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SOME PHYSICAL IMPLICATIONS OF THE VECTORS OF ZERO NORM

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Abstract. Using the geometry of the zero-norm vectors, it is shown that at the cosmological scale resolution, the removing of the singularity of the Schwarzchild metric implies, based on the Kruskal coordinates, Rindler-type dynamics, while at the microscopic scale resolution it implies the Yamamoto form of the spin coordinates and implicitly an egalitarian relationship of uncertainly relationships.

Keywords: Schwarzchild metric, Kruskal coordinates, Rindler motions.

1. Introduction

Sobczyk conducted a comprehensive study on the geometry of zero-norm vectors in [1]. This paper will demonstrate how to characterize unique behaviors of complex systems across different scale resolutions [2]: at the cosmological scale, we will analyze Rindler-type dynamics by removing the singularity of the Schwarzschild metric using Kruskal coordinates, while at the microscopic scale, we will employ Yamamoto's formulation of spin coordinates and an egalitarian interpretation of the uncertainty relations.

2. Removing the singularity of the Schwarzchild metric based on special zero-norm vectors

Let’s consider the Schwarzchild metric in the form [3]:

$$ds^2 = \left(1 - \frac{n}{r}\right) c^2 dt^2 - \left(1 - \frac{n}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\,\theta dl^2) \tag{1}$$

the meanings of the quantities involved in (1) being the usual ones from [2,3].

In this context, Kruskal [4] proposed a coordinate system that eliminates the Schwarzschild singularity. Thus, the new coordinates *u* and *v* are defined to satisfy the following equations:

$$u^2 - v^2 = (cT)^2 \left(\frac{r}{n} - 1\right) exp\frac{r}{n} \tag{2}$$

and

$$\frac{2uv}{u^2 + v^2} = tan\,h\left(\frac{ct}{n}\right) \tag{3}$$

where *T* is some constant. The metric (1) can be written in the form:

$$ds^2 = \frac{4n^3}{r(cT)^2} exp\left(-\frac{r}{n}\right)(dv^2 - du^2) - r^2 d\Omega^2 \tag{4}$$

According to Equation (2), the parameter, *r,* depend on $(u^2 - v^2)$. Thus, that r^2 is positive definite and continuous, without any discontinuities and the metric remains non-singular:

$$u^2 r^2 > v^2 - (cT)^2 \tag{5}$$

As a result, throughout the time interval

$$0 < v < cT \tag{6}$$

and for real values of u, the metric remains a well-defined, smooth and finite function dependent on *u.* Indeed, even $g_{\theta\theta}\ and\ g_{\varphi\varphi}$ remain non-zero at $u = 0$, ensuring that as we approach the coordinate origin $u \longrightarrow 0$, there is no restriction preventing continuation into negative values of *r*.

As a result, the space defined by (4) is devoid of singularities and comprises two identical bands, one for $u > 0$ and other for $u < 0$, which transition smoothly into each other at the ramification point $u = 0$. The bands separate at $v = cT$, and the metric exhibits a true singularity at

$$u = \pm\sqrt{v^2 - c^2T^2} \tag{7}$$

corresponding to $r = 0$. Nonetheless, even in this particular case, the metric remains free of singularities at $u = v$, a condition that corresponds specifically to the Schwarzschild radius *n*.

It is especially fascinating to note that the Kruskal transformation mechanism clearly illustrates how singularities are eliminated by modifying specific parameters [4].

$$X = (cT)^2 \left(\frac{r}{n} - 1\right) exp\left(\frac{r}{n}\right),$$
$$Y = (cT)^2 \sin h \left(\frac{ct}{n}\right)\left(\frac{r}{n} - 1\right) exp\left(\frac{r}{n}\right) \tag{8}$$
$$Z = (cT)^2 \cos h \left(\frac{ct}{n}\right)\left(\frac{r}{n} - 1\right) exp\left(\frac{r}{n}\right)$$

satisfying the identity

$$X^2 + Y^2 - Z^2 = 0 \tag{9}$$

Now, the components of the null vector can be considered to be precisely the Kruskal coordinates.

They play a pivotal role in the analytical representation of spin [5] and, as will be shown, are essential for evaluating the significance of the used coordinates.

Their overall handling would correspond to specific requirements concerning the clarification of coordinates and their role in Kruskal transformations, as well as the existence of particular internal symmetries within the field equations, taking their quantum implications into account. Furthermore, the identical coordinates can be associated with Rindler's movements [2,3,6].

3. Egalitarian uncertainly relations based on the zero-norm vectors

Let's consider the vector $\vec{v}$ of components (x_1, x_2, x_3) whose norm satisfies the relation (zero norm vectors):

$$x_1^2 + x_2^2 + x_3^2 = 0 \tag{10}$$

which gives the norm of the null vectors. It is reasonable to assume that the most general invariance group is defined by three rotation parameters, with its infinitesimal generators expressed as:

$$M_1 = -i\left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right)$$
$$M_2 = -i\left(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}\right) \tag{11}$$
$$M_3 = -i\left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}\right)$$

which indeed is the case. Yamamoto [5] emphasized this observation, showing that the infinitesimal generators in form (10) are equivalent to those derived for:

$$x_1 = \rho \sin \omega, x_2 = -\rho \cos \omega, x_3 = -i\rho \tag{12}$$

which yields:

$$M_1 = \cos \omega \ \rho \frac{\partial}{\partial \rho} - \sin \omega \frac{\partial}{\partial \omega}$$
$$M_2 = \sin \omega \ \rho \frac{\partial}{\partial \rho} + \cos \omega \frac{\partial}{\partial \omega} \tag{13}$$
$$M_3 = -i \frac{\partial}{\partial \omega}$$

The operators (13), when applied to the spin "eigenfunctions":

$$v_+ = \sqrt{\rho} e^{\frac{i\omega}{2}}, \qquad v_- = \sqrt{\rho} e^{-\frac{i\omega}{2}} \tag{14}$$

yield results consistent with the action of the Pauli matrices, in accordance with the following relations:

$$\begin{bmatrix} M_1 & v_+ \\ M_1 & v_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_+ \\ v_- \end{bmatrix}$$
$$\begin{bmatrix} M_2 & v_+ \\ M_2 & v_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \cdot \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \tag{15}$$
$$\begin{bmatrix} M_3 & v_+ \\ M_3 & v_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_+ \\ v_- \end{bmatrix}$$

It can be demonstrated directly that the operators (13) conform to the same algebraic structure as Pauli's matrices.

The infinitesimal transformations (13) and (11) provide little insight into this group. Consequently, we shall seek finite transformations that are created by the infinitesimal ones. In this context, we will express operators (13) in a manner that highlights their isomorphism to the Barbilian group [2].

We can define the new operators as linear combinations of:

$$X_1 = M_1 - iM_2$$
$$X_2 = M_3 \tag{16}$$
$$X_3 = -(M_1 + iM_2)$$

and they satisfy the same structure relations as the infinitesimal generators of the Barbilian group [2].

By considering α and $\bar{\alpha}$ as group variables, with $\alpha = v_+$, from (14), the operators (16) can also be expressed as:

$$Y_1 = \bar{\alpha} \frac{\partial}{\partial \bar{\alpha}}, \ \ Y_2 = \frac{1}{2}\left(\alpha \frac{\partial}{\partial \alpha} - \bar{\alpha} \frac{\partial}{\partial \bar{\alpha}}\right), \qquad Y_3 = -\alpha \frac{\partial}{\partial \alpha} \tag{17}$$

To determine the finite transformations induced by these infinitesimal transformations, we will derive the invariant functions of the operator [7]:

$$U = \mu Y_1 + \nu Y_2 + \lambda Y_3, \qquad \mu, \nu, \lambda - constants$$

as solutions of the equation $U\psi = 0$:

$$\left(\mu\bar{\alpha} + \frac{\nu}{2}\alpha\right)\frac{\partial\psi}{\partial\alpha} - \left(\frac{\nu}{2}\bar{\alpha} + \lambda\alpha\right)\frac{\partial\psi}{\partial\bar{\alpha}} = 0 \tag{18}$$

with the characteristic system:

$$\frac{d\alpha}{\mu\bar{\alpha} + \frac{\nu}{2}\alpha} = -\frac{d\bar{\alpha}}{\lambda\alpha + \frac{\nu}{2}\bar{\alpha}} \tag{19}$$

We now seek two additional constants, *m* and *n*, such that the linear combination derived from (19):

$$\frac{md\alpha - nd\bar{\alpha}}{m\left(\frac{\nu}{2}\alpha + \mu\bar{\alpha}\right) - n\left(\lambda\alpha + \frac{\nu}{2}\bar{\alpha}\right)} = d\tau \tag{20}$$

is a total differential. This condition holds only under the following constraints:

$$\left(\frac{\nu}{2} - \rho\right)m - \lambda n = 0$$

$$\mu m - \left(\frac{\nu}{2} - \rho\right)n = 0 \tag{21}$$

where ρ is an arbitrary factor. This system is valid only for the specific values of ρ given by:

$$\rho_1 = \frac{\nu}{2} + \sqrt{\lambda\mu}, \;\; \rho_2 = \frac{\nu}{2} - \sqrt{\lambda\mu}$$

For the first value, (20) yields the integral:

$$\sqrt{\lambda}\alpha + \sqrt{\mu}\bar{\alpha} = \left(\sqrt{\lambda}\alpha_0 + \sqrt{\mu}\bar{\alpha}_0\right)e^{\left(\frac{\nu}{2}+\sqrt{\lambda\mu}\right)\tau} \tag{22}$$

and for the second, the integral:

$$\sqrt{\lambda}\alpha - \sqrt{\mu}\bar{\alpha} = \left(\sqrt{\lambda}\alpha_0 - \sqrt{\mu}\bar{\alpha}_0\right)e^{\left(\frac{\nu}{2}-\sqrt{\lambda\mu}\right)\tau} \tag{23}$$

where $\alpha_0 \; and \; \bar{\alpha}_0$ are the values of $\alpha \; and \; \bar{\alpha} \; for \; \tau = 0$. Relations (21) and (22) give:

$$\alpha = e^{\frac{\nu}{2}}\left(\alpha_0 \cosh\sqrt{\lambda\mu}\tau + \sqrt{\frac{\mu}{\lambda}}\bar{\alpha}_0 \sinh\sqrt{\lambda\mu}\tau\right)$$

$$\bar{\alpha} = e^{\frac{\nu}{2}}\left(\alpha_0\sqrt{\frac{\lambda}{\mu}}\sinh\sqrt{\lambda\mu}\tau + \bar{\alpha}_0 \cosh\sqrt{\lambda\mu}\tau\right) \tag{24}$$

For (24) to be meaningful, it is required that λ be the complex conjugate of μ: $\bar{\lambda} = \mu = \gamma e^{i\phi}$ while ν must be real-and we set $\nu = 0$. Therefore, the finite transformations of the group (16), generated by the infinitesimal transformations, are explicitly described by the *unimodular* group:

$$\alpha = \alpha_0 \cosh \gamma\tau + \bar{\alpha}_0 \sinh \gamma\, \tau$$
$$\bar{\alpha} = \alpha_0 e^{-i\phi} \sinh \gamma\tau + \bar{\alpha}_0 \cosh \gamma\tau \tag{25}$$

The choice of α as the group variable is deliberate. This notation is typically used to denote the complex amplitude of harmonic oscillators, specifically representing the eigenvalue associated with the annihilation operator in the formalism of second quantization [8]. The transformation (25) was initially introduced by Stoler [9] as part of the generalization of the eigenstates of the annihilation operator, specifically in the formulation of coherent states. If α_0 represents, for example, an oscillator in a state that satisfies the minimum uncertainty principle for the position and momentum, the state α is characterized by the following uncertainty relation:

$$(\Delta p)^2(\Delta q)^2 = \frac{1}{4}(1 + \sinh^2 \gamma\tau \sin^2\phi) \tag{26}$$

where Δp and Δq denote the variations in momentum and coordinate of the oscillator, respectively, with the Planck constant set to unity.

As a result, the variable α can be interpreted as the complex amplitude of a bosonic field.

The isomorphism between the group (16) and the Barbilian group [2] indicates that the latter is equally applicable to variables associated with such a field, and, as we will demonstrate, this indeed holds true.

The group generated by (25) acting on the variables x and y can be obtained from (12) by using the following identifications:

$$x_1 = 2xy$$
$$x_2 = y^2 - x^2 \tag{27}$$
$$x_3 = -i(x^2 + y^2)$$

which results in:

$$x = \sqrt{\rho}\cos\frac{\omega}{2}$$
$$y = \sqrt{\rho}\sin\frac{\omega}{2} \tag{28}$$

As one can see, $\alpha_0 = x + iy$ and introducing $\alpha' = x' + iy'$ into (25), one obtains:

$$x' = (\cosh\gamma\tau + \sinh\gamma\tau\cos\phi)x + \sinh\gamma\tau\sin\phi\, y$$
$$y' = \sinh\gamma\tau\sin\phi\, x + (\cosh\gamma\tau - \sinh\gamma\tau\cos\phi)y \tag{29}$$

These transformations reduce to the transformations:

$$x' = x\cosh u + y\sinh u$$
$$y' = x\sinh u + y\cosh u \tag{30}$$

for $\phi = \frac{\pi}{2}$ and $\gamma\tau = u$, but the transformations induced on components Cartan-Whittaker (27) do not reduce to those given in [2].

We are especially focused on the physical significance of the spinor variables $(\alpha), \bar{\alpha}$: each of which represents a bosonic field. For further information on second quantization, Stoler's transformation, and generalized coherent states, consult [9]. Currently, we understand that addressing the angular variables in conjunction with the measured coordinates directly imparts spatial significance to the latter, via the stationary hyperbolic metric of N. Ionescu Pallas. By disregarding angular variables and focusing solely on a "internal" group, the measured quantities represent certain field amplitudes [10].

4. Conclusions

The primary conclusions of this study are as follows:

i) the use of zero-norm vectors based on Kruskal coordinates implies, at cosmological scale resolutions, the removing of the singularity in the Schwarzschild metric. In such a framework, various dynamics compatible with those of the Rindler type are highlighted;

ii) The use of zero-norm vectors utilizing Kruskal coordinates suggests that, at cosmic scale resolutions, the singularity in the Schwarzschild metric is eliminated. This approach emphasizes diverse dynamics akin to Rindler type; the use of zero-norm vectors at microscopic resolutions necessitates both the Yamamoto representation of spin coordinates and an equitable formulation of uncertainty relations. Furthermore, the presence of a background elucidated as a bosonic field is emphasized. We remind you that several models from General Relativity utilize such a field [3,10].

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