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COMPUTATION WITHOUT SINGULARITY FOR LOGARITHMS OF HOMOGENEOUS MATRICES AND ORTHOGONAL DUAL TENSORS

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Abstract. This paper presents a systematic approach to the elements of the Lie algebra of rigid body displacements, denoted as $se(3)$, by computing the logarithm of elements from the Lie group $SE(3)$. The methodology is entirely based on tensor calculus and explicitly addresses cases where the rotation component within an $SE(3)$ matrix is symmetric. This development generalizes the classical computation of logarithms for orthogonal matrices in $SO(3)$, which correspond to skew-symmetric matrices in the Lie algebra $so(3)$. A Rodrigues-type formulation is provided for the exponential map into $se(3)$ of $SE(3)$, which is shown to be surjective. Additionally, we propose a procedure for evaluating its multivalued inverse. These methodologies are extended to other parameterizations of rigid body displacements: orthogonal dual tensors. The calculations are presented in closed form and are free from singularities.

Keywords: Lie algebra, Lie group, orthogonal matrices, skew-symmetric matrices, homogenous matrices.

1. Introduction

Identifying the screw axis of a finite motion of a rigid body is a fundamental challenge in the field of rigid body kinematics [1-7]. This article introduces a structured method for calculating the logarithm of homogeneous transformation matrices using algebraic and vector techniques. These methodologies are also extended to other parameterizations of rigid body displacements: orthogonal dual tensors [8-13]. The calculations are presented in closed form and are free from singularities. Section 2 provides the mathematical foundations for the study. It describes how to calculate the logarithm of orthogonal eigenvectors–elements of the Lie group SO (3) without relying on eigenvalue decomposition. Specifically, for symmetric orthogonal tensors, two separate approaches are presented: an algebraic method and a purely vector approach. Section 3 extends these strategies to homogeneous transformation matrices in the Lie group SE (3). The proposed method first addresses the orthogonal (rotational) component and then applies vector operations to complete the calculation of the logarithm. Special attention is given to cases where the rotation matrix is symmetric, allowing for the reuse of algorithms from Section 2. This section also establishes a link between exponential and logarithmic mappings within this Lie group. Section 4 presents the parameterization of rigid body motions using orthogonal dual tensors. The Appendix includes preliminary mathematical concepts related to dual numbers, dual vectors, and dual tensors, which serve as algebraic tools for representing rigid body motion.

2. Preliminaries

2.1. The exponential of a skew-symmetric tensor

We begin by introducing the notations used throughout this section:

- the space of three-dimensional vectors, denoted $\boldsymbol{V}_3$;
- the second-order Euclidean tensors, denoted $\mathbf{L}(\boldsymbol{V}_3, \boldsymbol{V}_3)$;
- the set of proper orthogonal tensors, denoted $SO(3) = \{\boldsymbol{R} \in \mathbf{L}(\boldsymbol{V}_3, \boldsymbol{V}_3); \mid \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}_3, \det \boldsymbol{R} = 1\}$;
- the Lie algebra of skew-symmetric tensor, denoted $so(3) = \{\widetilde{\boldsymbol{\omega}} \in \mathbf{L}(\boldsymbol{V}_3, \boldsymbol{V}_3); \mid \widetilde{\boldsymbol{\omega}}^{\mathrm{T}} = -\widetilde{\boldsymbol{\omega}}\}$.

The pair $(SO(3), \cdot)$ is a Lie group and $so(3)$ serves as its associated Lie algebra.

Lemma 1. *The exponential map:*

$$exp: so(3) \rightarrow SO(3), exp(\widetilde{\boldsymbol{\omega}}) = \boldsymbol{I}_3 + \sum_{k=1}^{\infty} \frac{1}{k!} \widetilde{\boldsymbol{\omega}}^k \tag{1}$$

is both well-defined and surjective.

Proof. To demonstrate that the exponential map indeed maps elements from $so(3)$ to $SO(3)$, let $\widetilde{\boldsymbol{\omega}} \in so(3)$. We define $\mathbf{R} = exp\ (\widetilde{\boldsymbol{\omega}})$. It is known that for any matrix $\mathbf{R}$, the transpose of its exponential satisfies: $\boldsymbol{R}^{\mathrm{T}} = exp\ (-\widetilde{\boldsymbol{\omega}})$. Then $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I}_3$. Using that $\det[exp\ (\widetilde{\boldsymbol{\omega}})] = exp\ [\mathrm{Tr}(\widetilde{\boldsymbol{\omega}})] = 1$, it results $\det \boldsymbol{R} = 1$. The map $exp: so(3) \rightarrow SO(3)$, is well defined.

It is known that the characteristic polynomial of a skew symmetric tensor $\widetilde{\boldsymbol{\omega}}$ is:

$$p_{\widetilde{\omega}}(\lambda) = \lambda^3 + \omega^2 \lambda \tag{2}$$

where $\omega = \| \boldsymbol{\omega} \|$ denotes the Euclidean norm of vector $\boldsymbol{\omega}$ associated to the skew- symmetric tensor $\widetilde{\boldsymbol{\omega}}$. It results that $\widetilde{\boldsymbol{\omega}}^3 = -\omega^2 \widetilde{\boldsymbol{\omega}}$. By making all possible computations in Eq. (1), it results that:

$$\boldsymbol{R} = exp(\widetilde{\boldsymbol{\omega}}) = \boldsymbol{I}_3 + \frac{\sin\omega}{\omega} \widetilde{\boldsymbol{\omega}} + \frac{1 - \cos\omega}{\omega^2} \widetilde{\boldsymbol{\omega}}^2 \tag{3}$$

Also, it results:

$$\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}_3 - \frac{\sin\omega}{\omega} \widetilde{\boldsymbol{\omega}} + \frac{1 - \cos\omega}{\omega^2} \widetilde{\boldsymbol{\omega}}^2 \tag{4}$$

Assume that $\boldsymbol{R} \in SO(3)$. It is a known spectral property that the eigenvalues of such a rotation matrix are $\{1, e^{i\varphi}, e^{-i\varphi}\}$, with $\varphi \in [0, 2\pi)$. Let $\mathbf{u}$ be the unit eigenvector corresponding to the eigenvalue 1 and let $\widetilde{\boldsymbol{u}} \in so(3)$ be the skew-symmetric tensor constructed from $\mathbf{u}$.

We aim to demonstrate that:

$$exp(\varphi \widetilde{\boldsymbol{u}}) = \boldsymbol{R}. \tag{5}$$

Using the Rodrigues formula from Eq. (3) for the exponential map of a skew-symmetric tensor, we have:

$$exp(\varphi \widetilde{\boldsymbol{u}}) = \mathbf{I}_3 + \sin(\varphi)\, \widetilde{\boldsymbol{u}} + (1 - \cos(\varphi)) \widetilde{\boldsymbol{u}}^2 = \boldsymbol{Q} \tag{6}$$

Since $\mathbf{u}$ is also an eigenvector of $\boldsymbol{Q}$ associated to the eigenvalue 1, and both $\boldsymbol{R}$ and $\boldsymbol{Q}$ share same characteristic polynomials, it follows that:

$$\boldsymbol{R} = \boldsymbol{Q} \tag{7}$$

This confirms that the matrix exponential of $\varphi\tilde{\boldsymbol{u}}$ indeed reconstructs the original rotation matrix $\boldsymbol{R}$, thus completing the proof.■

2.2. Calculation of the logarithm of a proper orthogonal tensor

Lemma 2. *Let* $\boldsymbol{R} \in SO(3)$ *be a proper orthogonal tensor. Then*:

(i) *The trace of* $\boldsymbol{R}$ *satisfies*:

$$\mathrm{tr}(\boldsymbol{R}) \in [-1,3] \tag{8}$$

(ii) *If* $\mathrm{tr}(\boldsymbol{R}) = 3$*, then*

$$\boldsymbol{R} = \boldsymbol{I}_3 \tag{9}$$

(iii) *The equality* $\mathrm{tr}(\boldsymbol{R}) = -1$ *holds if and only if* $\boldsymbol{R}$ *is symmetric and distinct from the identity, i.e.,*

$$\mathrm{tr}(\boldsymbol{R}) = -1 \Leftrightarrow \boldsymbol{R} = \boldsymbol{R}^{\mathrm{T}},\ \boldsymbol{R} \neq \boldsymbol{I}_3. \tag{10}$$

Proof. (i) From the Rodrigues formula Eqs. (3) and (6), the trace of a proper orthogonal matrix $\boldsymbol{R}$ can be expressed as:

$$\mathrm{tr}(\boldsymbol{R}) = 1 + 2\cos\varphi \tag{11}$$

Since the cosine function is bounded in the interval $[-1,1]$, it follows directly that:

$$\mathrm{tr}(\boldsymbol{R}) \in [-1,3]$$

(ii) If $\mathrm{tr}\ (\boldsymbol{R}) = 3$, then $\cos\ \varphi = 1$, implying that all eigenvalues of $\boldsymbol{R}$ equal 1. Therefore, $\boldsymbol{R}$ must be the identity tensor:

$$\boldsymbol{R} = \boldsymbol{I}_3 \tag{12}$$

(iii) When $\mathrm{tr}(\boldsymbol{R}) = -1$, it implies $\cos\ \varphi = -1$, and hence $\sin\ \varphi = 0$. Using the Rodrigues expansion from Eq. (6), we obtain:

$$\boldsymbol{R} = \boldsymbol{I}_3 + 2\tilde{\boldsymbol{u}}^2 \tag{13}$$

where $\tilde{\boldsymbol{u}}$ is the skew-symmetric tensor associated with the rotation axis. This form implies that $\boldsymbol{R}$ is symmetric. Since $\mathrm{tr}(\boldsymbol{R}) = -1 \neq 3$, it must differ from the identity. Therefore:

$$\boldsymbol{R} = \boldsymbol{R}^{\mathrm{T}},\ \boldsymbol{R} \neq \boldsymbol{I}_3\,■ \tag{14}$$

The exponential of a skew-symmetric tensor is not an injective map, since $exp\ (\varphi\widetilde{\boldsymbol{u}}) = exp\ [-(2\pi - \varphi)\widetilde{\boldsymbol{u}}]$, as it results from Eq. (6). Its inverse will be a multivalued map:

$$\log: SO(3) \to so(3),\ \log(\boldsymbol{R}) = \{\ \widetilde{\boldsymbol{\omega}} \in so(3) \mid exp(\widetilde{\boldsymbol{\omega}}) = \boldsymbol{R}\ \} \tag{15}$$

We are concerned about offering an expression for this multivalued map. We proved that for any $\boldsymbol{R} \in SO(3)$ there exists $\widetilde{\boldsymbol{\omega}} \in so(3)$ such as:

$$\begin{aligned} \boldsymbol{R} &= \boldsymbol{I}_3 + \frac{\sin\omega}{\omega}\widetilde{\boldsymbol{\omega}} + \frac{1-\cos\omega}{\omega^2}\widetilde{\boldsymbol{\omega}}^2 = \\ &= \boldsymbol{I}_3 + \sin\varphi\widetilde{\boldsymbol{u}} + (1-\cos\varphi)\widetilde{\boldsymbol{u}}^2, \widetilde{\boldsymbol{\omega}} = \varphi\widetilde{\boldsymbol{u}}, \varphi \in [0,2\pi), u = 1 \end{aligned} \tag{16}$$

From Eq. (16) it results:

$$\begin{cases} \cos\varphi = \dfrac{\mathrm{tr}\boldsymbol{R} - 1}{2} \\ \widetilde{\boldsymbol{u}}\sin\varphi = \dfrac{1}{2}(\boldsymbol{R} - \boldsymbol{R}^T) \end{cases} \tag{17}$$

If $\boldsymbol{R} \neq \boldsymbol{R}^T$ the tensor $\boldsymbol{R}$ is not symmetric. From **Lemma 3** it results $\mathrm{tr}\boldsymbol{R} \in (-1,3)$ and $\log\ \boldsymbol{R} \supset \{\widetilde{\boldsymbol{\omega}}_{\mathbf{1}}, \widetilde{\boldsymbol{\omega}}_{\mathbf{2}}\}$, where $\widetilde{\boldsymbol{\omega}}_k = \varphi_k\widetilde{\boldsymbol{u}}_k \in so(3), \varphi_k \in [0,2\pi)$ and $\widetilde{\boldsymbol{u}}_k \in so(3), u = 1, k \in \{1,2\}$. They are:

$$\begin{cases} \varphi_1 = \arccos\dfrac{\mathrm{tr}\boldsymbol{R} - 1}{2}, \widetilde{\boldsymbol{u}}_1 = \dfrac{1}{\sqrt{(1+\mathrm{tr}\boldsymbol{R})(3-\mathrm{tr}\boldsymbol{R})}}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) \\ \varphi_2 = 2\pi - \arccos\dfrac{\mathrm{tr}\boldsymbol{R} - 1}{2}, \widetilde{\boldsymbol{u}}_2 = \dfrac{-1}{\sqrt{(1+\mathrm{tr}\boldsymbol{R})(3-\mathrm{tr}\boldsymbol{R})}}(\boldsymbol{R} - \boldsymbol{R}^T) \end{cases} \tag{18}$$

It should be emphasized that the set $\log\boldsymbol{R}$ is inherently infinite due to the multivalued nature of the inverse exponential mapping. A general representation for the elements belonging to $\log\boldsymbol{R}$ is given by:

$$\begin{cases} \varphi_1 = 2n\pi + \arccos\dfrac{\mathrm{tr}\boldsymbol{R} - 1}{2}, \widetilde{\boldsymbol{u}}_1 = \dfrac{1}{\sqrt{(1+\mathrm{tr}\boldsymbol{R})(3-\mathrm{tr}\boldsymbol{R})}}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) \\ \varphi_2 = 2n\pi - \arccos\dfrac{\mathrm{tr}\boldsymbol{R} - 1}{2}, \widetilde{\boldsymbol{u}}_2 = \dfrac{-1}{\sqrt{(1+\mathrm{tr}\boldsymbol{R})(3-\mathrm{tr}\boldsymbol{R})}}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right) \end{cases}, n \in \mathbb{Z} \tag{19}$$

If $\boldsymbol{R} = \boldsymbol{I}_3$, then $\log\ \boldsymbol{R} = 0$. If the proper orthogonal tensor $\boldsymbol{R}$ is symmetric, two algorithms are proposed for computing its logarithm: one is algebraic, and the other is vectorial.

2.2.1. Algebraic computing method for calculating the logarithm of a proper orthogonal symmetric tensor

The first method offered for computing the logarithm of a skew-symmetric orthogonal proper tensor uses its matrix representation in a positive oriented orthonormal base; let

$$\boldsymbol{R} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{20}$$

be a matrix that the orthogonality condition $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}_3$, has unit determinant $\det \boldsymbol{R} = 1$, and is symmetric, i.e., $\boldsymbol{R} = \boldsymbol{R}^T$. From elementary computations it results:

$$(a)\ \alpha_{ij} = \alpha_{ji}, i, j = \overline{1,3},$$

$$(b) \begin{cases} \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 = 1 \\ \alpha_{12}^2 + \alpha_{22}^2 + \alpha_{23}^2 = 1, \\ \alpha_{13}^2 + \alpha_{23}^2 + \alpha_{33}^2 = 1 \end{cases} \tag{21}$$

$$(c) \begin{cases} \alpha_{12}(\alpha_{11} + \alpha_{22}) + \alpha_{13}\alpha_{23} = 0 \\ \alpha_{13}(\alpha_{11} + \alpha_{33}) + \alpha_{12}\alpha_{23} = 0 \\ \alpha_{23}(\alpha_{22} + \alpha_{33}) + \alpha_{13}\alpha_{12} = 0 \end{cases}$$

The eigenvalues of $\boldsymbol{R}$ are $\{1, -1, -1\}$, so $\varphi = \pi$. We must determine unit vector **u** and $\log \boldsymbol{R} \supset \{\pm\pi\widetilde{\boldsymbol{u}}\}$ is computed. We seek for a skew-symmetric matrix

$$\widetilde{\boldsymbol{u}} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}, u_1^2 + u_2^2 + u_3^2 = 1 \tag{22}$$

such as $\boldsymbol{R} = \boldsymbol{I}_3 + 2\widetilde{\boldsymbol{u}}^2$. Making all computations, it results:

a)
$$\begin{cases} u_2^2 + u_3^2 = \dfrac{(1 - \alpha_{11})}{2} \\ u_1^2 + u_3^2 = \dfrac{(1 - \alpha_{22})}{2}, \\ u_1^2 + u_2^2 = \dfrac{(1 - \alpha_{33})}{2} \end{cases} \tag{23}$$

b)
$$\begin{cases} u_2 u_3 = \alpha_{23} \\ u_1 u_3 = \alpha_{13}, \\ u_1 u_2 = \alpha_{12} \end{cases} \tag{24}$$

c)
$$u_1^2 + u_2^2 + u_3^2 = 1. \tag{25}$$

Considering that $\text{tr}\boldsymbol{R} = \alpha_{11} + \alpha_{12} + \alpha_{13} = -1$ and Eqs. (21), (22) the solution to system (16) is:

$$(u_1, u_2, u_3) = \pm\left(\text{sgn}(\alpha_{23})\sqrt{\frac{1+\alpha_{11}}{2}}, \text{sgn}(\alpha_{13})\sqrt{\frac{1+\alpha_{22}}{2}}, \text{sgn}(\alpha_{12})\sqrt{\frac{1+\alpha_{33}}{2}}\right) \tag{26}$$

Then $\log \boldsymbol{R} \supset \{\pm\pi\tilde{\boldsymbol{u}}\}$, where:

$$\tilde{\boldsymbol{u}} = \begin{pmatrix} 0 & -\text{sgn}(\alpha_{12})\sqrt{\frac{1+\alpha_{33}}{2}} & \text{sgn}(\alpha_{13})\sqrt{\frac{1+\alpha_{22}}{2}} \\ \text{sgn}(\alpha_{12})\sqrt{\frac{1+\alpha_{33}}{2}} & 0 & -\text{sgn}(\alpha_{23})\sqrt{\frac{1+\alpha_{11}}{2}} \\ -\text{sgn}(\alpha_{13})\sqrt{\frac{1+\alpha_{22}}{2}} & \text{sgn}(\alpha_{23})\sqrt{\frac{1+\alpha_{11}}{2}} & 0 \end{pmatrix}. \tag{27}$$

2.2.2. Computing the logarithm of the proper orthogonal symmetric tensor by vectorial methods

Determining the logarithm of a proper orthogonal tensor $\boldsymbol{R} = exp\tilde{\boldsymbol{\omega}}$ involves finding a specific vector $\boldsymbol{\omega}$ such as:

$$\text{(a) } \boldsymbol{R\omega} = \boldsymbol{\omega} \quad (b) \cos\omega = \frac{\text{tr}\boldsymbol{R} - 1}{2} \quad \text{(c) } \frac{\tilde{\boldsymbol{\omega}}}{\omega}\sin\omega = \frac{1}{2}(\mathbf{R} - \mathbf{R}^T) \tag{28}$$

In the special case where $\boldsymbol{R} = \boldsymbol{R}^T$, it follows that $\sin(\omega) = 0$, which implies that directly extracting the skew-symmetric tensor $\tilde{\boldsymbol{\omega}}$ becomes non-trivial. To address this, we return to Eq. 17 and seek a unit vector $\mathbf{u} \in \boldsymbol{V}_3$ that satisfies the eigenvalue equation $\boldsymbol{R}\mathbf{u} = \mathbf{u}$.

It is well established that a symmetric orthogonal matrix $\boldsymbol{R}$ possesses the eigenvalue spectrum $\{1, -1, -1\}$. The eigenvalue 1 is associated with two antiparallel unit eigenvectors, namely $\mathbf{u}$ and $-\mathbf{u}$. The eigenvalue -1, on the other hand, corresponds to a two-dimensional invariant subspace of $\boldsymbol{V}_3$ that is orthogonal to the direction defined by $\mathbf{u}$.

Lemma 3. *Let $\boldsymbol{R} \in SO(3)$ be a proper orthogonal tensor that is also symmetri*c, *with $\boldsymbol{R} \neq \boldsymbol{I}_3$.Then, for any non-zero vector $\mathbf{a} \in \boldsymbol{V}_3$, one of the following mutually exclusive situations must occur:*

(i) $$\boldsymbol{R}\mathbf{a} = \mathbf{a} \tag{29}$$

(ii)
$$\boldsymbol{R}\mathbf{a} = -\mathbf{a} \tag{30}$$

(iii)
$$\frac{\boldsymbol{R}\mathbf{a} + \mathbf{a}}{\| \boldsymbol{R}\mathbf{a} + \mathbf{a} \|} = \pm\mathbf{u} \tag{31}$$

Proof. Consider $\mathbf{a} \in \boldsymbol{V}_3$, $\mathbf{a} \neq 0$. There are three possible cases:

(i) If $\mathbf{a}$ lies in the direction of the eigenvector $\mathbf{u}$, then it must satisfy $\boldsymbol{R}\mathbf{a} = \mathbf{a}$ due to the spectral property of $\boldsymbol{R}$.

(ii) If $\mathbf{a}$ belongs to the invariant subspace associated with the eigenvalue -1, then by symmetry, $\boldsymbol{R}\mathbf{a} = -\mathbf{a}$.

(iii) If $\mathbf{a}$ does not fall into either of the two categories above, then the sum $\boldsymbol{R}\mathbf{a} + \mathbf{a} \neq \mathbf{0}$. Since $\boldsymbol{R}^2 = \mathbf{I}_3$, it follows that:

$$\boldsymbol{R}(\boldsymbol{R}\mathbf{a} + \mathbf{a}) = \boldsymbol{R}\mathbf{a} + \mathbf{a} \tag{32}$$

implying that $\boldsymbol{R}\mathbf{a} + \mathbf{a}$ is itself an eigenvector corresponding to eigenvalue 1.

Hence, it must be collinear with $\mathbf{u}$, leading to the relation:

$$\frac{\boldsymbol{R}\mathbf{a} + \mathbf{a}}{\| \boldsymbol{R}\mathbf{a} + \mathbf{a} \|} = \pm\mathbf{u} \tag{33}$$

This completes the proof.■

Lemma 3 provides a procedure for determining the logarithm of a symmetric proper orthogonal tensor $\boldsymbol{R}$, based on the identification of the eigenvector $\mathbf{u}$ corresponding to the eigenvalue 1. Since the logarithm depends only on the direction of this vector, an algorithmic approach can be employed. One begins by selecting an arbitrary non-zero vector $\mathbf{a}$. If this vector satisfies $\boldsymbol{R}\mathbf{a} = \mathbf{a}$, then $\mathbf{a}$ is already an eigenvector associated with the eigenvalue 1, and the direction of $\mathbf{u}$ is given directly by normalizing $\mathbf{a}$.

$$\mathbf{u} = \frac{\mathbf{a}}{\| \mathbf{a} \|} \tag{34}$$

If instead $\boldsymbol{R}\mathbf{a} + \mathbf{a} \neq \mathbf{0}$, then the vector $\boldsymbol{R}\mathbf{a} + \mathbf{a}$ is collinear with $\mathbf{u}$, and we recover:

$$\mathbf{u} = \pm\frac{\boldsymbol{R}\mathbf{a} + \mathbf{a}}{\| \boldsymbol{R}\mathbf{a} + \mathbf{a} \|} \tag{35}$$

If, however, $\boldsymbol{R}\mathbf{a} = -\mathbf{a}$, then $\mathbf{a}$ lies entirely within the eigenspace corresponding to the eigenvalue -1, and a second vector $\mathbf{b}$, linearly independent of $\mathbf{a}$, must be chosen.

In the exceptional case where $\boldsymbol{R}\mathbf{b} = -\mathbf{b}$ as well, both vectors $\mathbf{a}$ and $\mathbf{b}$ belong to the same twodimensional invariant subspace, which is orthogonal to the direction of $\mathbf{u}$. In this case:

$$\mathbf{u} = \pm \frac{\mathbf{a} \times \mathbf{b}}{\| \mathbf{a} \times \mathbf{b} \|} \tag{36}$$

provides the desired unit eigenvector. Once $\mathbf{u}$ is determined, the logarithm of $\boldsymbol{R}$ is given by:

$$\log(\boldsymbol{R}) \supset \{\pm \pi \tilde{\boldsymbol{u}}\} \tag{37}$$

where $\tilde{\boldsymbol{u}}$ is the skew-symmetric tensor corresponding to the unit vector $\mathbf{u}$.

3. Computation of the logarithm of the homogenous matrices

Consider the transformations $\rho: \boldsymbol{V}_3 \to \boldsymbol{V}_3, \rho(\mathbf{x}) = \boldsymbol{R}\mathbf{x} + \mathbf{t}$, where $\boldsymbol{R}$ is a proper orthogonal tensor representing a rotation and $\mathbf{t}$ a translational vector in $\boldsymbol{V}_3$. This set forms a Lie group under the operation of composition and is known as the special Euclidean group $SE(3)$. Every rigid body displacement —can be described by such a transformation: the tensor $\boldsymbol{R}$ encodes the rotational component of the motion, while the vector $\mathbf{t}$ represents the translation. Such transformations admit a standard matrix representation as a homogeneous transformation matrix of size 4×4 square matrix:

$$\mathbf{A} = \begin{pmatrix} \boldsymbol{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}, \boldsymbol{R} \in SO(3), \mathbf{t} \in \boldsymbol{V}_3 \tag{38}$$

The Lie algebra $se(3)$ associated with the Euclidean Lie group $SE(3)$ consists of all matrices 4×4 of the form:

$$\boldsymbol{\Omega} = \begin{pmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{pmatrix}, \tilde{\boldsymbol{\omega}} \in so(3), \mathbf{v} \in \boldsymbol{V}_3. \tag{39}$$

Lemma 4. *The exponential map exp:* $se(3) \to SE(3), exp(\boldsymbol{\Omega}) = \boldsymbol{I}_4 + \sum_{k=1}^{\infty} \frac{1}{k!} \boldsymbol{\Omega}^k$, *is well defined and surjective.*

Proof. Let $\boldsymbol{\Omega} \in se(3)$ and by simple computations it results that:

$$\boldsymbol{\Omega} = \begin{pmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0 \end{pmatrix} \Rightarrow exp(\boldsymbol{\Omega}) = \boldsymbol{I}_4 + \sum_{k=1}^{\infty} \frac{1}{k!} \boldsymbol{\Omega}^k = \begin{pmatrix} exp(\tilde{\boldsymbol{\omega}}) & dexp_{\tilde{\boldsymbol{\omega}}} \mathbf{v} \\ 0 & 1 \end{pmatrix},$$
$$dexp_{\tilde{\boldsymbol{\omega}}} = \boldsymbol{I}_3 + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} \tilde{\boldsymbol{\omega}}^k \tag{40}$$

It follows directly that the exponential map *exp*: $se(3) \to SE(3)$ is well defined. Since it has already been established that the exponential map $exp: so(3) \to SO(3)$ is surjective, it remains only to show that tensor $dexp_{\tilde{\boldsymbol{\omega}}}$ is inversable. A key observation is that tensor $dexp_{\tilde{\boldsymbol{\omega}}}$ can be expressed as the integral:

$$dexp_{\tilde{\boldsymbol{\omega}}} = \int_0^1 exp(\tilde{\boldsymbol{\omega}}t)\, dt \tag{41}$$

Considering that $exp(\tilde{\boldsymbol{\omega}}t) = \boldsymbol{I}_3 + \frac{\sin \omega t}{\omega}\tilde{\boldsymbol{\omega}} + \frac{1-\cos \omega t}{\omega^2}\tilde{\boldsymbol{\omega}}^2$, it results:

$$dexp_{\tilde{\boldsymbol{\omega}}} = \boldsymbol{I}_3 + \frac{1-\cos\omega}{\omega^2}\tilde{\boldsymbol{\omega}} + \frac{\omega - \sin\omega}{\omega^3}\tilde{\boldsymbol{\omega}}^2 \tag{42}$$

By seeking its possible inverse of the form $dexp_{\tilde{\boldsymbol{\omega}}}{}^{-1} = \boldsymbol{I}_3 + \alpha\tilde{\boldsymbol{\omega}} + \beta\tilde{\boldsymbol{\omega}}^2, \alpha, \beta \in \mathbb{R}$, it results:

$$\begin{gathered}(dexp_{\tilde{\boldsymbol{\omega}}}d)(exp_{\tilde{\boldsymbol{\omega}}}{}^{-1}) = \boldsymbol{I}_3 \Rightarrow \\ \Rightarrow \boldsymbol{I}_3 + \left[\frac{1-\cos\omega}{\omega^2} + \alpha - \alpha\frac{\omega-\sin\omega}{\omega} - \beta(1-\cos\omega)\right]\tilde{\boldsymbol{\omega}} + \\ + \left[\frac{\omega-\sin\omega}{\omega^3} + \alpha\frac{1-\cos\omega}{\omega^2} + \beta - \beta\frac{\omega-\sin\omega}{\omega}\right]\tilde{\boldsymbol{\omega}}^2 = \boldsymbol{0}_3\end{gathered} \tag{43}$$

As the determinant of the linear system with unknowns α and β:

$$\begin{cases}\dfrac{1-\cos\omega}{\omega^2} + \alpha - \alpha\dfrac{\omega-\sin\omega}{\omega} - \beta(1-\cos\omega) = 0 \\ \dfrac{\omega-\sin\omega}{\omega^3} + \alpha\dfrac{1-\cos\omega}{\omega^2} + \beta - \beta\dfrac{\omega-\sin\omega}{\omega} = 0\end{cases} \tag{44}$$

is $\Delta = \frac{2(1-\cos\omega)}{\omega^2} \neq 0$, it results that tensor $dexp_{\tilde{\boldsymbol{\omega}}}$ is inversable and after computations, its inverse is:

$$dexp_{\tilde{\boldsymbol{\omega}}}{}^{-1} = \boldsymbol{I}_3 - \frac{1}{2}\tilde{\boldsymbol{\omega}} + \left(1 - \frac{\omega}{2}\cot\frac{\omega}{2}\right)\frac{\tilde{\boldsymbol{\omega}}^2}{\omega^2} \tag{45}$$

The proof is finalized.■

Conclusion. The logarithm of a homogenous matrix $\mathbf{g} = \begin{pmatrix}\boldsymbol{R} & \mathbf{t} \\ 0 & 1\end{pmatrix} \in SE(3)$ is a set of matrices $\boldsymbol{\Omega} = \begin{pmatrix}\tilde{\boldsymbol{\omega}} & \mathbf{v} \\ 0 & 0\end{pmatrix} \in se(3)$, where:

$$\begin{cases}\tilde{\boldsymbol{\omega}} \in \log \boldsymbol{R} \\ \mathbf{v} = \left[\boldsymbol{I}_3 - \dfrac{1}{2}\tilde{\boldsymbol{\omega}} + \left(1 - \dfrac{\omega}{2}\cot\dfrac{\omega}{2}\right)\dfrac{\tilde{\boldsymbol{\omega}}^2}{\omega^2}\right]\mathbf{t}\end{cases} \tag{46}$$

Moreover, when $\boldsymbol{R}$ is symmetric and belongs to $SO(3)$, its logarithm can be determined using one of the algorithms developed in Section 2, which rely on extracting the eigenvector corresponding to the eigenvalue 1. If tensor $\boldsymbol{R}$ is arbitrary symmetric proper orthogonal, then $\omega = \pi$, so vector $\mathbf{v}$ has the expression:

$$\mathbf{v} = \left[\boldsymbol{I}_3 - \frac{\pi}{2}\widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{u}}^2\right]\mathbf{t} \tag{47}$$

where $\widetilde{\boldsymbol{u}}$ is the skew-symmetric tensor corresponding to the unit eigenvector $\mathbf{u}$ of proper orthogonal tensor $\boldsymbol{R}$.

One of the most fundamental results in spatial kinematics is the Mozzi-Chasles theorem [8]: the most general rigid body displacement can be described as a screw displacement, which consists of a translation along a line followed (or preceded) by a rotation about that line. The vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ completely characterize the screw parameters of rigid body displacement. A directed line with the following vector equation represents the screw axis:

$$\mathbf{r} = \frac{\boldsymbol{\omega} \times \mathbf{v}}{\|\boldsymbol{\omega}\|^2} + \lambda\boldsymbol{\omega}, \lambda \in \mathbb{R}_+ \tag{48}$$

The rotation angle around this axis is:

$$\alpha = \|\boldsymbol{\omega}\| \tag{49}$$

and the translation vector is:

$$\mathbf{d} = \frac{\boldsymbol{\omega} \cdot \mathbf{v}}{\|\boldsymbol{\omega}\|^2}\boldsymbol{\omega} \tag{50}$$

4. Rigid Body Displacement Parameterization by Orthogonal Dual Tensors

Appendix presents mathematical preliminaries of dual algebra needed for the dual parameterization of rigid body displacements.

Let be the orthogonal dual tensor set denoted by:

$$\underline{SO}(3) = \left\{\underline{\boldsymbol{R}} \in \mathrm{L}\left(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3\right) \mid \underline{\boldsymbol{R}}^T = \underline{\boldsymbol{I}}, \det\underline{\boldsymbol{R}} = 1\right\} \tag{51}$$

with $\underline{\boldsymbol{I}}$ being the unit orthogonal dual tensor. The internal structure of any orthogonal dual tensor $\underline{\boldsymbol{R}} \in \underline{SO}(3)$ is presented in a series of results that were detailed in our previous work [9].

Theorem 1. *(Structure theorem): For any* $\underline{\boldsymbol{R}} \in \underline{SO}3)$, *a unique decomposition is viable*

$$\underline{\boldsymbol{R}} = (\boldsymbol{I} + \varepsilon\widetilde{\boldsymbol{t}})\boldsymbol{R} \tag{52}$$

where $\boldsymbol{R} \in SO(3)$ *and* $\boldsymbol{t} \in \boldsymbol{V}_3$ *are called structural invariants.*

Theorem 2. *(Representation theorem): For any orthogonal dual tensor* $\underline{\boldsymbol{R}}$ *defined as in Eq.* (52), *a dual number* $\underline{\alpha} = \alpha + \varepsilon d$ *and a dual unit vector* $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ *can be computed to have* [9]:

$$\underline{\boldsymbol{R}}(\underline{\alpha}, \underline{\mathbf{u}}) = \underline{\boldsymbol{I}} + \sin\underline{\alpha}\underline{\widetilde{\mathbf{u}}} + (1 - \cos\underline{\alpha})\underline{\widetilde{\mathbf{u}}}^2 = exp(\underline{\alpha}\underline{\widetilde{\mathbf{u}}}). \tag{53}$$

The computational formulas for $\alpha, \mathbf{u}, d, \mathbf{u}_0$, *are*:

$$\alpha = \operatorname{atan2}\left(\pm\sqrt{(1 + \operatorname{tr}\boldsymbol{R})(3 - \operatorname{tr}\boldsymbol{R})}; \operatorname{tr}\boldsymbol{R} - 1\right) \tag{54}$$

$$\mathbf{u} = \begin{cases} \pm\operatorname{vect}\left(\dfrac{1}{\sqrt{(1 + \operatorname{tr}\boldsymbol{R})(3 - \operatorname{tr}\boldsymbol{R})}}\left(\boldsymbol{R} - \boldsymbol{R}^{\mathrm{T}}\right)\right), \text{when } \operatorname{tr}\boldsymbol{R} \in (-1,3) \\ \dfrac{\boldsymbol{R}\mathbf{v} + \mathbf{v}}{\|\mathbf{R}\mathbf{v} + \mathbf{v}\|}\ \forall \mathbf{v} \in \boldsymbol{V}_3, \text{when } \operatorname{tr}\boldsymbol{R} = -1 (\boldsymbol{R} \text{ is symmetric}) \\ \dfrac{\mathbf{t}}{\|\mathbf{t}\|}, \text{when } \operatorname{tr}\boldsymbol{R} = 3 (\boldsymbol{R} = \mathbf{I}) \end{cases} \tag{55}$$

$$d = \mathbf{t} \cdot \mathbf{u} \tag{56}$$

$$\mathbf{u}_0 = \begin{cases} \dfrac{1}{2}\mathbf{t} \times \mathbf{u} + \dfrac{1}{2}\cot\dfrac{\alpha}{2}\mathbf{u} \times (\mathbf{t} \times \mathbf{u}), \alpha \neq 0 \\ \dfrac{1}{2}\mathbf{t} \times \mathbf{u}, \alpha = 0 \end{cases} \tag{57}$$

Both parameters $\underline{\alpha}$ and $\underline{\mathbf{u}}$ are called the natural invariants of $\underline{\boldsymbol{R}}$. The unit dual vector $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon\mathbf{u}_0$ gives the Plücker representation of the Mozzi-Chalses axis [10], while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translation distance d. If $\underline{\alpha} \in \mathbb{R}$, there is the case of a rotation parameterization, while for $\underline{\alpha} \in \varepsilon\mathbb{R}$, the parameterization describes a translation.

The Lie algebra of the Lie group $\underline{SO}(3)$ is the skew-symmetric dual tensor set denoted by $\underline{so}(3) = \{\underline{\widetilde{\boldsymbol{\omega}}} \in \mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3) \mid \underline{\widetilde{\boldsymbol{\omega}}} = -\underline{\widetilde{\boldsymbol{\omega}}}^{\mathrm{T}}\}$, where the internal mapping is $\langle \underline{\widetilde{\boldsymbol{\omega}}}_1, \underline{\widetilde{\boldsymbol{\omega}}}_2 \rangle = \widetilde{\underline{\boldsymbol{\omega}}_1\underline{\boldsymbol{\omega}}_2}$. The Lie algebra $\underline{so}(3)$ is isomorphic to the Lie algebra of dual vectors $\underline{\mathbf{V}}_3$, having as internal operation the cross product of dual vectors.

The link between the Lie algebra $\underline{so}(3)$, the Lie group $\underline{SO}(3)$, and the exponential map is given below.

Theorem 3. *The mapping*:

$$exp: so(3) \to \underline{SO}(3), exp(\underline{\widetilde{\boldsymbol{\omega}}}) = \sum_{\mathrm{k}=0}^{\infty} \frac{\underline{\widetilde{\boldsymbol{\omega}}}^{\mathrm{k}}}{\mathrm{k}!}. \tag{58}$$

is well-defined and surjective. It takes place the following closed form formula [14]:

$$exp(\underline{\widetilde{\boldsymbol{\omega}}}) = \underline{\boldsymbol{I}} + \text{sinc}\underline{\omega}\underline{\widetilde{\boldsymbol{\omega}}} + \frac{1}{2}\text{sinc}^2\frac{\omega}{2}\underline{\widetilde{\boldsymbol{\omega}}}^2, \tag{59}$$

where:

$$\underline{\omega} = \sqrt{-\frac{1}{2}\text{tr}\underline{\widetilde{\boldsymbol{\omega}}}^2} = \left|\text{vect}\underline{\widetilde{\boldsymbol{\omega}}}\right|, \tag{60}$$

$$\text{sinc}\underline{\omega} = \begin{cases} \frac{\sin\underline{\omega}}{\underline{\omega}}, \text{ if Re}\underline{\omega} \neq 0 \\ 1 \\ \quad \text{if Re}\underline{\omega} = 0 \end{cases}. \tag{61}$$

Proof of Theorem 3. Eq. (58) is a new shape of Eq. (53). ■

Theorem 4. *(Isomorphism theorem)* [4]: *The special Euclidean group* $(SE(3),\cdot)$ *and* $(\underline{SO}(3),\cdot)$ *are connected via the isomorphism of the Lie groups*

$$\begin{gathered} \Phi: SE(3) \to \underline{SO}(3) \\ \Phi(\boldsymbol{g}) = (\boldsymbol{I} + \varepsilon\tilde{\boldsymbol{t}})\boldsymbol{R} \end{gathered} \tag{62}$$

where $\boldsymbol{g} = \begin{bmatrix} \boldsymbol{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}, \boldsymbol{R} \in SO(3), \boldsymbol{\rho} \in \boldsymbol{V}_3$.

The Lie algebra $(se(3),\cdot)$ and $(\underline{\boldsymbol{V}}_3,\times)$ are connected via the isomorphism

$$\begin{gathered} \varphi: se(3) \to \underline{\boldsymbol{V}}_3 \\ \varphi(\hat{\xi}) = \boldsymbol{\omega} + \varepsilon\mathbf{v} \end{gathered} \tag{63}$$

where $\hat{\xi} = \begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}, \widetilde{\boldsymbol{\omega}} \in so(3), \mathbf{v} \in \boldsymbol{V}_3$.

Proof of Theorem 4. For any $\boldsymbol{g}_1, \boldsymbol{g}_2 \in SE(3)$, the map defined in Eq. (62) yields:

$$\Phi(\boldsymbol{g}_1\boldsymbol{g}_2) = \Phi(\boldsymbol{g}_1)\Phi(\boldsymbol{g}_2) \tag{64}$$

Let $\underline{\boldsymbol{R}} \in \underline{SO}(3)$. Based on **Theorem 1**, which ensures a unique decomposition, we can conclude that the only choice for $\boldsymbol{g}$, such that $\Phi(\boldsymbol{g}) = \underline{\boldsymbol{R}}$, is $\boldsymbol{g} = \begin{bmatrix} \boldsymbol{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$.

This underlines that Φ is a bijection and keeps all the internal operations, where $\boldsymbol{R}$ and $\boldsymbol{\rho}$ are denoted as structural invariant of orthogonal dual tensor $\underline{\boldsymbol{R}}$.

For any $\hat{\xi}_1, \hat{\xi}_2 \in \mathfrak{se}(3)$, the mapping defined by Eq. (63) verifies the identity

$$\varphi([\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2]) = \varphi(\hat{\boldsymbol{\xi}}_1) \times \varphi(\hat{\boldsymbol{\xi}}_2). \tag{65}$$

For any $\underline{\boldsymbol{\omega}} \in \underline{\boldsymbol{V}}_3, \underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon \mathbf{v}$, there is only determined $\hat{\xi} = \begin{bmatrix} \widetilde{\boldsymbol{\omega}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$ such that $\varphi(\hat{\xi}) = \underline{\boldsymbol{\omega}}$. Thus, φ is a bijective mapping.

Remark 1. The inverse of Φ is:

$$\Phi^{-1}: \underline{SO}(3) \leftrightarrow SE(3); \Phi^{-1}(\underline{\boldsymbol{R}}) = \begin{bmatrix} \boldsymbol{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \tag{66}$$

where $\boldsymbol{R} = \mathrm{Re}(\underline{\boldsymbol{R}}), \mathbf{t} = \mathrm{vect}\big(\mathrm{Du}(\underline{\boldsymbol{R}})\boldsymbol{R}^{\mathrm{T}}\big)$.

The dual skew-symmetric tensor $\underline{\widetilde{\boldsymbol{\omega}}} = \underline{\alpha}\, \underline{\widetilde{\boldsymbol{u}}}$, which has a dual angle $\underline{\alpha}$ and a unit dual vector $\underline{\mathbf{u}}$ as described in **Theorem 2**, represents the logarithm of the dual orthogonal tensor $\underline{\boldsymbol{R}} = (\boldsymbol{I} + \varepsilon \widetilde{\boldsymbol{t}})\boldsymbol{R}$. The dual vector $\underline{\boldsymbol{\omega}} = \underline{\alpha}\, \underline{\mathbf{u}}$ corresponding to $\underline{\widetilde{\boldsymbol{\omega}}}$, known as the Euler dual vector, fully characterizes the screw parameters associated with the rigid body displacement.

5. Numerical example

We illustrate the proposed method by computing the logarithm of two homogeneous transformation matrices: the first involving a non-symmetric orthogonal component, and the second containing a symmetric orthogonal component. In both cases, we will present only the principal solution corresponding to the rotation angle $\varphi \in [0,2\pi)$. The complete set of logarithmic solutions can be derived using the general expressions provided in Eq. (19).

1. Let:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{t} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \tag{67}$$

By applying Eq. (18) to compute the logarithm of the rotation tensor $\boldsymbol{R}$, and using Eq. (45) to evaluate the inverse of the matrix $\mathbf{T}$, we obtain the logarithm of the homogeneous matrix $\mathbf{A}$ as $\log(\mathbf{A}) \supset \{\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2\}$, where:

$$\boldsymbol{\Omega}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -0.7854 \\ 0 & 1 & 0 & 0.7854 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{\Omega}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2.3561 \\ 0 & -1 & 0 & 2.3561 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{68}$$

2. Let:

$$\mathbf{g} = \begin{bmatrix} -0.(3) & 0.(6) & 0.(6) & 0 \\ 0.(6) & -0.(3) & -1 & 1 \\ 0.(6) & 0.(6) & -0.(3) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \boldsymbol{R} = \begin{bmatrix} -0.(3) & 0.(6) & 0.(6) \\ 0.(6) & -0.(3) & 0.(6) \\ 0.(6) & 0.(6) & -0.(3) \end{bmatrix}$$

$$\text{and } \mathbf{t} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{69}$$

By computing the logarithm of the rotation tensor $\boldsymbol{R}$ according to the procedure outlined in Subsection 2.2.2, and evaluating the translational component $\boldsymbol{v}$ using Eq. (47), we obtain the logarithm of the homogeneous matrix $\mathbf{g}$ as $\log(\mathbf{g}) \supset \{\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2\}$, where:

$$\boldsymbol{\Omega}_1 = \begin{bmatrix} 0 & -0.5773 & 0.5773 & 1.2402 \\ 0.5773 & 0 & -0.5773 & 0.(3) \\ -0.5773 & 0.5773 & 0 & -0.5735 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{\Omega}_2$$

$$= \begin{bmatrix} 0 & 0.5773 & -0.5773 & -0.5735 \\ -0.5773 & 0 & 0.5773 & 0.(3) \\ 0.5773 & -0.5773 & 0 & 1.2402 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{70}$$

By applying **Theorem 1** and **Theorem 2** along with Eqs. (52) to (57), the same results are obtained.

6. Conclusions

Building on the tensorial framework developed in Section 2, Section 3 presents a general method for computing the logarithm of homogeneous transformation matrices. This approach addresses both generic and symmetric cases of the rotational component. The methodologies are further extended to other parameterizations of rigid body displacements: orthogonal dual tensors. The calculations are provided in closed form and are free from singularities. This proposed methodology has significant applications in rigid body kinematics, especially for determining the associated screw axis and translational vector of a finite rigid motion when it is expressed as a direct affine isometry.

Appendix

7. Mathematical preliminaries of dual algebra

7.1. Dual numbers

Let the set of real dual numbers:

$$\underline{\mathbb{R}} = \mathbb{R} + \varepsilon\mathbb{R} = \{\underline{a} = a + \varepsilon a_0 \mid a, a_0 \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\} \tag{A1}$$

In (A1) on denote $a = \mathrm{Re}(\underline{a})$ the real part of dual number $\underline{a}$, and $a_0 = Du(\underline{a})$ ths dual part. The sum and product of a dual numbers generate a commutative ring with zero divisors structure for $\underline{\mathbb{R}}$. The inverse of a dual number, denoted by $\underline{a}^{-1} \in \mathbb{R}$, exists if and only if $\mathrm{Re}(\underline{a}) \neq 0$ and can be computed using the $\underline{a}^{-1} = \frac{1}{\underline{a}} = \frac{1}{a} - \varepsilon\frac{a_0}{a^2}$ formulas.

7.2. Dual functions

Any differentiable real function $f: \mathbb{I} \subset \mathbb{R} \to \mathbb{R}, f = f(a)$ is extended of dual function by $f: \underline{\mathbb{I}} \subset \underline{\mathbb{R}} \to \underline{\mathbb{R}}; f(\underline{a}) = f(a) + \varepsilon a_0 f'(a)$.

Based on the previous property, we can compute: $\cos(\underline{a}) = \cos(a) - \varepsilon a_0 \sin(a)$; $\sin(\underline{a}) = \sin(a) + \varepsilon a_0 \cos(a)$; $\sqrt[n]{(\underline{a})} = \sqrt[n]{(a)} + \varepsilon\frac{a_0}{n\sqrt[n]{(a)^{n-1}}}$; $\tan(\underline{a}) = \tan(a) + \varepsilon\frac{a_0}{\cos^2(a)}$; $\arctan(\underline{a}) = \arctan(a) + \varepsilon\frac{a_0}{1+(a)^2}$; $\mathrm{arcos}(\underline{a}) = \mathrm{arcos}(a) - \varepsilon\frac{a_0}{\sqrt{1-(a)^2}}$; $\mathrm{atan2}(\underline{b}, \underline{a}) = \mathrm{atan2}(b, a) + \varepsilon\frac{b_0 a - b a_0}{a^2+b^2}$.

7.3. Dual Vectors

The linear vector in the Euclidean there-dimensional space is denoted by $\boldsymbol{V}_3$. The set of dual vectors is defined as:

$$\underline{\boldsymbol{V}}_3 = \boldsymbol{V}_3 + \varepsilon\boldsymbol{V}_3 = \{\underline{\mathbf{a}} = \mathbf{a} + \varepsilon\mathbf{a}_0; \mathbf{a}, \mathbf{a}_0 \in \boldsymbol{V}_3, \varepsilon^2 = 0, \varepsilon \neq 0\} \tag{A2}$$

In (A2) $\mathbf{a} = \mathrm{Re}(\underline{\mathbf{a}})$ is the real part of $\underline{\mathbf{a}}$ and $\mathbf{a}_0 = Du(\underline{\mathbf{a}})$ the dual part of dual vector, respectively. Three vector products are considered: scalar product (denoted by $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$), cross product (denoted by $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$) and triple scalar product (denoted by $\langle\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}\rangle = \underline{\mathbf{a}} \cdot (\underline{\mathbf{b}} \times \underline{\mathbf{c}})$. Regarding algebraic structure, $(\underline{\boldsymbol{V}}_3, +, \cdot\, \underline{\mathbb{R}})$ is a free $\underline{\mathbb{R}}$-module [14].

For dual vector $\underline{\mathbf{a}} \in \underline{\boldsymbol{V}}_3$, the magnitude of $\underline{\mathbf{a}}$, denoted by $|\underline{\mathbf{a}}|$, is the dual number which fulfills $|\underline{\mathbf{a}}| \cdot |\underline{\mathbf{a}}| = \underline{\mathbf{a}} \cdot \underline{\mathbf{a}}$ and can be computed using:

$$|\underline{\mathbf{a}}| = \begin{cases} \|\mathbf{a}\| + \varepsilon \dfrac{\mathbf{a}_0 \cdot \mathbf{a}}{\|\mathbf{a}\|}, \mathrm{Re}(\underline{\mathbf{a}}) \neq \mathbf{0} \\ \varepsilon \|\mathbf{a}_0\|, \mathrm{Re}(\underline{\mathbf{a}}) = \mathbf{0} \end{cases} \tag{A3}$$

where $\|\cdot\|$ is the Euclidean norm in $\boldsymbol{V}_3$. If $|\underline{\mathbf{a}}| = 1$ then $\underline{\mathbf{a}}$ is called unit dual vector.

Any dual vector $\underline{\mathbf{a}} \in \underline{V}_3$, with $\mathrm{Re}(\underline{\mathbf{a}}) \neq \underline{\mathbf{0}}$ can be associated with a directed line in the Euclidean three-dimensional space.

This directed line has the following parametric equation: $\mathbf{r} = \frac{\mathbf{a} \times \mathbf{a}_0}{\|\mathbf{a}\|^2} + \lambda \frac{\mathbf{a}}{\|\mathbf{a}\|}, \forall \lambda \in \mathbb{R}$. If $\mathrm{Re}(\underline{\mathbf{a}}) = \mathbf{0}$, the parametric equation is $\mathbf{r} = \mathbf{v} + \lambda \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|}, \forall \mathbf{v} \in V_3, \forall \lambda \in \mathbb{R}$.

7.4. Dual Tensors

An $\underline{\mathbb{R}}$-linear mapping of $\underline{\boldsymbol{V}}_3$ into $\underline{\boldsymbol{V}}_3$ is called a Euclidean dual tensor:

$$\underline{\boldsymbol{T}}(\underline{\lambda}_1 \underline{\mathbf{v}}_1 + \underline{\lambda}_2 \underline{\mathbf{v}}_2) = \underline{\lambda}_1 \underline{\boldsymbol{T}}(\underline{\mathbf{v}}_1) + \underline{\lambda}_2 \underline{\boldsymbol{T}}(\underline{\mathbf{v}}_2), \forall \underline{\lambda}_1, \underline{\lambda}_2 \in \underline{\mathbb{R}}, \forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in \underline{\boldsymbol{V}}_3. \tag{A4}$$

A Euclidean dual tensor will be called dual tensor and $\mathbf{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3)$ will denote the free $\mathbb{R}$-module of real Euclidean tensors. To simplify writing, we denote $\underline{T}(\underline{\mathbf{v}}) = \underline{T\mathbf{v}}$ for any tensor $\underline{\boldsymbol{T}} \in \mathbf{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3)$ and $\underline{\mathbf{v}} \in \underline{\boldsymbol{V}}_3$. A dual tensor $\underline{T} \in \mathbf{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3)$ can be decomposed in $\underline{\boldsymbol{T}} = \boldsymbol{T} + \varepsilon \boldsymbol{T}_0$, with $\boldsymbol{T}, \boldsymbol{T}_0 \in \mathbf{L}(\boldsymbol{V}_3, \boldsymbol{V}_3)$ are real tensors. The transposed dual tensor is denoted by $\underline{\boldsymbol{T}}^{\mathrm{T}}$ and is defined by: $\underline{\boldsymbol{T}}^{\mathrm{T}} = \boldsymbol{T}^{\mathrm{T}} + \varepsilon \boldsymbol{T}_{\mathbf{0}}^{T}$. $\forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \in \underline{\boldsymbol{V}}_{\mathbf{3}}$, $Re(\langle \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \rangle) \neq \mathbf{0}$ the determinant and trace of tensor $\underline{\boldsymbol{T}} \in \mathbf{L}(\underline{\boldsymbol{V}}_3, \underline{\boldsymbol{V}}_3)$ is, respectively:

$$\langle \underline{\boldsymbol{T}\mathbf{v}_1}, \underline{\boldsymbol{T}\mathbf{v}_2}, \underline{\boldsymbol{T}\mathbf{v}_3} \rangle = \det \underline{\boldsymbol{T}} \langle \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \rangle \tag{A5}$$

$$\mathrm{tr} \underline{\boldsymbol{T}} = \frac{\langle \underline{\boldsymbol{T}\mathbf{v}_1}, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \rangle + \langle \underline{\mathbf{v}}_1, \underline{\boldsymbol{T}\mathbf{v}_2}, \underline{\mathbf{v}}_3 \rangle + \langle \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\boldsymbol{T}\mathbf{v}_3} \rangle}{\langle \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3 \rangle} \tag{A6}$$

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