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THE INFLUENCE OF THE SHAFT MASS ON THE NATURAL PULSATION OF THE FREE AND FORCED TRANSVERSE VIBRATIONS AND HAVING A HOMOGENEOUS WHEEL MOUNTED AT THE END

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Abstract: *The paper aims to study the influence of the mass of the shafts on the natural pulsations of the free and forced transverse vibrations of an elastic system formed by a fixed shaft and a homogeneous wheel for two cases: free vibrations and forced vibrations due to harmonic disturbing forces. With the help of the professional program MATHCAD, the influence of the mass of the shafts on the natural pulsations was simulated for six cases in which the mass of the shaft is taken into account as: 5, 10, 25, 50, 75 and 100% of its real mass, for forced vibrations with frequencies very close to the first two natural pulsations p1 and p2.*

Keywords: *natural pulsations, free and forced vibrations, transverse vibrations of the shafts*

1. INTRODUCTION

To derive the differential equation of transverse vibrations of straight bars, we consider an element of this bar of length dx on which the distributed disturbing force f(x,t) perpendicular to the bar axis, the inertia force of the shaft fi(x,t) as well as the sectional stresses T(x,t) and the bending stresses M(x,t) act. We denote by w(x,t) the displacement along Oz of the element and by φ (x,t) the rotation of the section along the Oy axis as in figure 10.

We will derive the differential equation of the forced transverse vibrations under the action of the distributed disturbing force f(x,t) and free when this force is zero.

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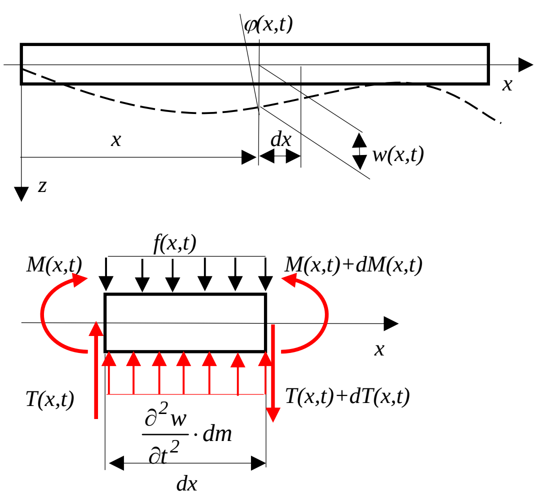

Fig. 1

The equation of the deformed mean fiber of the bar is written in terms of the bending stress Mi(x,t) and the second partial derivative of the displacement w(x,t) with respect to x according to the specialized literature [2,3,5,6], as well as the differential relations between the internal stresses (bending Mi(x,t) and shear T(x,t)) and the distributed external load f(x,t). Replacing the mass element and noting with:

$$\beta = \sqrt{\frac{\rho \cdot A}{E \cdot I_y}} \tag{1}$$

the differential equation of the forced transverse vibrations of straight bars is obtained:

$$\frac{\partial^4 w(x,t)}{\partial x^4} + \beta^2 \cdot \frac{\partial^2 w(x,t)}{\partial t^2} = \frac{f(x,t)}{E \cdot I_y} \tag{2}$$

Ecuaţia diferenţială a vibraţiilor transversale libere a barelor drepte se scrie:

$$\frac{\partial^4 w(x,t)}{\partial x^4} + \beta^2 \cdot \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \tag{3}$$

According to the technical literature [5,6] the solution of the differential equation (3) is obtained by the BERNOULLI-FOURIER method as the product between the displacement distribution function Y(x) and a harmonic time function $cos(pt - \varphi)$:

$$w(x,t) = Y(x) \cdot cos(pt - \varphi) \tag{4}$$

Substituting solution (4) into differential equation (3) we obtain the differential equation of the transverse vibrations of the bar:

$$\left[\frac{d^4Y(x)}{dx^4}-\beta^2\cdot p^2\cdot Y(x)\right]\cdot cos(pt-\varphi)=0 \qquad (5)$$

Since the harmonic function is non-zero, the homogeneous fourth-order differential equation of the displacement distribution form is obtained:

$$\frac{d^4Y(x)}{dx^4}-\beta^2\cdot p^2\cdot Y(x)=0 \qquad (6)$$

If the KRYLOV functions are used defined according to the techical literature [5,6] as :

$$\begin{cases} f_1(x)=\frac{1}{2}(ch\alpha x+cos\alpha x) \\ f_2(x)=\frac{1}{2}(sh\alpha x+sin\alpha x)\end{cases} \begin{cases} f_3(x)=\frac{1}{2}(ch\alpha x-cos\alpha x) \\ f_4(x)=\frac{1}{2}(sh\alpha x-sin\alpha x)\end{cases} \qquad (7)$$

The solution of the homogeneous differential equation (6) is in the form:

$$Y(x)=C_1\cdot f_1(x)+C_2\cdot f_2(x)+C_3\cdot f_3(x)+C_4\cdot f_4(x) \qquad (8)$$

The integration constants C1, …C4 are determined from the boundary conditions, if the values of the function Y(x) and its first three derivatives for the left end of the bar (x=0) are known:

$$\begin{cases} Y(0)=C_1\cdot f_1(0)+C_2\cdot f_2(0)+C_3\cdot f_3(0)+C_4\cdot f_4(0) \\ Y'(0)=\alpha\cdot C_1\cdot f_4(0)+\alpha\cdot C_2\cdot f_1(0)+\alpha\cdot C_3\cdot f_2(0)+\alpha\cdot C_3\cdot f_3(0) \\ Y''(0)=\alpha^2\cdot C_1\cdot f_3(0)+\alpha^2\cdot C_2\cdot f_4(0)+\alpha^2\cdot C_3\cdot f_1(0)+\alpha^2\cdot C_3\cdot f_2(0) \\ Y'''(0)=\alpha^3\cdot C_1\cdot f_2(0)+\alpha^3\cdot C_2\cdot f_3(0)+\alpha^3\cdot C_3\cdot f_4(0)+\alpha^3\cdot C_3\cdot f_1(0)\end{cases} \qquad (9)$$

The boundary conditions corresponding to the left end of the bar (x=0) and those corresponding to the right end of the bar (x=L) are the amplitudes of some characteristic quantities of transverse vibrations as follows:

$$w_1=Y(0);\quad w_2=Y(L) \qquad (10)$$

- displacements along the Oz axis of the bar ends:

$$\varphi_1=Y'(0);\quad \varphi_2=Y'(L) \qquad (11)$$

- rotations around the Oy axis of the bar ends:

$$M_1=-E\cdot I\cdot Y''(0);\quad M_2=-E\cdot I\cdot Y''(L) \qquad (12)$$

- bending forces along the Oy axis at the ends of the bar:

$$T_1=-E\cdot I\cdot Y'''(0);\quad T_2=-E\cdot I\cdot Y'''(L) \qquad (13)$$

- shear forces along the Oz axis at the ends of the bar.

Taking into account the properties of KRYLOV's functions (7):

$$f_1(0)=1;\quad f_2(0)=f_3(0)=f_4(0)=0$$

$$\begin{cases} f_1'(x)=\frac{\alpha}{2}(sh\alpha x-sin\alpha x)=\alpha\cdot f_4(x) \\ f_2'(x)=\frac{\alpha}{2}(ch\alpha x+cos\alpha x)=\alpha\cdot f_1(x) \\ f_3'(x)=\frac{\alpha}{2}(sh\alpha x+sin\alpha x)=\alpha\cdot f_2(x) \\ f_4'(x)=\frac{\alpha}{2}(ch\alpha x-cos\alpha x)=\alpha\cdot f_3(x)\end{cases} \qquad (14)$$

The following expressions of the constants C1, C2, C3 and C4 are obtained: :

$$\begin{cases} C_1=Y(0)=w_1;\ C_2=\frac{1}{\alpha}Y'(0)=\frac{\varphi_1}{\alpha}; \\ C_3=\frac{1}{\alpha^2}Y''(0)=-\frac{M_1}{\alpha^2\cdot E\cdot I}; \\ C_4=\frac{1}{\alpha^3}Y'''(0)=-\frac{T_1}{\alpha^3\cdot E\cdot I}\end{cases} \qquad (15)$$

By replacing the constants C1, C2, C3 and C4 given by relations (15), the solution of the differential equation of free transverse vibrations of straight bars is obtained, as well as the functions of rotations, bending and shear stresses:

$$\begin{cases} w(x)=\left[w_1\cdot f_1(x)+\varphi_1\cdot\frac{1}{\alpha}\cdot f_2(x)-M_1\cdot\frac{1}{\alpha^2E\cdot I}\cdot f_3(x)-T_1\cdot\frac{1}{\alpha^3\cdot E\cdot I}\cdot f_4(x)\right]cos(pt-\varphi) \\ \varphi(x)=\left[w_1\cdot\alpha\cdot f_4(x)+\varphi_1\cdot f_1(x)-M_1\cdot\frac{1}{\alpha\cdot E\cdot I}\cdot f_2(x)-T_1\cdot\frac{1}{\alpha^2\cdot E\cdot I}\cdot f_3(x)\right]cos(pt-\varphi) \\ M(x)=\left[-w_1\cdot E\cdot I\cdot\alpha^2\cdot f_3(x)-\varphi_1\cdot E\cdot I\cdot\alpha\cdot f_4(x)+M_1\cdot f_1(x)+T_1\cdot\frac{1}{\alpha}\cdot f_2(x)\right]cos(pt-\varphi) \\ T(x)=\left[-w_1\cdot E\cdot I\cdot\alpha^3\cdot f_2(x)-\varphi_1\cdot E\cdot I\cdot\alpha^2\cdot f_3(x)+M_1\cdot\alpha\cdot f_4(x)+T_1\cdot f_1(x)\right]cos(pt-\varphi)\end{cases} \qquad (16)$$

Using expressions (10), (11), (12) and (13) and substituting x=L in relations (17), we obtain the relations between the amplitudes of the four quantities at the ends of the bar as follows [1,4] : (17)

$$\begin{Bmatrix} w_2 \\ \frac{\varphi_2}{\alpha} \\ -\frac{M_2}{\alpha^2\cdot E\cdot I} \\ -\frac{T_2}{\alpha^3\cdot E\cdot I}\end{Bmatrix}=\begin{bmatrix} f_1(L) & f_2(L) & f_3(L) & f_4(L) \\ f_4(L) & f_1(L) & f_2(L) & f_3(L) \\ f_3(L) & f_4(L) & f_1(L) & f_2(L) \\ f_2(L) & f_3(L) & f_4(L) & f_1(L)\end{bmatrix}\cdot\begin{Bmatrix} w_1 \\ \frac{\varphi_1}{\alpha} \\ -\frac{M_1}{\alpha^2\cdot E\cdot I} \\ -\frac{T_1}{\alpha^3\cdot E\cdot I}\end{Bmatrix}$$

To determine the natural pulsations of the free transverse vibrations of straight bars, the boundary conditions of the bar are introduced into the matrix equation (19) and the condition for the existence of solutions is written. By numerically solving the equation with the unknowns $x_n=\alpha_n L$ using the MATHCAD program, the natural pulsations are obtained:

$$p_n = \frac{\alpha_n^2}{\beta} = \frac{x_n^2}{L^2\cdot\beta}. \qquad (18)$$

CASE A:

Free vibrations of a shaft embedded at one end and with a homogeneous wheel at the other end

A shaft of diameter d and length L embedded at one end with constant cross-section, modulus of elasticity E and density ρ is considered (fig. 2). A homogeneous wheel of radius R and mass M is mounted at the free end of the shaft [1,4] .

The first four pulsations and the corresponding functions of free transverse vibrations for the first mode of free vibrations with pulsation p1 or the distribution functions of displacements w(x), φ(x) and sectional stresses M(x), T(x) will be determined using parametric calculus method[1,4] .

The numerical values of the parameters are given*:* *d=50 mm; E=200·10*^{9} *Pa; L=0.5m,* ρ*=8000 kg/m*^{3}*; M=10 kg; R=0.3 m.*

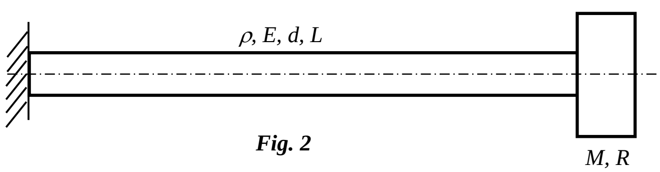

Fig. 2

To calculate the natural pulsations of the free transverse vibrations of the shaft, the following boundary conditions of the bar are introduced into the matrix equation (17):

$$\begin{cases} w_1=0;\quad \varphi_1=0;\quad M_1\neq 0;\ T_1\neq 0 \\ w_2\neq 0;\quad \varphi_2\neq 0; \\ M_2=-J\cdot\left.\dfrac{\partial^2\varphi}{\partial t^2}\right|_{x=L}=J\cdot\omega^2\cdot\varphi_2; \\ T_2=-M\cdot\left.\dfrac{\partial^2 w}{\partial t^2}\right|_{x=L}=M\cdot\omega^2\cdot w_2 \end{cases} \qquad (19)$$

The matrix equation (17) is written in this case as follows:

$$\begin{Bmatrix} w_2 \\ \dfrac{\varphi_2}{\alpha} \\ -\dfrac{J\omega^2}{\alpha^2 EI}\cdot\varphi_2 \\ -\dfrac{M\omega^2}{\alpha^3 EI}\cdot w_2 \end{Bmatrix} = \begin{bmatrix} f_1(L) & f_2(L) & f_3(L) & f_4(L) \\ f_4(L) & f_1(L) & f_2(L) & f_3(L) \\ f_3(L) & f_4(L) & f_1(L) & f_2(L) \\ f_2(L) & f_3(L) & f_4(L) & f_1(L) \end{bmatrix}\cdot\begin{Bmatrix} 0 \\ 0 \\ -\dfrac{M_1}{\alpha^2 EI} \\ -\dfrac{T_1}{\alpha^3 EI} \end{Bmatrix} \qquad (20)$$

To calculate the natural pulsations, the unknowns w2 and φ2 are eliminated from equations I, IV, respectively II, III and a homogeneous system of two equations is obtained with M1 and T1 as unknowns:

$$\begin{cases} \left(-\dfrac{M_1}{\alpha^2 EI}\right)\left(\dfrac{Mp^2}{\alpha^3 EI}\cdot f_3(L)+f_4(L)\right)+ \\ +\left(-\dfrac{T_1}{\alpha^3 EI}\right)\left(\dfrac{Mp^2}{\alpha^3 EI}\cdot f_4(L)+f_2(L)\right)=0 \\ \left(-\dfrac{M_1}{\alpha^2 EI}\right)\left(\dfrac{Jp^2}{\alpha\ EI}\cdot f_2(L)+f_1(L)\right)+ \\ +\left(-\dfrac{T_1}{\alpha^3 EI}\right)\left(\dfrac{Jp^2}{\alpha\ EI}\cdot f_3(L)+f_2(L)\right)=0 \end{cases} \qquad (21)$$

A homogeneous system of two equations was thus obtained, which can be written in matrix form as follows:

$$\begin{bmatrix} \dfrac{Mp^2}{\alpha^3 EI}\cdot f_3(L)+f_4(L) & \dfrac{Mp^2}{\alpha^3 EI}\cdot f_4(L)+f_1(L) \\ \dfrac{Jp^2}{\alpha\ EI} f_2(L)+f_1(L) & \dfrac{Jp^2}{\alpha\ EI} f_3(L)+f_2(L) \end{bmatrix}\cdot\begin{Bmatrix} -\dfrac{M_1}{\alpha^2 EI} \\ -\dfrac{T_1}{\alpha^3 EI} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (22)$$

The homogeneous system of equations (22) admits nonzero solutions if its determinant is zero:

$$F(L)=\begin{vmatrix} \dfrac{Mp^2}{\alpha^3 EI}\cdot f_3(L)+f_4(L) & \dfrac{Mp^2}{\alpha^3 EI}\cdot f_4(L)+f_1(L) \\ \dfrac{Jp^2}{\alpha\ EI} f_2(L)+f_1(L) & \dfrac{Jp^2}{\alpha\ EI} f_3(L)+f_2(L) \end{vmatrix}=0 \qquad (23)$$

The following notations are introduced into the determinant expression (23):

$$\alpha^2=\beta\cdot p=\sqrt{\frac{\rho\cdot A}{E\cdot I}}\cdot p;\quad \alpha^4 L=\frac{\rho\cdot A\cdot L}{E\cdot I}p^2$$
$$\Rightarrow \alpha^3 E\cdot I=\frac{\rho\cdot A\cdot L}{\alpha L}\cdot p^2=\frac{m_b}{\alpha L}\cdot p^2;$$
$$where:\ m_b=\rho\cdot A\cdot L;\quad J_b=m_b L^2$$
$$\Rightarrow \frac{Mp^2}{\alpha^3 EI}=\frac{M}{m_b}\cdot\alpha L;\quad \frac{J\,p^2}{\alpha\cdot EI}=\frac{J}{J_b}\cdot\alpha^3 L^3 \qquad (24)$$

The transcendental equation F(L)=0 can only be solved numerically using the root function in MATHCAD and the following results are obtained:

$$\begin{array}{ll} x1 := \mathrm{root}(F(x),x,1,2) & x1 = 1.329641 \\ x2 := \mathrm{root}(F(x),x,4,5) & x2 = 4.812596 \\ x3 := \mathrm{root}(F(x),x,7,8) & x3 = 7.953182 \\ x4 := \mathrm{root}(F(x),x,11,12) & x4 = 11.07888 \end{array} \qquad (25)$$

For the given values of the parameters*:* *d=50 mm; E=200·10*^{9} *Pa; L=0.5m,* ρ*=8000 kg/m*^{3}*; M=10 kg; R=0.3* m

the required values for the first four natural pulsations of the free vibrations are obtained:

$$p1 := \frac{x1^2}{L^2 \cdot \beta} \qquad p1 = 530.383553$$

$$p2 := \frac{x2^2}{L^2 \cdot \beta} \qquad p2 = 6.948324 \times 10^3$$

$$p3 := \frac{x3^2}{L^2 \cdot \beta} \qquad p3 = 1.897593 \times 10^4 \qquad (26)$$

$$p4 := \frac{x4^2}{L^2 \cdot \beta} \qquad p4 = 3.682248 \times 10^4$$

The distribution functions of the displacements w(x), φ(x) and of the stresses M(x), T(x), corresponding to the free transverse vibrations for the first mode of free vibrations (with pulsation p1) are written taking into account the boundary conditions w1=0, φ1=0, depending on the amplitudes M1 and T1 using relations (16) as follows:

$$\begin{cases} w(x) = -\dfrac{M_1}{\alpha^2 EI} \cdot f_3(x) - \dfrac{T_1}{\alpha^3 EI} \cdot f_4(x) \\ \dfrac{\varphi(x)}{\alpha} = -\dfrac{M_1}{\alpha^2 EI} \cdot f_2(x) - \dfrac{T_1}{\alpha^3 EI} \cdot f_3(x) \end{cases}$$

$$\begin{cases} M_m(x) = -EI \cdot \left(\dfrac{d^2 w(x)}{dx^2} \right) \\ T(x) = -EI \cdot \left(\dfrac{d^3 w(x)}{dx^3} \right) \end{cases} \qquad (27)$$

Using the MATHCAD program, figure 3 represents the distribution functions of the bar deformations and figure 4 represents the distribution functions of the bending moment Mm(x) and the shear force T(x) for vibration mode no. 1 (with pulsation p1)

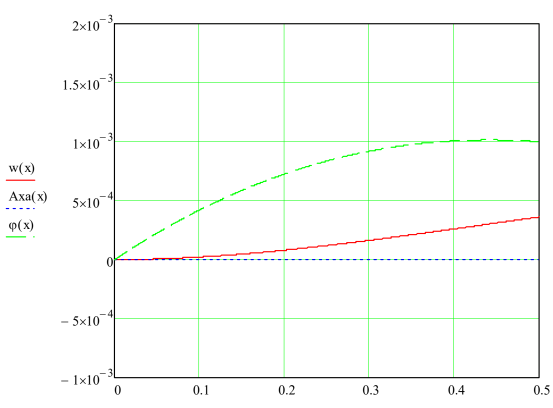

Fig. 3

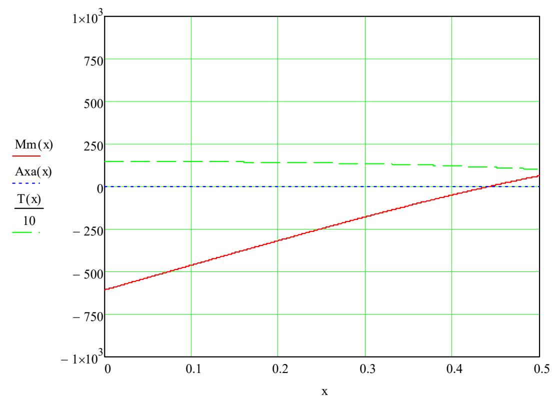

Fig. 4

CASE B:

Forced vibrations of a shaft embedded at one end with a homogeneous wheel at the other end, on which a harmonic force $F_0 \cdot \cos(\omega t)$ acts

We consider the same shaft of diameter d and length L embedded at one end having constant section, modulus of elasticity E and density ρ (fig. 5).

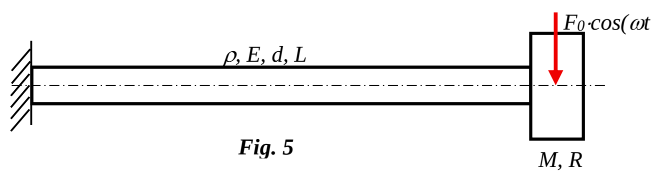

Fig. 5

At the free end of the shaft is fixed a homogeneous wheel of radius R and mass M, on which a harmonic force $F_0 \cdot \cos(\omega t)$ acts having four pulsations with values very close to the values of the first four natural pulsations of the shaft calculated previously: ω1=1.1 p1; ω2=1.1 p1; ω3=1.1 p3 and ω4=1.1 p4 .

The distribution functions of the displacements w(x), φ(x) and of the sectional stresses M(x), T(x) and the values of the dynamic factor of the bending moments in the embedment due to the forced vibrations for the given values of the four pulsations ω1=1.1 p1; ω2=1.1 p1; ω3=1.1 p3 and ω4=1.1 p4 will be determined.

For the study of forced transverse vibrations, the boundary conditions of the bar are introduced in the matrix equation (17) and it is taken into account that the vibration pulsation is ω:

$$\begin{cases} w_1 = 0; \quad \varphi_1 = 0; \quad M_1 \neq 0; \quad T_1 \neq 0 \\ w_2 \neq 0; \quad \varphi_2 \neq 0; \\ M_2 = -J \cdot \left. \dfrac{\partial^2 \varphi}{\partial t^2} \right|_{x=L} = J \cdot \omega^2 \cdot \varphi_2; \\ T_2 = F_0 - M \cdot \left. \dfrac{\partial^2 w}{\partial t^2} \right|_{x=L} = F_0 + M \cdot \omega^2 \cdot w_2 \end{cases} \qquad (28)$$

The matrix equation (17) is written in this case:

$$\begin{Bmatrix} w_2 \\ \dfrac{\varphi_2}{\alpha} \\ -\dfrac{J\omega^2}{\alpha^2 EI}\cdot\varphi_2 \\ -\dfrac{F_0}{\alpha^3 EI}-\dfrac{M\omega^2}{\alpha^3 EI}\cdot w_2 \end{Bmatrix}=\begin{bmatrix} f_1(L) & f_2(L) & f_3(L) & f_4(L) \\ f_4(L) & f_1(L) & f_2(L) & f_3(L) \\ f_3(L) & f_4(L) & f_1(L) & f_2(L) \\ f_2(L) & f_3(L) & f_4(L) & f_1(L) \end{bmatrix}\cdot\begin{Bmatrix} 0 \\ 0 \\ -\dfrac{M_1}{\alpha^2 EI} \\ -\dfrac{T_1}{\alpha^3 EI} \end{Bmatrix} \tag{29}$$

In order to calculate the unknowns M1 and T1, the unknowns w2 and $\varphi 2$ are eliminated from the four equations and a system of two nonhomogeneous equations is obtained:

$$\begin{cases} -\dfrac{M_1}{\alpha^2 EI}\left(\dfrac{M\omega^2}{\alpha^3 EI}\cdot f_3(L)+f_4(L)\right)- \\ -\dfrac{T_1}{\alpha^3 EI}\left(\dfrac{M\omega^2}{\alpha^3 EI}\cdot f_4(L)+f_1(L)\right)=-\dfrac{F_0}{\alpha^3 EI} \\ -\dfrac{M_1}{\alpha^2 EI}\left(\dfrac{J\omega^2}{\alpha\ EI}\cdot f_2(L)+f_1(L)\right)- \\ -\dfrac{T_1}{\alpha^3 EI}\left(\dfrac{J\omega^2}{\alpha\ EI}\cdot f_3(L)+f_2(L)\right)=0 \end{cases} \tag{30}$$

The nonhomogeneous system of equations (30) can also be written in matrix form:

$$\begin{Bmatrix} -\dfrac{F_0}{\alpha^3 EI} \\ 0 \end{Bmatrix}=\begin{bmatrix} \dfrac{M\omega^2}{\alpha^3 EI}\cdot f_3(L)+f_4(L) & \dfrac{M\omega^2}{\alpha^3 EI}\cdot f_4(L)+f_1(L) \\ \dfrac{J\omega^2}{\alpha\ EI} f_2(L)+f_1(L) & \dfrac{J\omega^2}{\alpha\ EI} f_3(L)+f_2(L) \end{bmatrix}\cdot\begin{Bmatrix} -\dfrac{M_1}{\alpha^2 EI} \\ -\dfrac{T_1}{\alpha^3 EI} \end{Bmatrix}; \tag{31}$$

Solve equation (31) using the CRAMER method:

$$\Delta=F(L)=\begin{vmatrix} \dfrac{M\omega^2}{\alpha^3 EI}\cdot f_3(L)+f_4(L) & f_1(L)+\dfrac{M\omega^2}{\alpha^3 EI}\cdot f_4(L) \\ \dfrac{J\omega^2}{\alpha\ EI} f_2(L)+f_1(L) & \dfrac{J\omega^2}{\alpha\ EI} f_3(L)+f_2(L) \end{vmatrix} \tag{32}$$

$$\Delta_1=\begin{vmatrix} -\dfrac{F_0}{\alpha^3 EI} & f_1(L)+\dfrac{M\omega^2}{\alpha^3 EI}\cdot f_4(L) \\ 0 & \dfrac{J\omega^2}{\alpha\ EI} f_3(L)+f_2(L) \end{vmatrix}$$

$$\Delta_1=-\frac{F_0}{\alpha^3 EI}\cdot\left(\frac{J\omega^2}{\alpha\ EI} f_3(L)+f_2(L)\right) \tag{33}$$

This gives the expression for the unknown M1:

$$\Rightarrow -\frac{M_1}{\alpha^2 EI}=-\frac{F_0}{F(L)\cdot\alpha^3 EI}\cdot\left(\frac{J\omega^2}{\alpha\ EI} f_3(L)+f_2(L)\right) \tag{34}$$

$$\Delta_2=\begin{vmatrix} \dfrac{M\omega^2}{\alpha^3 EI}\cdot f_3(L)+f_4(L) & -\dfrac{F_0}{\alpha^3 EI} \\ \dfrac{J\omega^2}{\alpha\ EI} f_2(L)+f_1(L) & 0 \end{vmatrix}$$

$$\Delta_2=\frac{F_0}{\alpha^3 EI}\cdot\left(\frac{J\omega^2}{\alpha\ EI} f_2(L)+f_1(L)\right) \tag{35}$$

This gives also the expression for the unknown T1:

$$\Rightarrow -\frac{T_1}{\alpha^3 EI}=\frac{F_0}{F(L)\cdot\alpha^3 EI}\cdot\left(\frac{J\omega^2}{\alpha\ EI} f_2(L)+f_1(L)\right) \tag{36}$$

The functions of the four corresponding quantities: w(x), φ(x), Mm(x) and T(x), are written using formulas (16) in which $\alpha_0^2=\beta\cdot\omega$ introduced instead of în loc de $\alpha^2=\beta\cdot p$.

$$\begin{cases} w(x)=-\dfrac{M_1}{\alpha^2 EI}\cdot f_3(x)-\dfrac{T_1}{\alpha^3 EI}\cdot f_4(x) \\ \dfrac{\varphi(x)}{\alpha}=-\dfrac{M_1}{\alpha^2 EI}\cdot f_2(x)-\dfrac{T_1}{\alpha^3 EI}\cdot f_3(x) \\ M_m(x)=-EI\cdot\left(\dfrac{d^2 w(x)}{dx^2}\right) \\ T(x)=-EI\cdot\left(\dfrac{d^3 w(x)}{dx^3}\right) \end{cases} \tag{37}$$

Numerical results obtained

Using the MATHCAD program for the numerical values of the parameters: d=50 mm; E=200 GPa; L=0.5m, ρ=8000 kg/m3; M=10 kg; R=0.3 m, the numerical values in Table 1 are obtained.

4.1. For the first case of loading with the harmonic force having the pulsation ω1=1.1p1, the bar deformations w(x), φ(x), respectively the stress functions M(x), T(x) were obtained in figures 6 and 7.

4.2. For the second case of loading with the harmonic force having the pulsation ω2=1.1p2, the bar deformations w(x), φ(x) respectively the stress functions M(x) , T(x) were obtained in figures 8 and 9.

4.3. For the third case of loading with the harmonic force having the pulsation ω3=1.1p3, the bar deformations w(x), φ(x), respectively the stress functions M(x), T(x) were obtained in fig. 10 and 11.

4.4. For the fourth case of loading with the harmonic force having the pulsation ω4=1.1p4, the bar deformations w(x), φ(x), respectively the stress functions M(x), T(x) were obtained in fig. 12 and 13.

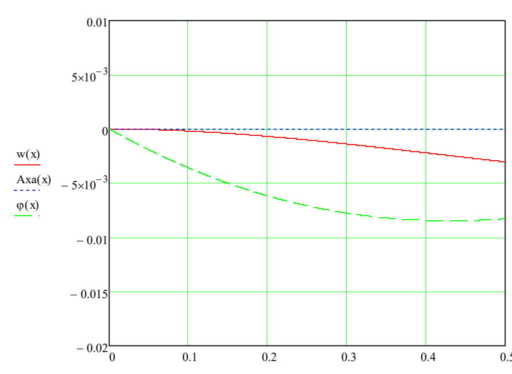

Fig. 6

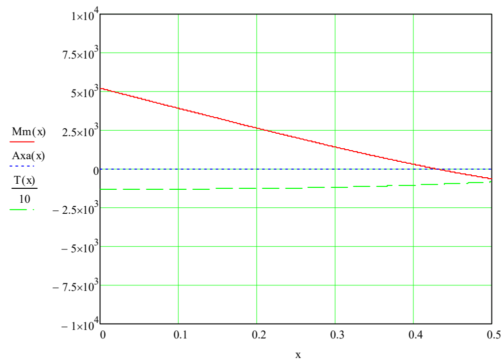

Fig. 7

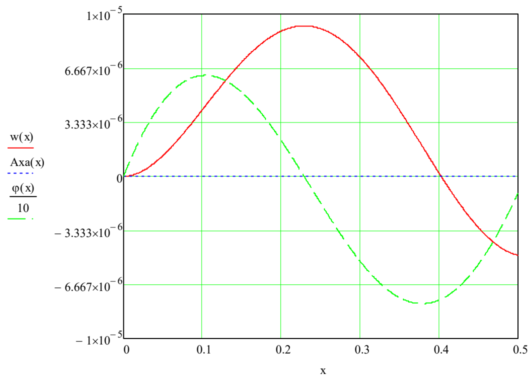

Fig. 8

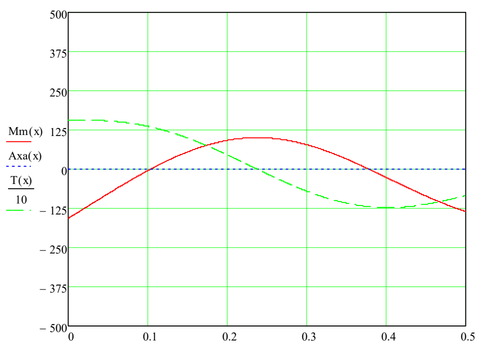

Fig. 9

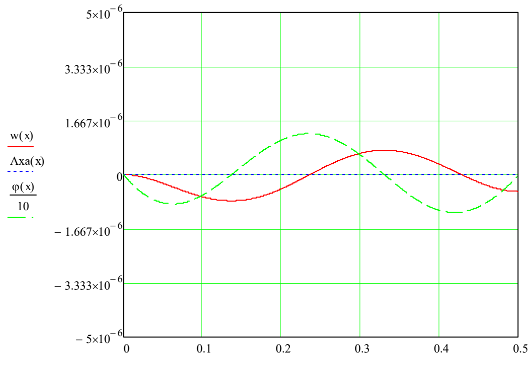

Fig. 10

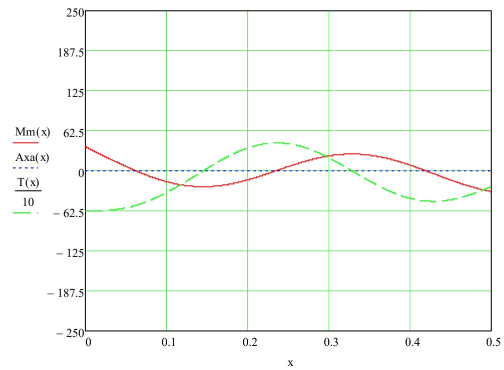

Fig. 11

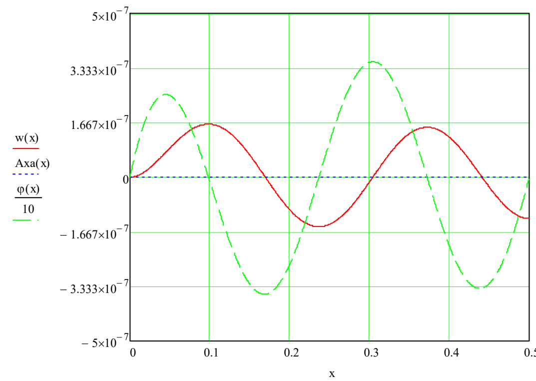

Fig. 12

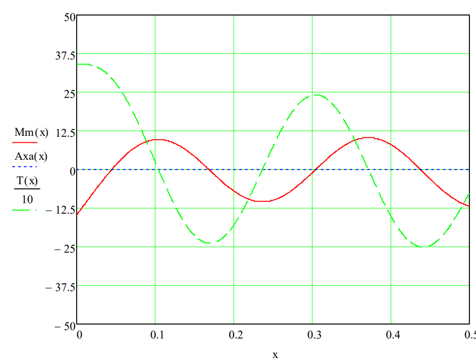

Fig. 13

Table 1

Natural mode of vibration	Natural pulsation value p (rad/s)	Forced vibration pulsation ω=1,1p (rad/s)	M1 (Nm)	T1 (N)	w2 (mm)	φ2 (rad)	M2 (Nm)	T2 (N)	The dynamic factor Ψ

1	530,38	583,42	5.216,98	-12.944,4	-3,05	$-8\cdot10^{-3}$	-635,5	-8.382,3	5,217
2	6.848,32	7.643,15	-156,433	1569,8	$-4,8\cdot10^{-3}$	$-1\cdot10^{-5}$	-135,2	-841,5	0,156
3	18.975,93	20.873,53	37,48	-625,44	$-5,16\cdot10^{-4}$	$-3,3\cdot10^{-7}$	-32,4	-247,9	0.037
4	36.822,48	40.504,72	-14,7	341,9	$1,2\cdot10^{-4}$	$-3,2\cdot10^{-8}$	-11,8	-75,5	0,015

5. DYNAMIC VIBRATION FACTOR

The dynamic vibration factor is defined as the absolute value of the ratio between the value of the embedment moment M1 and a reference value equal to the moment of force F0 if this force acts statically at the end of the bar:

$$\Psi = \frac{M_1(\omega)}{F_0 \cdot L}; \qquad \alpha_0(\omega) = \sqrt{\beta \cdot \omega}$$

$$M_1(\omega) = \frac{F_0}{F(L)\cdot\alpha_0(\omega)} \cdot \left(\frac{J \cdot \omega^2}{\alpha_0(\omega)\cdot EI} f_3(L) + f_2(L) \right)$$

Using the MATHCAD program, the variation of the dynamic factor as a function of the pulsation ω of the disturbing force was represented in figure 14. The numerical results obtained for the values of the dynamic factor corresponding to the four pulsations ω of the external force very close to the first four natural pulsations of the tree are:

$$\Psi(1.1\cdot p1) = 5.216981$$
$$\Psi(1.1\cdot p2) = 0.156417$$
$$\Psi(1.1\cdot p3) = 0.037484$$
$$\Psi(1.1\cdot p4) = 0.014714$$

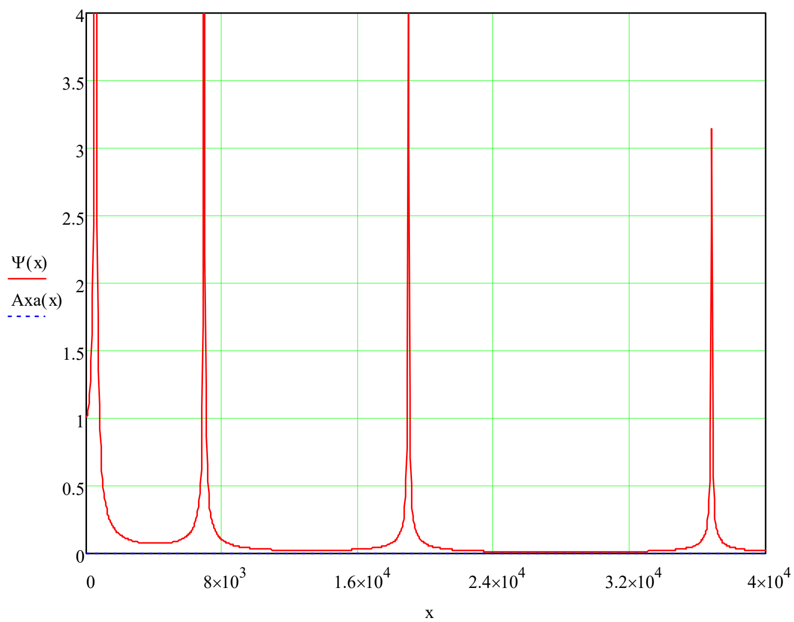

Fig.14

6. INFLUENCE OF THE SHAFT MASS ON THE NATURAL PULSATES

For forced vibrations with pulsations $\omega_1=1,1\cdot p_1$ and $\omega_2=1,1\cdot p_2$ the influence of the shaft mass on the quantities M1, T1, M2, T2, w2, φ2 and Ψ was simulated, using the professional calculation program MATHCAD, for six cases in which we took into consideration the shaft mass as: 100%, 75%, 50%, 25%, 10% and 5% of its total mass, the results obtained being given in Table 2 and Table 3 respectively.

Table 2

Masa arborelui	Natural pulsation value p_1 (rad/s)	Forced vibration pulsation $\omega_1=1,1p_1$ (rad/s)	M1 (Nm)	T1 (N)	w2 (mm)	φ2 (rad)	M2 (Nm)	T2 (N)	Factorul dimanic Ψ
100%	530,38	583,42	5.217	-12.944	-3,05	$-8,3\cdot10^{-3}$	-635,5	-8.382	5,22
75%	474,48	521,92	5.015	-12.012	-3	$-8,3\cdot10^{-3}$	-509	-8.451	5,01
50%	401,08	441,19	4.788	-10.970	-2,93	$-8,3\cdot10^{-3}$	-364	-8.499	4,79
25%	294,36	323,80	4.531	-9.800	-2,85	$-8,3\cdot10^{-3}$	-196	-8.514	4,53
10%	190,64	209,70	4.360	-9.025	-2,81	$-8,3\cdot10^{-3}$	-82	-8.499	4,36
5%	135,91	149,50	4.300	-8.755	-2,79	$-8,3\cdot10^{-3}$	-42	-8.489	4,30

Table 3

Masa arborelui	Natural pulsation value p_1 (rad/s)	Forced vibration pulsation $\omega_1=1{,}1p_1$ (rad/s)	M1 (Nm)	T1 (N)	w2 (mm)	φ2 (rad)	M2 (Nm)	T2 (N)	Factorul dimanic Ψ
100%	6.848,32	7.643,15	-156,43	1.569,80	$-4{,}8\cdot10^{-3}$	$-10\cdot10^{-6}$	-135,2	-841,5	0,156
75%	6.902,08	7.592,29	-94,77	947,59	$-0{,}002\cdot10^{-3}$	$-6{,}3\cdot10^{-6}$	-82,3	-527,3	0,095
50%	6.848,26	7.533,09	-45,59	453,88	$-0{,}001\cdot10^{-3}$	$-3{,}1\cdot10^{-6}$	-39,8	-263,3	0,045
25%	6.785,47	7.464,02	-12,41	122,89	$-0{,}003\cdot10^{-4}$	$-0{,}87\cdot10^{-6}$	-10,89	-74,6	0,012
10%	6.742,79	7.417,07	-2,09	20,68	$-0{,}005\cdot10^{-5}$	$-0{,}15\cdot10^{-6}$	-1,84	-12,9	0,002
5%	6.727,63	7.400,04	-0,53	5,26	$-0{,}001\cdot10^{-5}$	$-0{,}03\cdot10^{-6}$	-0,47	-3,33	0,0005

7. ANALYSIS OF RESULTS - CONCLUSIONS

Analyzing the numerical results obtained in **Table 1,** the following conclusions can be drawn:

- the amplitude of the moments (M1) and the cutting forces (T1) in the embedment decrease in absolute value compared to the case of the pulsation p1;
- the amplitude of the displacements (w2) and rotations (φ2) in the shaft end decrease greatly in absolute value compared to the case of the pulsation p1;
- the values of the dynamic factor (Ψ) corresponding to the four pulsations are increasingly lower, being similar to the amplitude values in the diagram of the frequency spectra of the vibrations w(ω).

Analyzing the numerical results obtained in **Table 2** and **Table 3**, the following conclusions can be drawn regarding the influence of the shaft mass:

- the values of the natural pulsations ω1 and ω2 decrease with the decrease of the shaft mass in the conditions in which the modulus of elasticity of the shaft edge does not change;
- the moments (M1) and the cutting forces (T1) in the embedment decrease in value at higher rates in the case of pulsation ω2 compared to the case of pulsation ω1;
- for the case of pulsation ω1 the amplitude of the displacements (w2) at the shaft end decreases very little and the amplitude of the rotations (φ2) remains practically constant;
- for the case of pulsation ω2 the amplitude of the displacements (w2) and the rotations (φ2) at the shaft end decrease a lot;
- for the case of pulsation ω1 the amplitude of the moments (M2) at the shaft end decreases and the amplitude and of the cutting forces (T2) remains practically constant;
- for the case of pulsation ω2, the amplitude of the moments (M2) and the cutting forces (T2) at the end of the shaft decrease**.**

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