

ESTIMATION OF MEASUREMENT ERRORS IN RADAR TRAJECTORIES USING THE LEAST SQUARES METHOD

ANDREI SIDOROV^{1,*}, TATIANA IGONINA² and VLADIMIR KESELMAN²

¹*Alexandru Ioan Cuza University of Iași, Faculty of Economics and Business Administration
Bulevardul Carol I nr. 22, 700505, Iași, Romania*

²*RTU MIREA – Russian Technological University, Department of higher mathematics No. 1, Institute of Artificial
Intelligence, 78 Vernadsky Avenue, 119454, Moscow, Russia*

**Corresponding author; e-mail: aa@mirrors.cc*

SIDOROV ANDREI, IGONINA
TATIANA and KESELMAN
VLADIMIR

ESTIMATION OF MEASUREMENT
ERRORS IN RADAR TRAJECTORIES
USING THE LEAST SQUARES
METHOD

BULLETIN OF THE POLYTECHNIC
INSTITUTE OF IAȘI, 2025, 71(75), 3-4,
7-28;

DOI: 10.2478/bipmf-2025-0007

Received: May 9, 2026

Accepted: May 25, 2026

Publication: July 15, 2026

© 2025 Sidorov A. et al.

This is an open access article licensed
under the Creative Commons
Attribution-NonCommercial-
NoDerivatives 4.0 International
License (CC BY-NC-ND 4.0)

Abstract. Radar systems are widely used for the detection and tracking of moving objects such as aircraft, missiles, and other airborne targets. Accurate determination of object coordinates is a key task in radar tracking systems, since the reliability of trajectory analysis and further decision-making depends on the precision of these measurements. However, radar measurements are affected by random errors caused by noise, signal fluctuations, propagation conditions, and system limitations. Therefore, the statistical estimation of measurement errors is an important problem in radar data processing. The purpose of this study is to estimate the root mean square errors of radar coordinate measurements using statistical processing of trajectory data. The research focuses on the analysis of measurement deviations obtained from radar observations and on the application of mathematical methods for improving the accuracy of trajectory representation. The proposed approach is based on the approximation of measured radar trajectories using the least squares method. Polynomial functions are used to approximate the observed trajectory of a moving object, allowing the deterministic component of motion to be separated from random measurement deviations. The coefficients of the approximating polynomial are determined using the least squares technique, and the residual deviations between the measured data and the approximated trajectory are analyzed as random errors. Statistical methods are then applied to estimate the mean values, variances, and confidence intervals of these errors. The obtained results show that trajectory approximation allows effective smoothing of measurement noise and provides reliable estimates of coordinate measurement errors. The proposed method can be used for statistical analysis of radar tracking data and for evaluating the accuracy of radar measurement systems.

Keywords: radar tracking, radar measurements, trajectory approximation, least squares method, measurement error estimation, statistical data processing, root mean square error.

1. Introduction

Radar systems are widely used for the detection, tracking, and monitoring of moving objects in applications such as air traffic control, aerospace surveillance, autonomous transportation, and defense systems. A key function of radar systems is the measurement of object coordinates, which enables the determination of trajectories and motion parameters. The accuracy of these measurements directly affects the performance of radar tracking systems and trajectory estimation [1, 2].

Radar measurements are inevitably affected by random errors caused by thermal noise, signal fluctuations, environmental propagation conditions, and hardware limitations. These uncertainties lead to deviations between measured coordinates and the true trajectory of the observed object, making statistical error analysis and the development of reliable estimation methods essential tasks in radar data processing [3, 4].

Recent studies have proposed various approaches to improving radar trajectory estimation and reducing measurement errors, including polynomial trajectory fitting, least-squares-based estimation methods, and error compensation techniques for radar systems deployed on mobile and unmanned platforms [5-8].

One of the most widely used approaches for processing experimental measurement data is the least squares method, which determines an approximating function by minimizing the mean square deviation between observations and the mathematical model. Polynomial least-squares approximation is particularly effective for smoothing trajectory data and separating deterministic motion from random measurement noise. In addition, statistical analysis of approximation residuals enables the estimation of root mean square measurement errors and the construction of confidence intervals for accuracy assessment [9, 10].

The objective of this study is to develop a method for estimating the root mean square errors of coordinate measurements of moving airborne objects observed by radar systems. The proposed approach combines polynomial trajectory approximation based on the least squares method with statistical estimation of measurement errors and is implemented in a computer program.

The remainder of the paper is organized as follows. Section 2 describes the basic principles of measuring object coordinates using radar systems. Sections 3 and 4 present the approximation of simulated data using polynomial functions based on the least squares method and describe the corresponding approximation algorithm. Section 5 discusses statistical methods for estimating measurement errors, including the construction of confidence intervals for the root mean square deviation, and presents the algorithm for estimating the root mean square errors of coordinate measurements of moving airborne objects. The study is predominantly theoretical, and the proposed

approach is validated exclusively through numerical simulations without the use of real experimental data.

2. Methods for Measuring Object Coordinates Using Radar Systems

Radar is a field of radio electronics that provides information about objects through the reception and analysis of reflected radio waves [11, 12].

Typical radar targets include aircraft, missiles, ships, and ground vehicles. In addition to artificial objects, radar systems may also detect natural objects such as clouds or planetary bodies [13].

Technical systems designed to obtain information about objects through the reception and processing of radio waves are referred to as radar stations or radar systems.

The process of obtaining information about objects using radar systems can generally be divided into several main stages:

- detection of objects;
- measurement of coordinates and motion parameters;
- resolution;
- object recognition.

Detection involves determining the presence or absence of an object within a specified region of space while ensuring that the probabilities of false decisions remain within acceptable limits [11].

Measurement consists of estimating the coordinates and motion parameters of detected objects with minimal permissible error. In most radar systems, the spherical coordinates of a target are determined: range r , azimuth β , and elevation angle ε (Figure 1). The motion parameters of the object can also be obtained by calculating derivatives of these coordinates with respect to time [12].

Resolution refers to the ability of a radar system to detect and measure the parameters of a selected object in the presence of other objects located within the observation region [11].

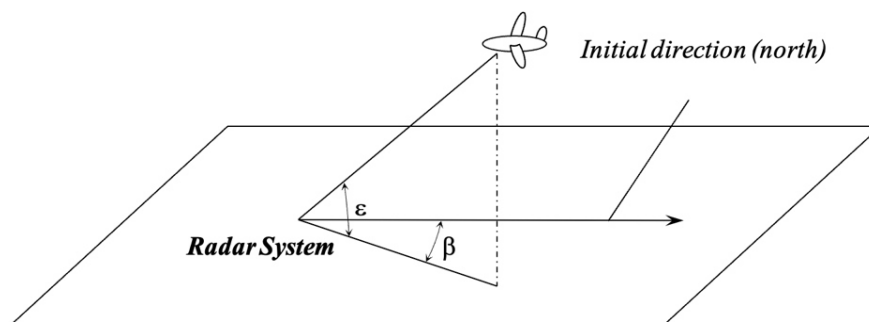


Fig. 1 – Spherical coordinates of the object.

Object recognition involves determining the class to which a detected object belongs, for example distinguishing between a civilian aircraft and a military aircraft, or between a fighter and a bomber (Kuzmin, 2000).

In radar systems, information about an object is obtained due to the disturbance of the electromagnetic field caused by the object, particularly through the emission or reflection of radio waves [11].

Several types of radio-wave emission associated with an object can be distinguished: secondary radiation, reradiation (retransmission), and intrinsic radiation. In the first two cases, the radar system transmits a powerful probing radio signal toward the object, whereas in the latter case external illumination of the object is not required.

Radar systems that rely on secondary radiation or reradiation are referred to as active radar, while systems that utilize the object's own electromagnetic emission are known as passive radar [13].

In the general case, an object does not necessarily emit or retransmit radio waves independently. Therefore, in many situations radar information is obtained using the phenomenon of secondary radiation. When the probing signal emitted by the radar system (the primary electromagnetic wave) impinges on the object, it induces forced oscillations of electric charges within it. As a result, the object behaves similarly to an antenna and generates its own electromagnetic field. This field forms a secondary, or reflected electromagnetic wave, which returns to the radar receiver and carries information about the object [12].

For example, the amplitude of the received signal partially characterizes the size and reflective properties of the object. The time delay between the transmitted probing signal and the received echo signal is used to determine the range to the object. The frequency of the received oscillations, due to the Doppler effect, provides information about the object's velocity. In addition, the polarization parameters of the reflected wave may be used to evaluate certain properties of the object, such as its shape or the ratio of its dimensions. Finally, the direction of arrival of the reflected wave contains information about the angular coordinates of the object, namely the azimuth and elevation angles [11].

Active radar systems can be classified as monostatic or bistatic, depending on the spatial arrangement of the transmitting and receiving equipment. In a monostatic radar system, the transmitting and receiving devices are located at the same site and usually share a common antenna system (Figure 2) [11, 12].

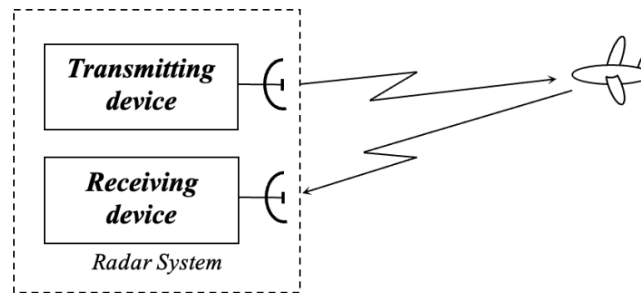


Fig. 2 – Combined radar station.

In a bistatic radar system, the transmitting and receiving units are separated by a certain distance d from each other (Figure 3). In this configuration, the transmitter illuminates the target, while the reflected signal is received at a spatially separated receiving station [11, 13].

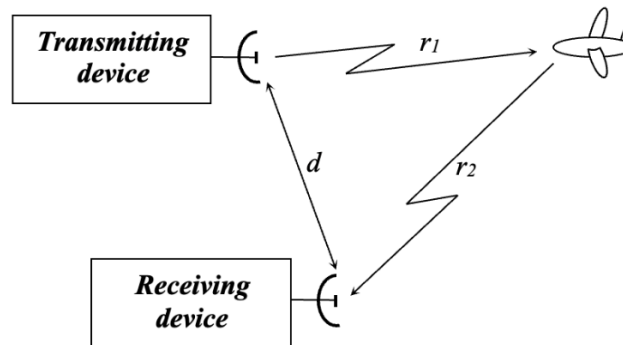


Fig. 3 – Distributed radar system.

In passive radar systems, the object is not illuminated by externally generated electromagnetic waves. Instead, electromagnetic radiation is produced by elements of the object itself. Such radiation may originate from heated components of the object (thermal emission in the infrared or millimeter-wave ranges), from onboard radio devices used for communication, navigation, or radar operation, as well as from oscillating charged particles in ionized regions of the atmosphere surrounding the object. Examples include radio emissions generated during rocket launches or nuclear explosions. Signal reception in passive radar may be performed by a single receiver or by several spatially separated receiving stations [11, 14].

When measuring object coordinates in radar systems, specific physical properties of radio-wave propagation are utilized. In free space, the propagation velocity of radio waves is constant and equal to approximately $c \approx 3 \cdot 10^8$ m/s, regardless of the polarization or frequency of the oscillations. Under ideal conditions, both the probing and the reflected signals propagate along straight-line trajectories without distortion of their waveform [11, 12].

The time delay t_d of the reflected signal relative to the transmitted probing signal (Figure 4) in a bistatic radar system is determined by the following relationship:

$$t_d = \frac{r_1 + r_2}{c} \quad (2.1)$$

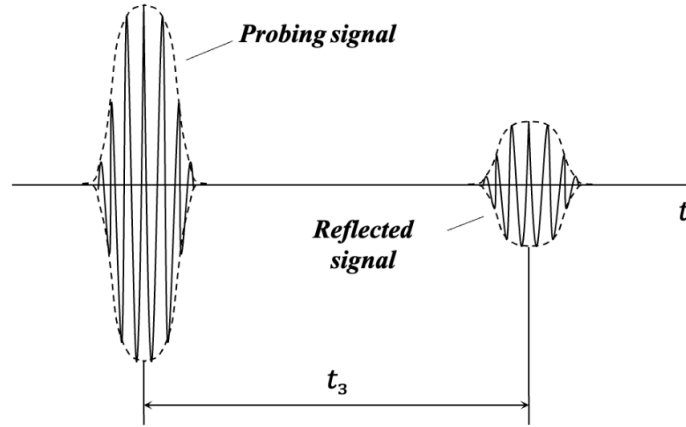


Fig. 4 – Explanation of the delay of the reflected signal.

where r_1 and r_2 represent the distances from the target to the transmitting and receiving stations, respectively (Figure 3) [11, 12].

For a monostatic radar system (Figure 2), the time delay is expressed as

$$t_d = \frac{2r}{c} \quad (2.2)$$

In this case, the range to the target is determined as

$$r = \frac{ct_d}{2} \quad (2.3)$$

or, in practical units,

$$r_{[km]} = 150 \cdot t_{d[ms]}$$

which provides a convenient relationship between the measured signal delay and the distance to the target [13, 14].

The concentration of radiated energy in a specific direction combined with directional reception significantly increases the effective range of a radar system. This configuration also enables the measurement of the angular coordinates of a target—azimuth and elevation angle—for example, by detecting the direction corresponding to the maximum reflected signal (Figure 5) [11, 12].

The beamwidth (or directional characteristic) of a radar antenna is determined by the ratio of the electromagnetic wavelength to the geometric dimensions of the antenna. Consequently, high directivity is achieved by increasing the physical size of the antenna and by using shorter wavelengths, such as decimeter, centimeter, or millimeter waves [13, 14].

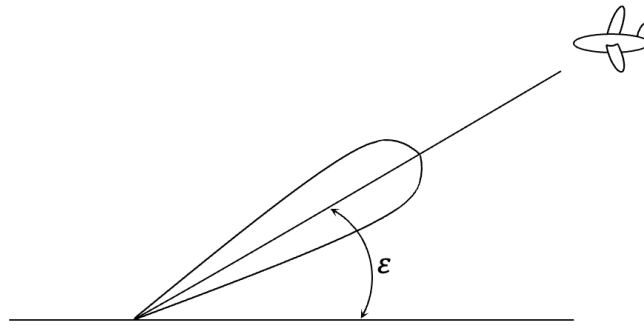


Fig. 5 – Measurement of the object's angular coordinate.

Thus, the range of a target in a radar system is calculated using formula (2.3) based on the time delay of the reflected signal and the propagation speed of electromagnetic waves. The angular coordinates of the target—azimuth β and elevation angle ε —are determined by the orientation of the main axis of a highly directional antenna at the moment when the reflected signal reaches its maximum.

The Cartesian coordinates X, Y, H of a target can be obtained from its measured spherical coordinates using the standard transformation formulas [11-13], as illustrated in Figure 6:

$$x = r \cos \varepsilon \cos \beta,$$

$$y = r \cos \varepsilon \sin \beta,$$

$$h = r \sin \varepsilon$$

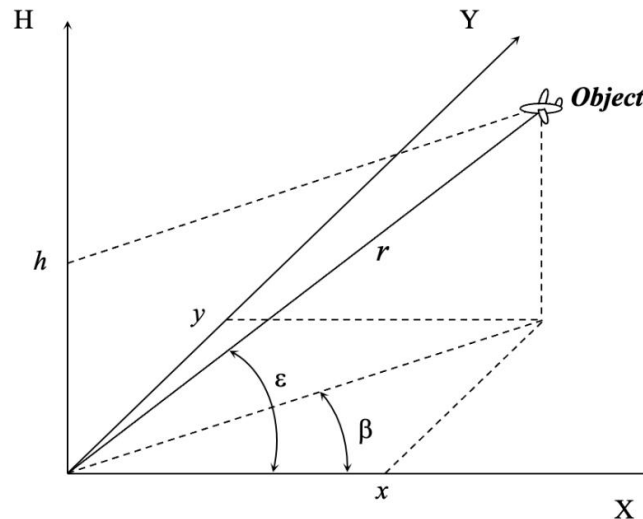


Fig. 6 – Object coordinates in a rectangular system.

If a radar system measures the coordinates of a moving aerial object at time instances t_i , ($i = 1, 2, 3, \dots, N$) and records them—either by storing them in a computer's memory or transmitting them to an external user—the resulting set of coordinates $\{(t_i, x_i, y_i, h_i), i = 1, 2, 3, \dots, N\}$ represents the

measured discrete trajectory of the object, defined at nodal points. Since each coordinate measurement contains random errors, the true trajectory of the object, when overlaid on the measured discrete trajectory recorded by the radar, may not pass exactly through these nodal points [11, 12].

One of the objectives of the analysis is to approximate the discrete trajectory using a polynomial of degree no higher than four. The approximation is performed using the least squares method, a standard numerical technique for fitting functions to discrete data [9, 15].

3. Least squares approximation of simulated data by a smooth curve

The problem of approximating a function $x(t)$, defined on the interval $[a, b]$ and square-integrable, by a polynomial of degree not higher than n using the least squares method consists in determining a polynomial approximation that provides the best fit to the function in the mean-square sense [9, 15]. The approximating polynomial is represented as:

$$\hat{P}_n(t) = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2 + \dots + \hat{a}_n t^n$$

The coefficients of the polynomial are chosen in such a way that the integral mean-square deviation between the function $x(t)$ and its approximation $\hat{P}_n(t)$ is minimized. In other words, among all polynomials $P_n(t)$ of degree not higher than n , the polynomial $\hat{P}_n(t)$ satisfies the condition.

$$\int_a^b (x(t) - \hat{P}_n(t))^2 dt \leq \int_a^b (x(t) - P_n(t))^2 dt$$

The sequence $q_k = t^{k-1}, k = 1, 2, 3, \dots, n+1$ of power functions serves as the natural basis in the space of polynomials, and the coefficients of the polynomial $\hat{P}_n(t)$, for which the deviation from the function $x(t)$ is minimized, are determined from a system of equations [9, 15]:

$$\sum_{i=0}^n \hat{a}_i \int_a^b t^{i+k-1} dt = \int_a^b t^{k-1} x(t) dt \quad (k = 1, 2, 3, \dots, n+1)$$

or

$$\sum_{i=0}^n \hat{a}_i \frac{(b^{i+k} - a^{i+k})}{i+k} = \int_a^b t^{k-1} x(t) dt$$

However, this system is inconvenient and can only be solved with moderate accuracy, since the determinant of the system grows without bound as n increases [9, 15].

The system of equations for finding the coefficients of the approximating polynomial becomes particularly simple if an orthogonal system of Legendre polynomials, defined on the interval $[-1; 1]$ is chosen as the basis.

$$L_n(t) = \frac{1}{2^n \cdot n!} \frac{d^n}{dt^n} (t^2 - 1)^n, L_0(t) \equiv 1$$

The orthogonality properties of the Legendre polynomials are expressed by the following formulas [9]:

$$\int_{-1}^1 (L_n(t))^2 dt = \frac{2}{2n+1}, \int_{-1}^1 L_n(t) L_m(t) dt = 0 \text{ where } m \neq n$$

and

$$\int_{-1}^1 t^k L_n(t) dt = 0 \quad (k = 0, 1, 2, \dots, n-1), \int_{-1}^1 t^n L_n(t) dt = \frac{2^{n+1} (n!)^2}{(2n+1)!}$$

For Legendre polynomials, the following recurrence relations hold:

$$(n+1)L_{n+1}(t) = (2n+1)tL_n(t) - nL_{n-1}(t)$$

From these relations, the first five polynomials can be determined:

$$L_0(t) = 1 \tag{3.1}$$

$$L_1(t) = t \tag{3.2}$$

$$L_2(t) = \frac{3t^2 - 1}{2} \tag{3.3}$$

$$L_3(t) = \frac{5t^3 - 3t}{2} \tag{3.4}$$

$$L_4(t) = \frac{35t^4 - 30t^2 + 3}{8} \tag{3.5}$$

If the best approximation $\hat{P}_n(t)$ on the interval $[-1; 1]$ is sought as a linear combination of Legendre polynomials:

$$\hat{P}_n(t) = \sum_{k=0}^n c_k L_k(t) \tag{3.6}$$

then, due to the orthogonality relations, the system of equations reduces to the following equations:

$$c_k = \frac{2k+1}{2} \int_{-1}^1 x(t) L_k(t) dt, \quad k = 0, 1, 2, \dots, n \tag{3.7}$$

Since the Legendre polynomials are defined on the interval $[-1; 1]$, to approximate a function $x(t)$ defined on an arbitrary interval $[a; b]$, it is necessary to perform the substitution:

$$\tau = \frac{2t - a - b}{b - a} \tag{3.8}$$

which makes it possible to transform the argument τ so that it varies over the interval $[-1; 1]$ when the argument t varies over the interval $[a; b]$. After approximating the function $x(\tau)$, it is necessary to return from the argument τ to the argument t using the formula:

$$t = \frac{(b-a)\tau + a + b}{2} \quad (3.9)$$

Consequently, formulas (3.1) - (3.9) define the algorithm for approximating the function by a polynomial using the least squares method.

In many cases, experimental data describing a process are recorded at discrete moments in time, as is the case for the discrete trajectory of an object measured by a radar system. To extend the theory of approximating a continuous function by polynomials using the least squares method to the discrete case, the discrete points should be connected by a continuous piecewise-linear function.

Let x_i ($i = 1, 2, 3, \dots, N$) be the values of a process (for example, the coordinates of an object) recorded at times t_i . Without loss of generality, assume that $t_1 \leq t_2 \leq t_3 \leq \dots \leq t_N$. A piecewise-linear function approximating the discrete data, denoted $x_l(t)$, is defined as the function connecting adjacent nodes with straight lines (Figure 7).

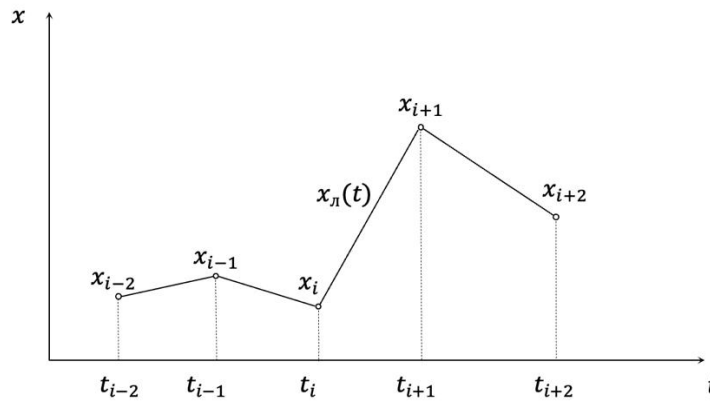


Fig. 7 – Illustration of a piecewise linear function.

The piecewise-linear function passing through the discrete nodes can be written as:

$$x_l(t) = x_i + \frac{x_{i+1} - x_i}{t_{i+1} - t_i} (t - t_i), t \in [t_i; t_{i+1}], i = 1, 2, 3, \dots, N - 1 \quad (3.10)$$

To approximate discrete experimental data by a polynomial using the least squares method, the approximation algorithm should be applied to the piecewise-linear function $x_l(t)$. Therefore, formulas (3.1) - (3.10) define the algorithm for approximating discrete simulated data by a polynomial using the least squares method.

4. Development of a least-squares polynomial approximation algorithm for a discrete trajectory

If a discrete trajectory of an object is given by the nodal points $\{(t_i, x_i, y_i, h_i), i = 1, 2, 3, \dots, N\}$, where t_i is the time instant and x_i, y_i, h_i are the Cartesian coordinates of the object at that moment, then the least squares approximation algorithm for the trajectory with respect to each coordinate is identical and consists of the following steps:

Step 1

The piecewise-linear function $x_l(t)$ is defined using formula (3.10).

Step 2

The time interval $[t_1; t_N]$ corresponding to the existence of the trajectory is transformed into the interval $[-1; 1]$ by replacing the variable t with τ according to a formula analogous to (3.8):

$$\tau = \frac{2t - t_1 - t_N}{t_N - t_1}$$

Step 3

Using formulas (3.7) and (3.1) - (3.5), the values of the coefficients c_k are calculated for $k = 0, 1, 2, 3, 4$ (according to the chosen condition, the discrete trajectory is approximated by a polynomial of degree not higher than four).

The evaluation of the integrals:

$$c_k = \int_{-1}^1 f_k(\tau) d\tau, f_k(\tau) = \frac{2k+1}{2} x_l(\tau) L_k(\tau)$$

on a computer is performed using Simpson's rule [9].

$$\int_{-1}^1 f_k(\tau) d\tau = \frac{\Delta}{3} (f_k(\tau_0) + 4f_k(\tau_1) + 2f_k(\tau_2) + 4f_k(\tau_3) + \dots + 4f_k(\tau_{2M-1}) + f_k(\tau_{2M})) + \frac{\Delta^5}{90} f_k^{(4)}(\xi)$$

Where:

$$\tau_i = -1 + i\Delta, i = 0, 1, 2, \dots, 2M, \Delta = \frac{1}{M}, -1 \leq \xi \leq 1$$

Step 4

Using formulas (3.6) and (3.1) - (3.5), the polynomial $\hat{P}_4(\tau)$ is calculated.

Step 5

The argument τ in the polynomial $\hat{P}_4(\tau)$ is replaced by the argument t using a formula analogous to (3.9), in order to return from the interval $[-1; 1]$ to the time interval $[t_1; t_N]$:

$$t = \frac{(t_N - t_1)\tau + t_1 + t_N}{2}$$

The polynomial $\hat{P}_4(t)$, which approximates the discrete trajectory of the object with respect to the coordinate x using the least squares method, has been constructed; denote it as $x(t) = \hat{P}_4(t)$. The construction of approximating polynomials for the coordinates y and h is carried out using the same algorithm. As a result, an approximation $(x(t), y(t), h(t), t_1 \leq t \leq t_N)$, of the discrete trajectory $\{(t_i, x_i, y_i, h_i), i = 1, 2, 3, \dots, N\}$ by polynomials using the least squares method is obtained.

5. Methods for Estimating the Characteristics of a Random Variable. Algorithm for Computing the Root Mean Square Errors of Object Trajectory Measurements

If a sample of values of a random variable Z ($z_1, z_2, \dots, z_N$), of size N is given, then the sample mean is defined as the following value:

$$\bar{z} = \sum_{i=1}^N z_i \quad (5.1)$$

and the sample variance is defined as the following value [9]:

$$S^2 = \frac{1}{N-1} \sum_{i=1}^N (z_i - \bar{z})^2 \quad (5.2)$$

For a normally distributed random variable Z , a confidence interval for the variance D_z can be constructed using the empirical variance S^2 . As noted in [9], a confidence estimates of the variance D_z of the random variable Z with reliability level $1 - \alpha$ is given by the interval:

$$\left(\frac{(N-1)S^2}{\chi_{1-\frac{\alpha}{2}}^2(m)}, \frac{(N-1)S^2}{\chi_{\frac{\alpha}{2}}^2(m)} \right)$$

i.e.

$$P \left(\frac{(N-1)S^2}{\chi_{1-\frac{\alpha}{2}}^2(m)} \leq D_z \leq \frac{(N-1)S^2}{\chi_{\frac{\alpha}{2}}^2(m)} \right) = 1 - \alpha \quad (5.3)$$

where P is the probability, and the numbers $\chi_{1-\frac{\alpha}{2}}^2(m)$ and $\chi_{\frac{\alpha}{2}}^2(m)$ are the quantiles of the χ^2 distribution with $m = N - 1$ degrees of freedom and are determined from reference tables.

For example, for the reliability level 0.98 ($\alpha = 0.02$) and the number of degrees of freedom $m = 29$ (sample size $N = m + 1 = 30$), the quantiles of the χ^2 distribution are taken from the tables [9], i.e., $\chi_{0.99}^2(29) = 49.6$, $\chi_{0.01}^2(29) = 14.3$. Therefore, the confidence interval for the variance is determined by the following relation:

$$P\left(\frac{(30-1)S^2}{49.6} \leq D_z \leq \frac{(30-1)S^2}{14.3}\right) = P(0.58S^2 \leq D_z \leq 2.03S^2) = 1 - \alpha = 0.98 \quad (5.4)$$

From the realization (5.2), the empirical standard deviation can be determined:

$$S = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (z_i - \bar{z})^2} \quad (5.5)$$

Since the standard deviation σ_z is the square root of the variance, $\sigma_z = \sqrt{D_z}$, then under the conditions of the example considered above, the confidence interval for the standard deviation obtained from relation (5.4) is given by the following expression:

$$P\left(\sqrt{0.58S^2} \leq \sigma_z \leq \sqrt{2.03S^2}\right) = P(0.76S \leq \sigma_z \leq 1.42S) = 0.98 \quad (5.6)$$

When the sample size N increases to 32 or more, the quantiles of the χ^2 distribution cannot be found in every reference book (for example, in the handbooks [9, 10] the quantiles of the χ^2 distribution are provided only for the number of degrees of freedom $m = N - 1 \leq 30$).

To construct a confidence interval for σ_z when the sample size $N \geq 32$ (the number of degrees of freedom of the χ^2 distribution $m = N - 1 \geq 31$), an approximate formula for the quantiles of the χ^2 distribution can be used [10]:

$$\chi_{1-\alpha}^2(m) \approx m \left(1 - \frac{2}{9m} + u_{1-\alpha} \sqrt{\frac{2}{9m}} \right)^3 \quad (5.7)$$

where $u_{1-\alpha}$ is the quantile of the normal distribution:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u_{1-\alpha}} e^{-\frac{1}{2}z^2} dz = 1 - \alpha \quad (5.8)$$

Formula (5.7), which relates the quantiles of the χ^2 distribution and the normal distribution, reflects the well-known property of the χ^2 distribution to approach the normal distribution as the number of degrees of freedom increases.

Using formula (5.7), a confidence interval for σ_z can be constructed for a sample size $N = 100$. For $m = N - 1 = 99$, $\alpha = 0.02$, the values $u_{1-\frac{\alpha}{2}} = u_{0.99} = 2.33$ and $u_{\frac{\alpha}{2}} = u_{0.01} = -2.33$ are obtained from tables of the normal distribution function:

$$\chi_{0.99}^2(99) \approx 99 \left(1 - \frac{2}{9 \cdot 99} + 2.33 \sqrt{\frac{2}{9 \cdot 99}} \right)^3 \approx 134.72$$

$$\chi_{0.01}^2(99) \approx 99 \left(1 - \frac{2}{9 \cdot 99} - 2.33 \sqrt{\frac{2}{9 \cdot 99}} \right)^3 \approx 69.17$$

From formula (5.3) obtain $P\left(\frac{99S^2}{134.72} \leq D_z \leq \frac{99S^2}{69.17}\right) = 0.98$. Taking the square root of all parts of the inequality in parentheses yields the confidence interval for the standard deviation $0.86S \leq \sigma_z \leq 1.2S$ with a reliability level of 0.98.

By performing similar calculations, confidence intervals for estimating the standard deviations can also be obtained from the empirical standard deviations for other sample sizes. Thus, for $N = 300$ and $N = 1000$, the intervals $P(0.91S \leq \sigma_z \leq 1.1S) = 0.98$ and $P(0.95S \leq \sigma_z \leq 1.05S) = 0.98$ are obtained, respectively.

As the sample size increases, the confidence interval decreases; that is, the empirical standard deviation S approaches the true standard deviation σ_z : $S \rightarrow \sigma_z$ as $N \rightarrow \infty$.

Suppose a discrete trajectory of an object is given by the nodal points $\{(t_i, x_i, y_i, h_i), i = 1, 2, 3, \dots, N\}$, and its approximation $(x(t), y(t), h(t), t_1 \leq t \leq t_N)$ is obtained using polynomials via the least squares method. The algorithm for calculating the empirical root-mean-square errors of the object's trajectory measurements along each coordinate x, y, h consists of using formula (5.5) with the difference that the mean value should be taken as the value of the approximating trajectory at the corresponding moment in time (since the task assumes that the true trajectory of the object coincides with the least squares approximation using polynomials of degree no higher than 4):

$$S_x = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - x(t_i))^2} \quad (5.9)$$

$$S_y = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (y_i - y(t_i))^2} \quad (5.10)$$

$$S_h = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (h_i - h(t_i))^2} \quad (5.11)$$

As follows from the estimates using confidence intervals, for a discrete trajectory with 30 samples, the relative errors in the estimation of the root-mean-square errors, calculated using formulas

(5.9)–(5.11) with a confidence level of 0.98, do not exceed 42%; for 100 samples, the relative errors do not exceed 20%; for 300 samples, 10%; and for 1000 samples, 5%.

The algorithms developed in the previous sections - the approximation of a discrete trajectory using a polynomial via the least squares method and the calculation of empirical root-mean-square errors of trajectory measurements - were used in the implementation of a computer program.

In the program, a discrete trajectory of the object's motion is specified, then approximated by a 4th-degree polynomial using the least squares method, after which the empirical root-mean-square deviation of the discrete trajectory from the approximating polynomial is calculated; this represents the measurement error of the object's coordinate - the result of the program. As an illustration, the program also outputs graphs of the discrete trajectory and the approximating polynomial.

As a deterministic test trajectory of the object's motion, consider a trajectory defined by 30 discrete points (t_i, x_i) lying on a 4th-degree curve:

$$x(t) = (t + 1)t(t - 1)(t - 2), x_i = x(t_i) \quad (5.12)$$

$$t_i = -1.25 + \frac{3.5 \cdot (i - 1)}{30 - 1}, i = 1, 2, 3, \dots, 30 \quad (5.13)$$

The discrete trajectory of the object's motion under consideration, defined by formulas (5.12) and (5.13), is shown in Figure 8.

The approximation of this discrete trajectory using a 4th-degree polynomial via the least squares method, performed by the developed program, is shown in Figure 9.

From Figures 8 and 9, it follows that the approximating polynomial passes exactly through the points of the deterministic discrete trajectory and, like the original curve defined by formula (5.12), has four roots: -1, 0, 1, 2. This result indicates that the program for approximating the discrete trajectory of the object's motion with a 4th-degree polynomial using the least squares method works correctly.

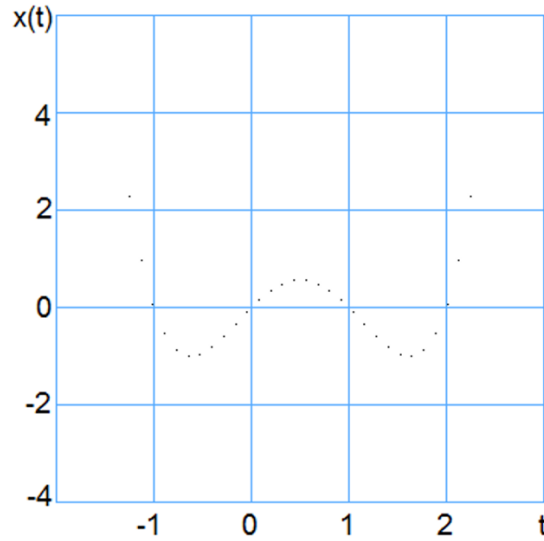


Fig. 8 – Discrete trajectory of an object.

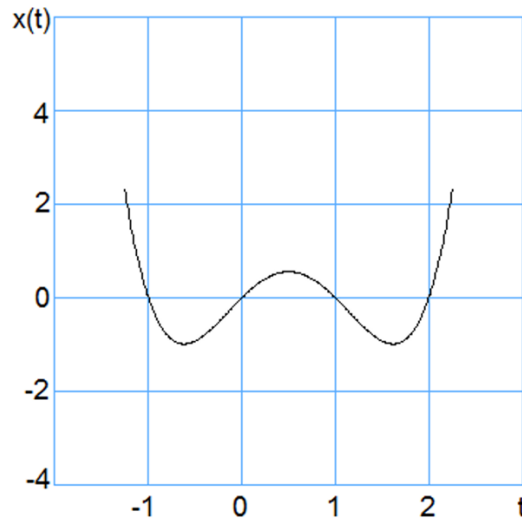


Fig. 9 – Approximation of the discrete trajectory using a 4th-degree polynomial.

As a random test trajectory of the object's motion, consider a trajectory defined by discrete points (t_i, x_i) obtained from the deterministic trajectory defined by formulas (5.12) and (5.13) with the addition of independent normally distributed random variables $\hat{x}_i$ with zero mean $M\hat{x} = 0$ and variance $D_{\hat{x}} = 0.025$ ($\sigma_{\hat{x}} = 0.5$), simulating random measurement errors:

$$x_i = x(t_i) + \hat{x}_i \quad (5.14)$$

where:

$$x(t) = (t + 1)t(t - 1)(t - 2) \quad (5.15)$$

$$\hat{x} \sim N(0; 0.025) \quad (5.16)$$

$$t_i = -1.25 + \frac{3.5 \cdot (i - 1)}{N - 1}, i = 1, 2, 3, \dots, N \quad (5.17)$$

Figure 10 shows the graphical results of the program: the random test discrete trajectory of the object's motion with a given number of points and the continuous approximating trajectory obtained using a 4th-degree polynomial via the least squares method.

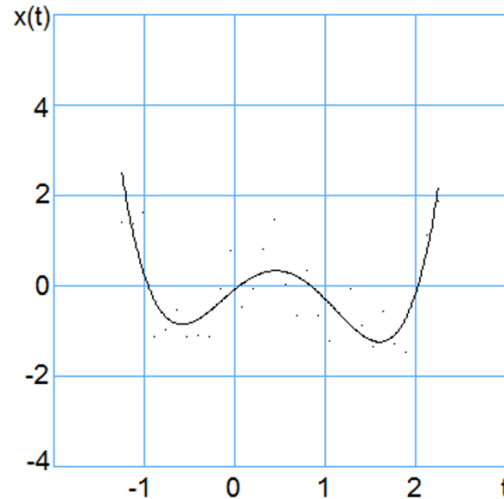


Fig. 10 – Approximation of the random test discrete trajectory of the object's motion (with points) using a 4th-degree polynomial (continuous curve).

Comparison of Figures 10 and 9 shows that the graph of the polynomial approximating the noisy trajectory is slightly shifted relative to the graph of the polynomial approximating the deterministic trajectory (without random measurement errors). This is explained by the small number of samples used to define the discrete trajectory of the object's motion.

Another result of the program is the value of the empirical root-mean-square deviation of the random discrete trajectory from the approximating polynomial, $S_x = 0.61$. Since in this example the random trajectory was simulated with a root-mean-square deviation from the deterministic trajectory of $\sigma_{\hat{x}} = 0.5$, the relative error in estimating the root-mean-square deviation is $\frac{0.61-0.5}{0.5} \cdot 100\% = 22\%$, which falls within the 42% confidence interval for a trajectory defined by $N = 30$ discrete points.

Figure 11 shows the graphical results of the program: a random test discrete trajectory of the object's motion with $N = 1000$ points and the continuous approximating trajectory obtained using a 4th-degree polynomial via the least squares method.

Comparison of Figures 11 and 9 shows that the graph of the polynomial approximating the noisy trajectory practically coincides with the graph of the polynomial approximating the deterministic trajectory (without random measurement errors). This is explained by the large number of samples used to define the discrete trajectory of the object's motion.

Another result of the program is the value of the empirical root-mean-square deviation of the random discrete trajectory from the approximating polynomial, $S_x = 0.502$. Since in this example the random trajectory was simulated with a root-mean-square deviation from the deterministic trajectory of $\sigma_{\hat{x}} = 0.5$, the relative error in estimating the root-mean-square deviation is $\frac{0.502-0.5}{0.5} \cdot 100\% = 4\%$, which falls within the 5% confidence interval for a trajectory defined by $N = 1000$ discrete points.

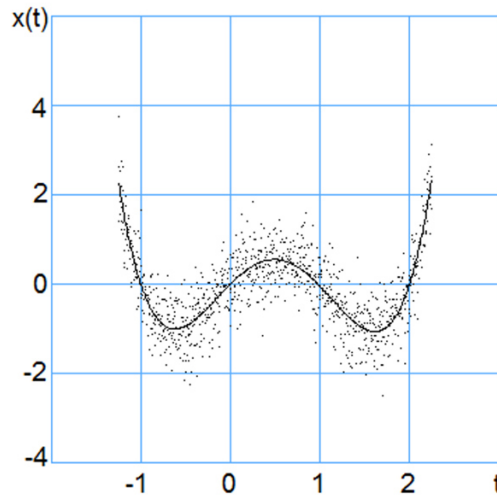


Fig. 11– Approximation of the random test discrete trajectory of the object's motion ($N = 1000$ points) using a 4th-degree polynomial (continuous curve).

The considered examples of the developed program's operation on test trajectories of object motion confirm the correctness of approximating discrete trajectories using 4th-degree polynomials via the least squares method, as well as the validity of the statistical estimates of the root-mean-square measurement errors of object coordinates, including those based on confidence interval theory.

As an illustrative simulation example, consider a discrete aircraft trajectory generated to imitate coordinate measurements obtained from a high-precision radar system. The trajectory data were synthetically generated for validation of the proposed approximation and error-estimation algorithms and do not correspond to real radar observations. One of the coordinate components of the simulated trajectory has the following values (t_i, x_i) $i = 1, 2, 3, \dots, 31$ (time t is given in seconds, coordinate x in meters):

$t_1=51.881,$	$x_1=37710.65;$	$t_{17}=55.786,$	$x_{17}=37532.27;$
$t_2=51.964,$	$x_2=37706.08;$	$t_{18}=56.320,$	$x_{18}=37509.40;$
$t_3=52.408,$	$x_3=37687.78;$	$t_{19}=56.441,$	$x_{19}=37504.83;$

$t_4=52.527,$	$x_4=37683.21;$	$t_{20}=56.971,$	$x_{20}=37486.54;$
$t_5=52.945,$	$x_5=37664.91;$	$t_{21}=57.079,$	$x_{21}=37481.96;$
$t_6=53.049,$	$x_6=37660.34;$	$t_{22}=57.621,$	$x_{22}=37463.67;$
$t_7=53.126,$	$x_7=37655.76;$	$t_{23}=57.727,$	$x_{23}=37459.09;$
$t_8=53.594,$	$x_8=37642.04;$	$t_{24}=58.266,$	$x_{24}=37431.65;$
$t_9=53.705,$	$x_9=37632.90;$	$t_{25}=58.388,$	$x_{25}=37431.65;$
$t_{10}=53.817,$	$x_{10}=37628.32;$	$t_{26}=58.928,$	$x_{26}=37404.21;$
$t_{11}=54.232,$	$x_{11}=37610.03;$	$t_{27}=59.041,$	$x_{27}=37399.63;$
$t_{12}=54.350,$	$x_{12}=37591.73;$	$t_{28}=59.565,$	$x_{28}=37376.76;$
$t_{13}=54.896,$	$x_{13}=37564.29;$	$t_{29}=59.684,$	$x_{29}=37372.19;$
$t_{14}=55.023,$	$x_{14}=37559.72;$	$t_{30}=60.224,$	$x_{30}=37349.32;$
$t_{15}=55.129,$	$x_{15}=37555.14;$	$t_{31}=60.376,$	$x_{31}=37344.75.$
$t_{16}=55.675,$	$x_{16}=37536.85;$		

Figure 12 shows the graphical results of the program: the discrete trajectory of the aircraft's motion with $N = 31$ points and the continuous approximating trajectory obtained using a 4th-degree polynomial via the least squares method. The synthetic trajectory was used solely to demonstrate the practical application of the proposed method and to illustrate the estimation procedure.

Another result of the program is the value of the empirical root-mean-square deviation of the random discrete trajectory from the approximating polynomial, $S_x = 4.1m$. Using formula (5.6) for the confidence interval, obtain:

$$P(0.76S_x \leq \sigma_x \leq 1.42S_x) = P(3.1 \leq \sigma_x \leq 5.8) = 0.98$$

Therefore, with a confidence level of 0.98, the estimate of the root-mean-square measurement error of the aircraft's coordinate x by the radar station falls within the following confidence interval: $3.1m \leq \sigma_x \leq 5.8m$.

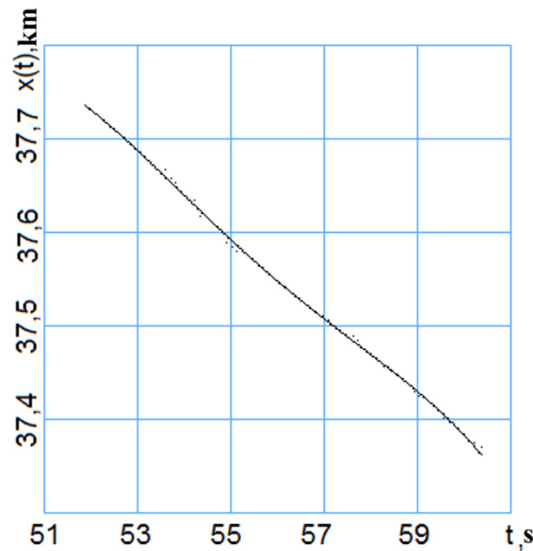


Fig. 12 – Approximation of the discrete trajectory of the aircraft's motion ($N = 31$ points) using a 4th-degree polynomial (continuous curve).

Thus, a computer program has been developed that implements the algorithm for approximating discrete trajectories of object motion using 4th-degree polynomials via the least squares method. The correctness of trajectory approximation has been confirmed on test discrete trajectories (both deterministic and random). The program also calculates the empirical root-mean-square measurement errors of object coordinates. Statistical experiments conducted using the developed program confirmed that the estimates obtained by the program fall within the theoretically constructed confidence intervals with a high degree of reliability.

An approximating polynomial was constructed for a synthetic aircraft trajectory generated to simulate coordinate measurements of a high-precision radar system. The root-mean-square measurement error was estimated and, with a confidence level of 0.98, was found to lie within the confidence interval: $3.1 \leq \sigma_x \leq 5.8$.

6. Conclusions

The study produced the following main results. A method was developed for radar stations measuring the coordinates of moving aerial objects to estimate the root-mean-square errors of these measurements. The algorithm is based on approximating the discrete trajectory of a moving object by a fourth-degree polynomial using the least squares method, followed by calculating the empirical root-mean-square deviation of the discrete trajectory from the approximating polynomial. Formulas were derived that allow the coefficients of the fourth-degree polynomial to be computed numerically on a computer for approximating the discrete trajectory of an object.

To assess the root-mean-square errors of coordinate measurements, confidence intervals were constructed, demonstrating that when a discrete trajectory is defined by 30 points, the relative errors of the estimates with a confidence level of 0.98 do not exceed 42%; for 100 points, the errors do not exceed 20%; for 300 points, 10%; and for 1000 points, 5%. A computer program implementing the algorithm for approximating discrete object trajectories with fourth-degree polynomials was developed. The performance of the proposed method was demonstrated on test trajectories. The program also calculates empirical root-mean-square errors of coordinate measurements, and statistical experiments conducted with the program showed that the obtained estimates lie within theoretically constructed confidence intervals.

An approximating polynomial was constructed for a synthetic aircraft trajectory generated to simulate coordinate measurements of a high-precision radar system. The root-mean-square measurement error was estimated and was found to lie within the corresponding confidence interval at a confidence level of 0.98. The practical significance of the developed method lies in its ability to assess the measurement accuracy of radar stations using only the data from the tested radar, without the need for additional high-precision measurement tools, such as optical instruments. This substantially reduces the time and material resources required for conducting radar station tests.

REFERENCES

- [1] Shirman Y.D., *Principles of Radar and Radio Navigation*, Sovetskoe Radio, Moscow, 1970, 320 p.
- [2] Sosulin Y.G., *Radar Systems*, Radio I Svyaz Publ., Moscow, 1992, 512 p.
- [3] Farina A., Studer F., *Radar Data Processing*, Vol. 1–2, Research Studies Press Ltd., London, 1993, 1050 p.
- [4] Zhdanyuk B.F., *Statistical Processing of Radar Measurements*, Sovetskoe Radio Publ., Moscow, 1978, 240 p.
- [5] Magu G., Lucaciu R., Isar A., *Improving the Targets' Trajectories Estimated by an Automotive Radar Sensor Using Polynomial Fitting*, Applied Sciences, 2021, Vol. 11, No. 1, 361, 15 p.
- [6] Chen L., Zeng X., Zhong S., Gong J., Yang X., *Trajectory Deviation Estimation Method for UAV-Borne Through-Wall Radar*, Remote Sensing, 2024, Vol. 16, No. 9, 1593, 18 p.
- [7] Shi W., Tong P., Bi X., *Moving-Least-Squares-Enhanced 3D Object Detection for 4D Millimeter-Wave Radar*, Remote Sensing, 2025, Vol. 17, No. 8, 1465, 20 p.
- [8] Zhang Y., Li H., Wang J., *A novel regularized least-squares estimator for range/bearings-only maneuvering target tracking with unknown noise statistics*, Digital Signal Processing, 2025, Vol. 150, 104500, 14 p.
- [9] Bronshtein I.N., Semendyaev K.A., *Handbook of Mathematics for Engineers and Students of Technical Universities*, Nauka Publ., Moscow, 1980, 1024 p.
- [10] Korn G.A., Korn T.M., *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill Book Company, New York, 1984, 1152 p.

- [11] Shirman Ya.D. (ed.), *Theoretical Foundations of Radar*, Sovetskoe Radio Publ., Moscow, 1970, 560 p.
- [12] Finkelstein M.I., *Fundamentals of Radar*, Radio I Svyaz Publ., Moscow, 1983, 536 p.
- [13] Sosulin Yu.G., *Theoretical Foundations of Radar and Radionavigation*, Radio I Svyaz Publ., Moscow, 1992, 304 p.
- [14] Kuzmin S.Z., *Digital Radar: Introduction to Theory*, Kvits Publ., Kyiv, 2000, 428 p.
- [15] Faddeev D.K., Faddeeva V.N., *Computational Methods of Linear Algebra*, Fizmatgiz Publ., Moscow, 1963, 352 p.