (2022), the study of asymptotic stability of the following nonlinear Korteweg-de Vries equation with time-delay on the internal feedback

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + a(x)u(x,t) + b(x)u(x,t-h) \\ = 0, \quad t > 0, x \in (0,L); \\ u(0,t) = u(L,t) = u_x(L,t) = 0, \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in (0,L); \\ u_x(x,t) = z_0(x,t), \quad x \in (0,L), t \in [-h,0], \end{cases} \tag{3}$$

was tackled. The semiglobal stability of (3), for any lengths, in the event that the following assumption is satisfied

$$\exists c_0 > 0, \; b(x) + c_0 \leq a(x) \quad \text{a.e in } \omega,$$

where ω is an open subset of $(0,L)$, has been proven, by using an observability inequality on the nonlinear system (3). Valein (2022) also proved the local exponential stability result in the case where $\operatorname{supp} b \not\subseteq \operatorname{supp} a$.

The objective of this paper is to contribute to the study of the Korteweg-de Vries equation with a time delay. All the latest works that investigate the question of stability regarding the KdV equation with time delay, study this problem without a source term. Therefore, we aim to study this topic in the presence of a saturated source term. The main contribution of this work is the development of a robust stabilization approach for the KdV equation that incorporates these saturation constraints. By employing C_0-semigroup theory and a carefully constructed Lyapunov functional, we demonstrate the exponential stability of the system. This extends the existing results in the literature by addressing the combined effects of delay and saturation, providing insights into the practical stabilization of constrained systems.

The article is organized as follows. In Section 2, the problem is stated. The well-posedness of system (1) in the presence of saturated source term is given in Section 3. Section 4 is devoted to exponential stability results. Finally, we give some conclusions in Section 5.

Notation: u_t, u_x and u_μ stand for the partial derivative of function u with respect to t, x and μ, respectively. Given $L > 0$, $\|\cdot\|_{L^2(0,L)}$ (respectively $\langle \cdot, \cdot \rangle$), denotes the norm (respectively the inner product) in $L^2(0,L)$. $H^1(0,L)$ denotes the set of all functions $u \in L^2(0,L)$ such that $u_x \in L^2(0,L)$. $H^2(0,L)$ denotes the set of all functions $u \in L^2(0,L)$ such that $u_x, u_{xx} \in L^2(0,L)$. $H^3(0,L)$ denotes the set of all functions $u \in L^2(0,L)$ such that $u_x, u_{xx}, u_{xxx} \in L^2(0,L)$.

The function $sat(\cdot)$ is the saturation function, and is defined as follows

$$sat(s) = \begin{cases} s, & \text{if } \|s\|_{L^2} \leq 1; \\ \frac{s}{\|s\|_{L^2}}, & \text{if } \|s\|_{L^2} \geq 1. \end{cases} \tag{4}$$

2. Problem statment

The aim of this paper is to study the following KdV equation with time-delay on the boundary feedback in the presence of saturated source term:

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) + u(x,t)u_x(x,t) \\ = -sat(a(x)u(x,t)), & t > 0, x \in (0,L); \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-h), & t > 0; \\ u(x,0) = u_0(x), & x \in (0,L); \\ u_x(0,t) = z_0(t) & t \in [-h,0]. \end{cases} \tag{5}$$

Here, $u(x,t)$ is the amplitude of the water wave at position x at time t, $h > 0$ is the delay, $L > 0$ is the length of the spatial domain, and $\alpha, \beta \neq 0$ are real constants. We assume that $a(\cdot) \in L^\infty((0,L))$ is a nonnegative function, where $\tilde{a} = a(x) \in L^\infty(0,L)$, satisfying

$$\begin{cases} a_1 \geq a = a(x) \geq a_0 > 0 \quad \text{on} \quad \omega \subseteq [0,L], \\ \omega \quad \text{is a nonempty open subset of} \quad [0,L]. \end{cases} \tag{6}$$

Remark 1 *The hypothesis $a = a(x) \in L^\infty((0,L))$ satisfying (6) is a classical assumption used to study the stabilization of the KdV equation (see Marx et al., 2017; Perla Menzala, Vasconcellos and Zuazua, 2002; Taboye and Laabissi, 2022).*

Moreover, we define the matrix M_1 by

$$M_1 = \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 - |\beta| \end{pmatrix} \tag{7}$$

where α, β are real constants. We will assume that α and β satisfy the following inequality

$$|\alpha| + |\beta| \leq 1. \tag{8}$$

According to Baudouin, Crépeau and Valein (2018), if (8) is satisfied, then the matrix M_1 is negative definite.

We introduce the Hilbert space $H = L^2(0,L)\times L^2(0,1)$, equipped with usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle = \int_0^L uu_1 dx + \int_0^1 zz_1 d\mu, \tag{9}$$

for all $(u,z),(u_1,z_1)\in H$ and its norm

$$\left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|^2 = \int_0^L u^2 dx + \int_0^1 z^2 d\mu.$$

Furthermore, let $\mathcal{H}$ be the Hilbert space of the initial and boundary date, defined as follows: $\mathcal{H} = L^2(0,L)\times L^2[-h,0]$. The Hilbert space $\mathcal{H}$ is equipped with the norm $\|\cdot\|_{\mathcal{H}}$, defined for all $(u,z)\in\mathcal{H}$ by

$$\|(u,z)\|^2_{\mathcal{H}} = \int_0^L u^2 dx + \int_{-h}^0 z^2(s) ds. \tag{10}$$

Now, we recall the definition of a mild solution, hence let us consider the following abstract system in a Hilbert space

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t) + f(t), \quad t>0, \\ u(0) = u_0, \end{cases} \tag{11}$$

where $\mathcal{A}$ is an infinitesimal generator of linear C_0-semigroup $(T(t))_{t\geq 0}$, defined on its domain $D(A)\subseteq H$, where H is a Hilbert space and $f\in L^1_{loc}([0,T],H)$.

Definition 1 *(Pazy, 1983, Definition 2.3) Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Let $u_0 \in H$ and $f\in L^1(0,T,H)$. Then the function $u\in \mathcal{C}([0,T],H)$, given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \quad 0\leq t\leq T, \tag{12}$$

is the unique mild solution of the initial value problem (11) *on* $[0,T]$.

Remark 2 *We recall that a strongly continuous semigroup is an operator-valued function $T(t)$ from $\mathbb{R}_+$ to $\mathcal{L}(H)$ that satisfies the following properties:*

- $T(0) = I$,
- $T(t+s) = T(t)T(s)$ *for all* $t,s\geq 0$,
- $\lim\limits_{t\to 0} T(t)x = x$ *for all* $x\in H$.

We recall in the following result the fact that the saturation function is Lipschitzian in $L^2(0,L)$.

Lemma 1 (Slemrod, 1989, Theorem 5.1) *For all* $(u,v) \in L^2(0,L)$*, we have*

$$\|sat(u) - sat(v)\|_{L^2(0,L)} \leq 3\|u-v\|_{L^2(0,L)}$$

The following Propositions will be needed.

Proposition 1 (Marx et al., 2017, Proposition 3.4) *Assume that a satisfies 6. If* $u \in L^2(0,T;H^1(0,L))$*, then* $sat(au) \in L^1(0,T;L^2(0,L))$ *and the map* $\psi : u \in L^2(0,T;H^1(0,L)) \mapsto sat(au) \in L^1(0,T;L^2(0,L))$ *is continuous.*

Proposition 2 (Rosier, 1997, Proposition 4.1) *Let* $u \in L^2(0,T;H^1(0,L))$*. Then* $uu_x \in L^1(0,T;L^2(0,L))$ *and the map*

$$\phi : u \in L^2(0,T;H^1(0,L)) \mapsto uu_x \in L^1(0,T;L^2(0,L))$$

is continuous. Moreover, there exists $K_1 > 0$ *such that, for any* u*,* $\tilde{u} \in L^2(0,T;H^1(0,L))$*, we have*

$$\begin{aligned}\int_0^T \|uu_x - \tilde{u}\tilde{u}_x\|_{L^2(0,L)} \leq & K_1\|u-\tilde{u}\|_{L^2(0,T;H^1(0,L))} \\ & \times (\|u\|_{L^2(0,T;H^1(0,L))} + \|\tilde{u}\|_{L^2(0,T;H^1(0,L))}).\end{aligned}$$

3. Well-posedness

The aim of this section is to prove the local well-posedness result of nonlinear system (5).

Before moving to the study of the existence and uniqueness of solutions for the nonlinear system (5), we reccal that Baudouin, Crépeau and Valein (2018) have previously demonstrated the existence and uniqueness of solution for the following linear system

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = f(x,t), \quad t>0, \quad x \in (0,L); \\ u(0,t) = u(L,t) = 0, \quad t > 0; \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-h), \quad t>0; \\ u(x,0) = u_0(x), \quad x \in (0,L); \\ u_x(0,t) = z_0(t), \qquad t \in [-h,0], \end{cases} \tag{13}$$

where f is the source term. To prove the well-posedness of (13), the authors referred to assume that the source term $f \in L^1(0,T,L^2(0,L))$.

Let us introduce the space $\mathcal{B} = \mathcal{C}([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ with $T > 0$. We equip the space $\mathcal{B}$ with the following norm

$$\|u\|_{\mathcal{B}} = \|u\|_{\mathcal{C}([0,T],L^2(0,L))} + \|u\|_{L^2(0,T,H^1(0,L))}.$$

Let us state the main result of this section.

Theorem 1 *Let $T > 0, L > 0$ and suppose that* (8) *holds. We also assume that $a \in L^\infty(0,L)$ is satisfying* (6). *Then there exists $r > 0$ and $K > 0$ such that for every $(u_0, z_0) \in H$, satisfying $\|(u_0,z_0)\|_H \le r$, there exists a unique $u \in \mathcal{B}$ of system* (5), *satisfying $\|u\|_{\mathcal{B}} \le K\|(u_0,z_0)\|_H$.*

Proof Take $(u_0, z_0) \in H$ such that $\|(u_0,z_0)\|_H \le r$ for $r > 0$ chosen small enough later on. Let us take $u \in \mathcal{B}$ and consider the following map

$$\begin{aligned} \chi : \mathcal{B} &\to \mathcal{B} \\ u &\mapsto \chi(u) = \tilde{u} \end{aligned}$$

where $\tilde{u}$ is the solution of the following system

$$\begin{cases} \tilde{u}_t(x,t) + \tilde{u}_x(x,t) + \tilde{u}_{xxx}(x,t) & \\ = -(u(x,t)u_x(x,t) + sat(au(x,t)), & t > 0, x \in (0,L); \\ \tilde{u}(0,t) = \tilde{u}(L,t) = 0, & t > 0 \\ \tilde{u}_x(L,t) = \alpha\tilde{u}_x(0,t) + \beta\tilde{u}_x(0,t-h), & t > 0; \\ \tilde{u}(x,0) = \tilde{u}_0(x), & x \in (0,L); \\ \tilde{u}_x(0,t) = z_0(t), & t \in [-h,0]. \end{cases} \tag{14}$$

Therefore, $u \in B$ is a solution of (5) if and only if u is a fixed point of χ. Let

$$f(x,t) = -u(x,t)u_x(x,t) - sat(au(x,t)).$$

From Proposition 2, if $u \in L^2(0,T,H^1(0,L))$, then $uu_x \in L^1(0,T,L^2(0,L)$ and, according to Proposition 1, if $u \in L^2(0,T,H^1(0,L))$, hence, $sat(au(x,t) \in L^1(0,T,L^2(0,L))$. Thus, $f(x,t) \in L^1(0,T,L^2(0,L))$. Consequently, from Baudouin, Crépeau (2018, Proposition 2), if (8) is satisfied, then there exists $C > 0$ such that

$$\begin{aligned} \|(u,z)\|^2_{\mathcal{C}([0,T],H)} \le & C\left(\|(u_0, z_0(-h\cdot))\|_H\right. \\ & \left.+ \|uu_x + sat(au)\|_{L^1(0,T,L^2(0,L))}\right). \end{aligned} \tag{15}$$

and

$$\|u_x\|^2_{L^2(0,T,L^2(0,L))} \leq C\left(\|(u_0,z_0(-h\cdot))\|_H + \|uu_x+sat(au)\|_{L^1(0,T,L^2(0,L))}\right). \tag{16}$$

Hence, from (15), (16), Proposition 2, and Lemma 1, we obtain

$$\begin{aligned}
\|\chi(u)\|_B &= \|\tilde{u}\|_B\\
&\leq C\left(\|(u_0,z_0(-h\cdot))\|_H + \int_0^T \|uu_x+sat(au)\|_{L^2(0,L)}dt\right)\\
&\leq C\left(\|(u_0,z_0(-h\cdot))\|_H + \int_0^T \|uu_x\|_{L^2(0,L)}dt + \int_0^T \|sat(au)\|_{L^2(0,L)}dt\right)\\
&\leq C\left(\|(u_0,z_0(-h\cdot))\|_H + K_1\|u\|^2_{\mathcal{B}}dt + 3\int_0^T \|au\|_{L^2(0,L)}dt\right)\\
&\leq C\left(\|(u_0,z_0(-h\cdot))\|_H + K_1\|u\|^2_{\mathcal{B}}dt + 3a_1\int_0^T \|u\|_{L^2(0,L)}dt\right)\\
&\leq C\left(\|(u_0,z_0(-h\cdot))\|_H + K_1\|u\|^2_{\mathcal{B}}dt + 3a_1\sqrt{T}\sqrt{L}\|u\|^2_{L^1(0,T,L^2(0,L))}\right)\\
&\leq C\left(\|(u_0,z_0(-h\cdot))\|_H + K_1\|u\|^2_{\mathcal{B}} + 3a_1\sqrt{T}\sqrt{L}\|u\|^2_{\mathcal{B}}\right)\\
&\leq CK_1K_2\left(\|(u_0,z_0(-h\cdot))\|_H + \|u\|^2_{\mathcal{B}}\right),
\end{aligned}$$

where $K_2 = 3a_1\sqrt{T}\sqrt{L} > 0$ and $K_1 > 0$ is given from Proposition 2. Therefore,

$$\|\chi(u)\|_{\mathcal{B}} \leq K\left(\|(u_0,z_0(-h\cdot))\|_H + \|u\|^2_{\mathcal{B}}\right),$$

where $K = C\times K_1\times K_2 = C\times K_1\times 3a_1\sqrt{T}\sqrt{L} > 0$.

Following the previous argument, we have

$$\begin{aligned}
&\|\chi(u_1)-\chi(u_2)\|_{\mathcal{B}}\\
&\leq C\left(\int_0^T \|-u_1u_{1,x}+u_2u_{2,x}-sat(au_1)+sat(au_1)\|_{L^2(0,L)}dt\right)\\
&\leq C\left(\int_0^T \|u_1u_{1,x}-u_2u_{2,x}\|_{L^2(0,L)}dt + \int_0^T \|sat(au_1)-sat(au_1)\|_{L^2(0,L)}dt\right)\\
&\leq C\left(K_1(\|u_1\|_{\mathcal{B}}+\|u_2\|_{\mathcal{B}})\|u_1-u_2\|_{\mathcal{B}} + K_2\|u_1-u_2\|_{L^1(0,T,L^2(0,L))}\right)\\
&\leq C\left(K_1(\|u_1\|_{\mathcal{B}}+\|u_2\|_{\mathcal{B}})\|u_1-u_2\|_{\mathcal{B}} + K_2\|u_1-u_2\|_{\mathcal{B}}\right)\\
&\leq K\left((\|u_1\|_{\mathcal{B}}+\|u_2\|_{\mathcal{B}})\|u_1-u_2\|_{\mathcal{B}} + \|u_1-u_2\|_{\mathcal{B}}\right)\\
&\leq 2K(\|u_1\|_{\mathcal{B}}+\|u_2\|_B)\|u_1-u_2\|_{\mathcal{B}}.
\end{aligned}$$

We restricted χ to the closed ball $\{u \in \mathcal{B}; \|u\|_{\mathcal{B}} \leq R\}$, where $R > 0$ is to be chosen later. Thus,

$$\|\chi(u)\|_B \leq K(r + R^2),$$

and

$$\|\chi(u_1) - \chi(u_2)\|_{\mathcal{B}} \leq 4KR\|u_1 - u_2\|_{\mathcal{B}}.$$

Hence, it is enough to take R and r satisfying

$$R < \frac{1}{4K} \text{ and } r < \frac{R}{4K}.$$

Therefore, $\|\chi(u)\|_{\mathcal{B}} \leq R$ and $\|\chi(u_1) - \chi(u_2)\|_{\mathcal{B}} < 4KR\|u_1 - u_2\|_{\mathcal{B}}$, with $4KR < 1$. Hence, we can apply the Banach fixed-point theorem and we deduce that the map χ has a unique fixed-point. Consequently, the nonlinear system (5) has a unique solution $u \in \mathcal{B}$. ∎

4. Exponential stability

The aim of this section is to prove the exponential stability of system (5). Before stating the main result of this section, let us consider the following energy

$$E(t) = \int_0^L u^2(x,t)dx + |\beta|h \int_0^1 u_x^2(0, t - h\mu)d\mu. \tag{17}$$

The following lemma establishes that energy (17) does not increase.

Lemma 2 *Assume that (8) holds and $a = a(x) \in L^\infty(0, L)$ satisfies (6). Let $(u_0, z_0(-h\cdot)) \in D(A)$ and $u \in L^2(0, T, H^1(0, L))$. Then, for any regular solution of (5), the following inequality is satisfied*

$$\frac{d}{dt}E(t) \leq \begin{pmatrix} u_x(0,t) \\ u_x(0,t-h) \end{pmatrix}^T M_1 \begin{pmatrix} u_x(0,t) \\ u_x(0,t-h) \end{pmatrix} \leq 0. \tag{18}$$

Proof Let us consider a regular solution of (5). By definition,

$$z(\mu, t) = u_x(0, t - h\mu),$$

hence we rewrite the energy (17) as follows

$$E(t) = \int_0^L u^2(x,t)dx + |\beta|h \int_0^1 z^2(\mu, t)d\mu.$$

By differentiating $E(\cdot)$, we get

$$\begin{aligned}\frac{d}{dt}E(t) &= 2\int_0^L uu_t dx + 2|\beta|h\int_0^1 zz_t d\mu \\ &= -2\int_0^L uu_x dx - 2\int_0^L uu_{xxx} dx - 2\int_0^L u^2u_x dx - 2\int_0^L sat(au)u dx \\ &\quad -2|\beta|\int_0^1 zz_\mu d\mu. \end{aligned} \tag{19}$$

After some integrations by parts, we obtain

$$-2\int_0^L uu_x dx = 0, \tag{20}$$

$$\begin{aligned}-2\int_0^L uu_{xxx} dx =& u_x^2(L,t) - u_x^2(0,t) \\ =& (\alpha u_x(0,t) + \beta z(1,t))^2 - u_x^2(0,t)\end{aligned} \tag{21}$$

$$-2\int_0^L u^2 u_x dx = 0, \tag{22}$$

and

$$\begin{aligned}-2|\beta|\int_0^1 zz_\mu d\mu =& -|\beta|\left[z^2(\mu,t)\right]_0^1 \\ =& -|\beta|\left[z^2(1,t) - z^2(0,t)\right] \\ =& |\beta|u_x^2(0,t) - |\beta|z^2(1,t).\end{aligned} \tag{23}$$

Using (4), (22), (21) and (23), we get

$$\begin{aligned}\frac{d}{dt}E(t) &= (\alpha u_x(0,t) + \beta z(1,t))^2 - u_x^2(0,t) - 2\int_0^L sat(au)u dx \\ &+ |\beta|u_x^2(0,t) - |\beta|z^2(1,t) \\ &= \alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)z(1,t) + \beta^2 z^2(1,t) - u_x^2(0,t) \\ &- 2\int_0^L sat(au)u dx + |\beta|u_x^2(0,t) - |\beta|z^2(1,t) \\ &= \left(\alpha^2 - 1 + |\beta|\right)u_x^2(0,t) + 2\alpha\beta u_x(0,t)z(1,t) \\ &+ \left((\beta^2 - |\beta|)\right)z^2(1,t)) - 2\int_0^L sat(au)u dx.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}E(t) &+ \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T (-M_1) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\ &= \left(\alpha^2 - 1 + |\beta|\right) u_x^2(0,t) + 2\alpha\beta u_x(0,t) z(1,t) \\ &+ \left(\beta^2 - |\beta|\right) z^2(1,t)) - 2\int_0^L sat(au)udx \\ &+ \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T (-M_1) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\ &= -2\int_0^L sat(au)udx.\end{aligned}$$

$\int_0^L sat(au)udx \geq 0$, indeed, if $\|au\|_{L^2} \leq 1$, then

$$sat(au)u = au^2 \geq 0.$$

If $\|au\|_{L^2} \geq 1$,

$$sat(au)u = \frac{au}{\|au\|_{L^2}}u = \frac{au^2}{\|au\|_{L^2}} \geq 0,$$

where $a = a(x)$ is a nonnegative function. Therefore,

$$\frac{d}{dt}E(t) + \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T (-M_1) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \leq 0.$$

Since $z(1,t) = u_x(0,t-h)$, and using (8), we conclude the proof. ∎

In order to prove the stability result for (5), we need to introduce the following Lyapunov function

$$V(t) = E(t) + \lambda V_1(t) + \gamma V_2(t), \tag{24}$$

where

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{|\beta|}{2}h\int_0^1 u_x^2(0,t-h\mu)d\mu,$$

$$V_1(t) = \int_0^L xu^2(x,t)dx, \tag{25}$$

and

$$V_2(t) = h\int_0^1 (1-\mu)u_x^2(0,t-h\mu)d\mu, \tag{26}$$

REMARK 3 *The Lyapunov functional, used in this study, is a classic choice, commonly employed in stability analyses of similar systems. It is positive definite and designed to decrease along the system's trajectories, ensuring stability. This functional captures the energy dynamics of the system and is well-suited for systems with boundary delays and nonlinearities. Additionally, it accounts for input saturation constraints, making it applicable to practical control systems. For more details, readers can consult the following references: Capistrano Filho et al. (2023); Taboye and Ennouari (2025); and Valein (2022).*

The following lemmas play an important role in demonstration of the exponential stability of the system (5).

LEMMA 3 *Assume that $a = a(x) \in L^\infty(0, L)$ satisfies (6), $\begin{pmatrix} u_0 & z_0(-h\cdot) \end{pmatrix} \in D(A)$ and $u \in L^2(0, T, H^1(0, L))$, then for any regular solution of (5), the following equality is satisfied*

$$\begin{aligned}\frac{d}{dt}V_1(t) =& L(\alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)u_x(0,t-h) + \beta^2 u_x^2(0,t-h)) \\ &+ \int_0^L u^2 dx - 3\int_0^L u_x^2 dx + \frac{2}{3}\int_0^L u^3 dx - 2\int_0^L x sat(au)u dx.\end{aligned} \tag{27}$$

PROOF Let us consider a regular solution, then differentiate $V_1(\cdot)$, so that we get

$$\begin{aligned}\frac{d}{dt}V_1(t) &= 2\int_0^L x u_t u dx \\ &= -2\int_0^L x u u_x dx - 2\int_0^L x u u_{xxx} dx - 2\int_0^L x u^2 u_x dx \\ &- 2\int_0^L x sat(au)u dx.\end{aligned}$$

After some integration by parts, we obtain

$$-2\int_0^L x u u_x dx = \int_0^L u^2 dx;$$

$$\begin{aligned}-2\int_0^L x u u_{xxx} dx &= L u_x^2(L,t) - 3\int_0^L u_x^2 dx \\ &= L(\alpha u_x(0,t) + \beta u_x(0,t-h))^2 - 3\int_0^L u_x^2 dx,\end{aligned}$$

and

$$-2\int_0^L xu^2u_x dx = \frac{2}{3}\int_0^L u^3 dx.$$

Using the last equations, we arrive at

$$\begin{aligned}\frac{d}{dt}V_1(t) &= \int_0^L u^2 dx + L(\alpha u_x(0,t)+\beta u_x(0,t-h))^2 \\ &- 3\int_0^L u_x^2 dx + \frac{2}{3}\int_0^L u^3 dx - 2\int_0^L xsat(au)udx \\ &= L(\alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)u_x(0,t-h) + \beta^2 u_x^2(0,t-h)) \\ &+ \int_0^L u^2 dx - 3\int_0^L u_x^2 dx + \frac{2}{3}\int_0^L u^3 dx - 2\int_0^L xsat(au)udx.\end{aligned}$$

■

Lemma 4 *Suppose that* $\left(u_0, \quad z_0(-h\cdot) \right) \in D(A)$ *and* $u \in L^2(0,T,H^1(0,L))$, *then for any regular solution of (5), the following equality is satisfied*

$$\frac{d}{dt}V_2(t) = -\int_0^1 u_x^2(0,t-h\mu)d\mu + u_x^2(0,t). \tag{28}$$

Proof Consider a regular solution, then differentiate $V_2(\cdot)$ and using integration by part, we obtain

$$\begin{aligned}\frac{d}{dt}V_2(t) =& 2h\int_0^1 (1-\mu)u_x(0,t-h\mu)\partial_t u_x(0,t-h\mu)d\mu \\ =& -2\int_0^1 (1-\mu)u_x(0,t-h\mu)\partial_\mu u_x(0,t-h\mu)d\mu \\ =& -\left[(1-\mu)u_x^2(0,t-h\mu)\right]_0^1 - \int_0^1 u_x^2(0,t-h\mu)d\mu \\ =& u_x^2(0,t) - \int_0^1 u_x^2(0,t-h\mu)d\mu.\end{aligned} \tag{29}$$

Therefore

$$\frac{d}{dt}V_2(t) = -\int_0^1 u_x^2(0,t-h\mu)d\mu + u_x^2(0,t),$$

which finishes the proof. ■

Now, we are able to state and prove the main result of this section.

Theorem 2 *Assume that $a = a(x) \in L^\infty(0, L)$ is satisfying (6), and $L < \pi\sqrt{3}$. Moreover, suppose that the assumption (8) is satisfied. Then, there exists $r > 0$, such that for every $(u_0, z_0) \in \mathcal{H}$, satisfying $\|(u_0, z_0)\|_{\mathcal{H}} \leq r$, there exist $\delta > 0$ and $M > 0$ such that*

$$E(t) \leq Me^{-2\delta t}E(0), \quad \forall t > 0, \tag{30}$$

where for λ and γ sufficiently small, the two positive constants δ and M satisfy the following inequality:

$$\delta \leq \min\left\{\frac{(9\pi^2 - 3L^2 - 2L^{\frac{3}{2}}r\pi^2)}{3L^2(1 + 2L\lambda)}\lambda, \frac{\gamma}{h(2\gamma + |\beta|)}\right\} \tag{31}$$

and

$$M \leq 1 + \max\left\{L\lambda, \frac{2\gamma}{|\beta|}\right\}.$$

Remark 4 *The Lyapunov function $V(\cdot)$ and the energy $E(\cdot)$ are equivalent. Indeed,*

$$E(t) \leq V(t) \leq M_1E(t) \quad \forall t > 0, \tag{32}$$

where $M_1 = 1 + \max\left\{L\lambda, \frac{2\gamma}{|\beta|}\right\} > 0$. Thanks to inequality (32), in order to prove the exponential stability of system (5), it is sufficient to show that for all $\delta > 0$,

$$\frac{d}{dt}V(t) + 2\delta V(t) \leq 0.$$

Proof Using Lemma 2, (27) and (28), we get

$$\begin{aligned}\frac{d}{dt}V(t) \leq &\frac{1}{2}Y^TM_1Y + \lambda\int_0^L u^2dx + \lambda L\alpha^2u_x^2(0,t)\\ &+2L\lambda\alpha\beta u_x(0,t)u_x(0,t-h) + L\lambda\beta^2u_x^2(0,t-h))\\ &-3\lambda\int_0^L u_x^2dx + \frac{2}{3}\lambda\int_0^L u^3dx - 2\lambda\int_0^L xsat(au)udx\\ &-\gamma\int_0^1 u_x^2(0,t-h\mu)d\mu + \gamma u_x^2(0,t).\end{aligned} \tag{33}$$

Since $x \in (0, L)$ and $sat(au)u \geq 0$, then $\int_0^L xsat(au)udx \geq 0$. Therefore

$$\begin{aligned}\frac{d}{dt}V(t) \leq &Y^T\left[\frac{1}{2}M_1 + M_2\right]Y + \lambda\int_0^L u^2dx\\ &-3\lambda\int_0^L u_x^2dx + \frac{2}{3}\lambda\int_0^L u^3dx - \gamma\int_0^1 u_x^2(0,t-h\mu)d\mu.\end{aligned} \tag{34}$$

Here, $Y = \begin{pmatrix} u_x(0,t) \\ u_x(0,t-h) \end{pmatrix}$ and $M_2 = \begin{pmatrix} L\lambda\alpha^2 + \gamma & L\lambda\alpha\beta \\ L\lambda\alpha\beta & L\lambda\beta^2 \end{pmatrix}$.

Now we calculate $2\delta V(t)$; we have

$$\begin{aligned} 2\delta V(t) =& 2\delta E(t) + 2\delta\lambda V_1(t) + 2\delta\gamma V_2(t) \\ =& \delta \int_0^L u^2 dx + \delta|\beta|h \int_0^1 u_x^2(0,t-h\mu)d\mu + 2\delta\lambda \int_0^L xu^2 dx \\ &+ 2\delta\gamma h \int_0^1 u_x^2(0,t-h\mu)d\mu - 2\delta\gamma h \int_0^1 \mu u_x^2(0,t-h\mu)d\mu \\ \leq& \delta \int_0^L u^2 dx + \delta|\beta|h \int_0^1 u_x^2(0,t-h\mu)d\mu \\ &+ 2\delta\lambda L \int_0^L u^2 dx + 2\delta\gamma h \int_0^1 u_x^2(0,t-h\mu)d\mu \end{aligned} . \tag{35}$$

We know that under the assumption (8), the matrix M_1 is negative definite. Then, by the continuity of the trace and determinant, we deduce that for λ and γ small enough, the matrix $\frac{1}{2}M_1 + M_2$ is negative definite. Hence, from (34) and (35), we deduce that

$$\begin{aligned} \frac{d}{dt}V(t) + 2\delta V(t) \leq& Y^T \left[\frac{1}{2}M_1 Y + M_2\right] Y - 3\lambda \int_0^L u_x^2 dx \\ &+ (\lambda + \delta + 2L\lambda\delta) \int_0^L u^2 dx + \frac{2}{3}\lambda \int_0^L u^3 dx \\ &+ (\delta|\beta|h + 2\gamma\delta h - \gamma) \int_0^1 u_x^2(0,t-h\mu)d\mu \\ \leq& -3\lambda \int_0^L u_x^2 dx + (\lambda + \delta + 2L\lambda\delta) \int_0^L u^2 dx + \frac{2}{3}\lambda \int_0^L u^3 dx \\ &+ (\delta|\beta|h + 2\gamma\delta h - \gamma) \int_0^1 u_x^2(0,t-h\mu)d\mu. \end{aligned} \tag{36}$$

Using Cauchy Schwarz inequality and the injection to $H_0^1(0,L)$ into $L^\infty(0,L)$, we obtain

$$\begin{aligned} \int_0^L u^3(x,t)dx &\leq \|u(\cdot,t)\|^2_{L^\infty(0,L)} \int_0^L u(x,t)dx \\ &\leq L\sqrt{L}\|u_x(\cdot,t)\|^2_{L^2(0,L)}\|u(\cdot,t)\|_{L^2(0,L)}. \end{aligned}$$

From Lemma 2, we deduce that $\|u(\cdot,t)\|_{L^2(0,L)} \leq r$, and so we have

$$\int_0^L u^3(x,t)dx \leq L^{\frac{3}{2}} r \|u_x(\cdot,t)\|^2_{L^2(0,L)}.$$

Therefore, using the Poincaré's inequality, we get

$$\begin{aligned}\frac{d}{dt}V(t) + 2\delta V(t) \leq &\left(\frac{L^2}{\pi^2}\left(\lambda+\delta+2L\lambda\delta\right) + \frac{2}{3}L^{\frac{3}{2}}r\lambda - 3\lambda\right)\int_0^L u_x^2 dx \\ &+ \left(\delta|\beta|h + 2\gamma\delta h - \gamma\right)\int_0^1 u_x^2(0,t-h\mu)d\mu.\end{aligned} \tag{37}$$

By assumption, $L < \pi\sqrt{3}$, then, according to Baudouin, Crépeau and Valein (2018), it is possible to choose r small enough to have $r < \frac{3(3\pi^2-L^2)}{2L^{\frac{3}{2}}\pi^2}$. Consequently, we can choose $\delta > 0$ such that (31) holds in order to obtain that

$$\frac{L^2}{\pi^2}\left(\lambda+\delta+2L\lambda\delta\right) + \frac{2}{3}L^{\frac{3}{2}}r\lambda - 3\lambda \leq 0 \text{ and } \delta|\beta|K + 2\gamma\delta K - \gamma(1-d) \leq 0,$$

and therefore

$$\frac{d}{dt}V(t) + 2\delta V(t) \leq 0 \quad \forall t \geq 0;$$

hence we deduce that

$$V(t) \leq Ce^{-2\delta t}V(0) \quad \forall t \geq 0.$$

Using (32), we get

$$E(t) \leq Ce^{-2\delta t}E(0) \quad \forall t \geq 0.$$

Using the density of $D(A)$, we conclude the proof by extending the result to any initial condition within $\mathcal{H}$. ∎

5. Conclusion

In this paper, we have proven the well-posedness and the local exponential stability of the nonlinear Korteweg-de Vries equation with time-delay on the boundary feedback with saturated source term. The well-posedness is obtained using Banach fixed-point theorem. The proof of the stabilization result is essentially based on the use of an appropriate Lyapunov functional with an estimate of the decay rate, but the length of the spatial domain L has to satisfy the condition $L < \pi\sqrt{3}$.

The global exponential stability of the nonlinear Korteweg-de Vries equation with time-delay on boundary feedback in the presence of saturation source term may be a potential area for future research.

Acknowledgements

We would like to thank the referees for the time they did devote to studying this paper.

References

Achhab, M. E. and Laabissi, M. (2002) Feedback stabilization of a class of distributed parameter systems with control constraints. *Systems & Control Letters* 45, 163–171.

Baudouin, L. Crépeau, E. and Valein, J. (2018) Two approaches for the stabilization of nonlinear kdv equation with boundary time-delay feedback. *IEEE Transactions on Automatic Control* 64, 1403–1414.

Capistrano-Filho, R. d. A,. Chentouf, B., de Sousa, L. S. and Gonzalez Martinez, V. H. (2023) Two stability results for the kawahara equation with a time-delayed boundary control. *Zeitschrift für angewandte Mathematik und Physik* **74**, 16.

Cerpa, E. (2014) Control of a Korteweg-de Vries equation: a tutorial. *Math. Control Relat. Fields* 4, 45–99.

Chen, Y., Zuo, Z. and Wang, Y. (2018) Stabilization of wave equation with boundary saturated control. In: *2018 37th Chinese Control Conference (CCC), IEEE* 726–731.

Coron, J.-M. (2007) *Control and Nonlinearity. American Mathematical Society* **136**.

Curtain, R. and Zwart, H. (2016) Stabilization of collocated systems by nonlinear boundary control. *Systems & Control Letters* 96, 11–14.

Jacob, B., Schwenninger, F. L. and Vorberg, L. A. (2020) Remarks on input-to-state stability of collocated systems with saturated feedback. *Mathematics of Control, Signals, and Systems* 32, 293–307.

Lasiecka, I. and Seidman, T. I. (2003) Strong stability of elastic control systems with dissipative saturating feedback. *Systems & Control Letters* 48, 243–252.

Linares, F. and Ponce, G. (2014) *Introduction to Nonlinear Dispersive Equations.* Springer.

Marx, S., Andrieu, V. and Prieur, C. (2017) Cone-bounded feedback laws for m-dissipative operators on Hilbert spaces. *Mathematics of Control, Signals, and Systems* 29, 1–32.

Marx, S., Cerpa, E., Prieur, C. and Andrieu, V. (2015) Stabilization of a linear Korteweg-de Vries equation with a saturated internal control. In: *2015 European Control Conference (ECC), IEEE*, 867–872.

Marx, S., Cerpa, E., Prieur, C. and Andrieu, V. (2016) Global stabilization of a korteweg-de vries equation with a distributed control saturated in l2-norm. *IFAC-PapersOnLine* 49, 122–127.

Marx, S., Cerpa, E., Prieur, C. and Andrieu, V. (2017) Global stabilization of a Korteweg–de Vries equation with saturating distributed control. *SIAM Journal on Control and Optimization* 55, 1452–1480.

Marx, S., Chitour, Y. and Prieur, C. (2018) Stability results for infinite-dimensional linear control systems subject to saturations. In: *2018 European Control Conference (ECC), IEEE*, 2995–3000.

Parada, H. (2022) Stability analysis of a Korteweg-de Vries equation with saturated boundary feedback. *IFAC-PapersOnLine* 55, 1–6.

Parada, H., Timimoun, C. and Valein, J. (2023) Stability results for the kdv equation with time-varying delay. *Systems & Control Letters* 177, 105547.

Pazy, A. (1983) *Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences* **44**, Springer Science & Business Media.

Perla Menzala, G., Vasconcellos, C. F. and Zuazua, E. (2002) Stabilization of the Korteweg-de Vries equation with localized damping. *Quarterly of Applied Mathematics* 60, 111–129.

Prieur, C., Tarbouriech, S. and Da Silva, J. M. G. (2016) Wave equation with cone-bounded control laws. *IEEE Transactions on Automatic Control* 61, 3452–3463.

Rosier, L. (1997) Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM: Control, Optimisation and Calculus of Variations* 2, 33–55.

Rosier, L. and Zhang, B.-Y. (2006) Global stabilization of the generalized Korteweg–de Vries equation posed on a finite domain. *SIAM Journal on Control and Optimization* 45, 927–956.

Rosier, L. and Zhang, B.-Y. (2009) Control and stabilization of the Korteweg-de Vries equation: recent progresses. *Journal of Systems Science and Complexity* 22, 647–682.

Seidman, T. I. and Li, H. (2001) A note on stabilization with saturating feedback. *Discrete and Continuous Dynamical Systems* 7, 319–328.

Sene, N. (2022) Numerical methods applied to a class of seir epidemic models described by the Caputo derivative. In: *Methods of Mathematical Modeling*, Elsevier, 23–40.

Sene, N. and Ndiaye, A. (2024) Existence and uniqueness study for partial neutral functional fractional differential equation under Caputo derivative. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)* 14, 208–219.

Slemrod, M. (1989) Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control. *Mathematics of Control, Signals and Systems* 2, 265–285.

Taboye, A. and Ennouari, T. (2024) Stabilization of linear KdV equation with boundary time-delay feedback and internal saturation. *Gulf Journal of Mathematics* 17, 72–86.

Taboye, A. and Ennouari, T. (2025) Stability result of the Kawahara equation with localized damping and time-varying delay. *Evolution Equations and Control Theory* 14, 36–56.

Taboye, A. M. and Laabissi, M. (2022) Exponential stabilization of a linear Korteweg-de Vries equation with input saturation. *Evolution Equations and Control Theory* 11, 1519–1532.

Tarbouriech, S., Garcia, G., da Silva Jr, J. M. G. and Queinnec, I. (2011) *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer Science & Business Media.

Valein, J. (2022) On the asymptotic stability of the Korteweg-de Vries equation with time-delayed internal feedback. *Mathematical Control and Related Fields*, **12**(3), 667–694.