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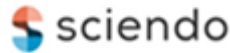
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APPROCHING OF THE LINE CONCENTRATED CONTACT AS A QUARTER SPACE PROBLEM - A REVIEW

BY

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Abstract. Concentrated contact frequently occurs between machine elements, leading to the development of various forms of damage: abrasive wear, adhesive wear, and seizure, as well as surface destruction through contact fatigue. Contact fatigue is a typical mode of damage to machine parts (bearings, gears, wheel-track contact, but also cams and valves) under strong and repeated Hertz contact pressure (Kohn and Silva, 2020).

Most concentrated contact problems, where bodies touch at a point or a line, are well described and solved using the assumptions and equations of Hertz's theory, which associate bodies with elastic half-spaces. In the case of cylindrical bodies of finite length, Hertz's equations do not correctly describe the state of stresses arising at the boundary. At non-coincidentally concentrated linear contact, these uncompensated stresses lead to the so-called 'end effect': the pressure increases near the edges of the cylinder of shorter length. These uncompensated stresses have to be taken into account in the geometrical and functional design of machine elements.

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In this paper, the main methods of solving the stresses arising at concentrated linear contact with finite length are presented, in approaching the problem as a quarter-space problem.

Keywords: concentrated contact, quarter space, non-Hertz contact.

1. Introduction

The concentrated elastic contact defined by Hertz involves several simplifying assumptions about the shape of the contact domain, idealization of bodies and surfaces. Thus, it is assumed that the contact domain is an ellipse, bodies are assimilated to elastic half-spaces, and surfaces are smooth. A particular case of concentrated elastic contact is linear Hertz contact (Fig. 1), where the contact area is rectangular, the pressure distribution is elliptical, and the maximum pressure is constant (Fig. 2). In the case of Hertz contact, the pressures are distributed according to an elliptical cylinder, i.e., the pressure evolves according to an ellipse in the frontal plane and remains constant in the axial plane. The Hertzian stress σ_H is selected as the critical stress in rolling contact fatigue.

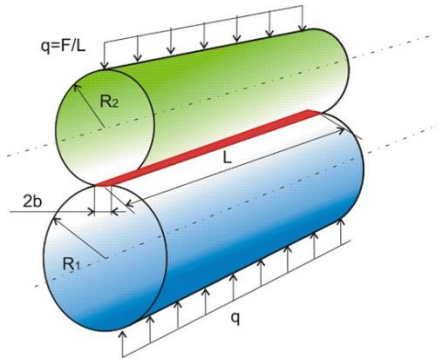


Fig. 1 – Hertz concentrated contact.

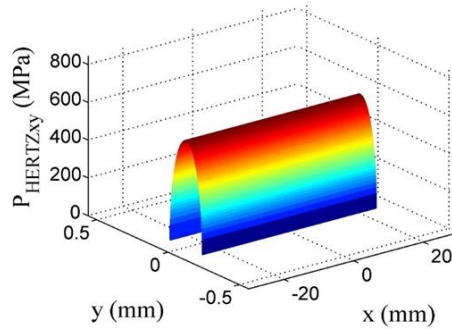


Fig. 2 – Hertz pressure distribution.

This model has a number of limitations (Crețu, 2009):

- no explanation for two possible origins of failure in RCF: surface and subsurface;
- no possible evaluation of the edge effect.

At the same time, the Hertz model results in the infinite length of bodies along the line of contact, which is obviously contrary to the contact between moving parts such as gears, cams, and a large part of roller bearings. In these applications, obviously, the bodies are of finite length and the resulting contact line is also finite. Due to the finite length of one of the cylinders, the shape of the contact area is no longer rectangular, but a specific "bone" shape (Fig. 3) appears

due to the end effect (Crețu, 2019). In the case of a finite length of one of the cylinders, the pressure at the ends increases as the short cylinder end approaches (Benchea *et al.*, 2014) (Fig. 4).

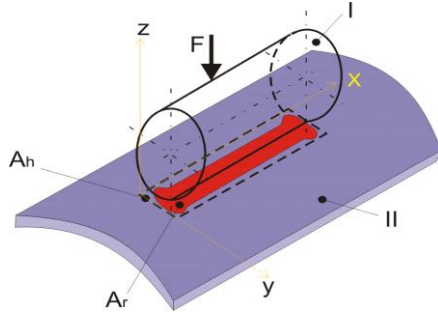


Fig. 3 – Non-Hertz concentrated contact.

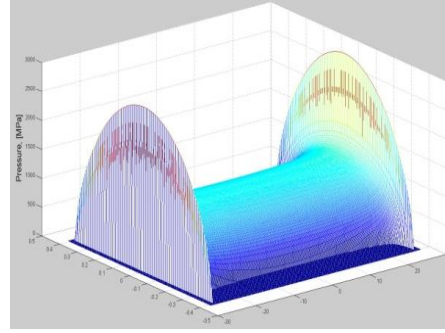


Fig. 4 – Non-Hertz pressure distribution.

The surface and subsurface stress distributions are the controlling factors of the rolling contact fatigue (Najjari and Guibault, 2014), and this is the reason why an accurate description of the stress fields near discontinuities are needed.

1.1. The equations of the elastic model

The five equations that define the elastic model of surface deformation form a system that for the case of non-Hertz contact has no analytical solution, so it is necessary to present the non-Hertz contact problem in discrete form.

The actual pressure distribution is approximated using a virtual pressure distribution.

The elastic model of surface deformation is defined by 5 linear equations:

1) geometrical equation of elastic contact

$$g_{ij} = h_{ij} + w_{ij} - \delta_0 \quad (1)$$

2) the integral equation of the normal displacement of the boundary of the elastic hemispace

$$w_{ij} = \sum_{k=0}^{N_x-1} \sum_{j=0}^{N_y-1} (K_{i-k,j-l} p_{kl}) \quad (2)$$

3) the balance equation

$$\Delta x \Delta y \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} p_{ij} = F \quad (3)$$

4) the reduced non-penetration equation

$$g_{ij} = 0, p_{ij} > 0, (i, j) \in A_r \quad (4)$$

5) the reduced non-adhesion equation

$$g_{ij} > 0, p_{ij} = 0, (i, j) \notin A_r \quad (5)$$

The equation of the stress state at a point $M(x, y, z)$ determined by application of the superposition of effects principle:

$$\sigma_{ij}(x, y, z) = \sum_{k=0}^{N_x-1} \sum_{l=0}^{N_y-1} C_{ijkl} p_{kl} \quad (6)$$

where $C_{ijkl}(x, y, z)$ corresponds to the component $\sigma_{ij}(x, y, z)$ determined by a unit pressure acting on the elementary rectangle (k, l) of the virtual domain discretization mesh.

2. Approaching of the linear concentrated contact stresses as a quarter space problem

Most concentrated contact problems, where bodies touch at a point, are well described and solved using the assumptions and equations of Hertz's theory. Even for some particular non-Hertz contact problems, this theory can be applied with satisfactory results. However, there are also cases where the Hertz equations do not correctly describe and give erroneous results for stresses arising in the case of bodies with rough surfaces, or, in the case of cylindrical bodies of finite length, where uncompensated stresses arise on the boundary. The magnitude of these stresses is not studied sufficiently to develop a design methodology capable of eliminating possible negative effects on durability.

A method of approaching this problem was proposed by Hetényi in 1960 (Hetényi, 1960; Hetényi, 1970), by solving loading problems with normal and tangential forces on the boundary of a quarter-space (Fig. 5).

Keer (Keer *et al.*, 1984) reduces the problem to solving two independent integral equations.

The methods proposed by Hetényi and Keer eliminate both normal and tangential stress, but at the cost of considerable computational effort. Based on observations made by Reusner (de Mul *et al.*, 1986) that the influence of normal stress is so small that it can be neglected, de Mul proposes for the cancellation of tangential stresses on finite free surfaces, a mirroring of the pressure distribution with respect to the front-end area.

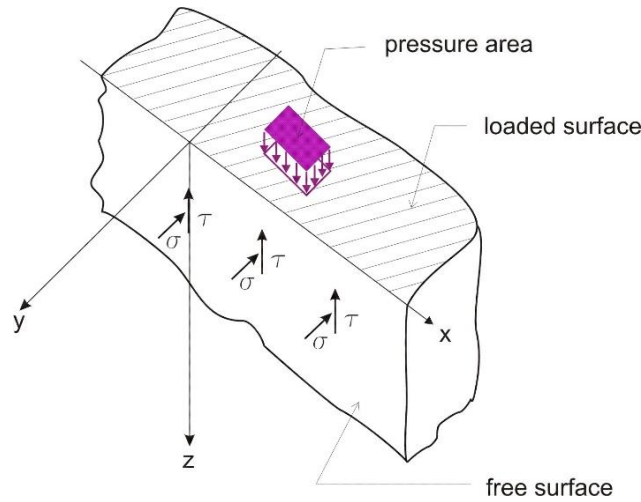


Fig. 5 – Internal stress on the free border of the quarter space.

In 2011, Guibault proposes the introduction of a correction factor ψ , by which he multiplies de Mul's virtual pressure distributions to achieve a super correction that includes the normal stress.

2.1. The Hetényi approach

In 1960, Hetényi, starting from Flamant's solutions for a half-plane, by removing the normal charge distribution $[\sigma_y]_{y=0}$ from the Ox boundary aims to obtain solutions for the quarter-plane (xOy). Consider an elastic half-plane loaded on the boundary with two concentrated normal forces P , acting at distance a from the vertical, according to Fig. 6.

For any point (x, z) the stresses are given by the equations:

$$\sigma_x = -\frac{2P}{\pi} \left\{ \frac{z(x+a)^2}{[z^2 + (x+a)^2]^2} + \frac{z(x-a)^2}{[z^2 + (x-a)^2]^2} \right\} \quad (7a)$$

$$\sigma_z = -\frac{2P}{\pi} \left\{ \frac{z^3}{[z^2 + (x+a)^2]^2} + \frac{z^3}{[z^2 + (x-a)^2]^2} \right\} \quad (7b)$$

$$\tau_{xz} = -\frac{2P}{\pi} \left\{ \frac{z^2(x+a)}{[z^2 + (x+a)^2]^2} + \frac{z^2(x-a)}{[z^2 + (x-a)^2]^2} \right\} \quad (7c)$$

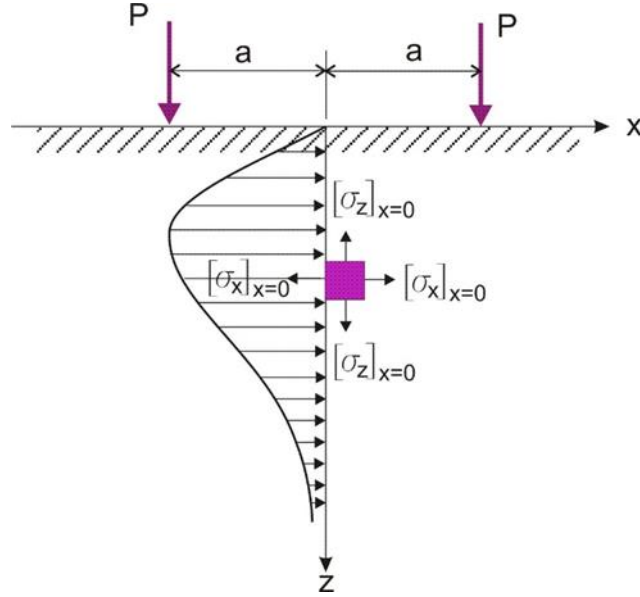


Fig. 6 – Initial loading condition used in deriving the solution for the quarter plane subjected to the action of a concentrated normal force.

For $x = 0$, the stresses have the following values:

$$[\sigma_x]_{x=0} = -\frac{4P}{\pi} \cdot \frac{za^2}{(z^2 + a^2)^2} \quad (8a)$$

$$[\sigma_z]_{x=0} = -\frac{4P}{\pi} \cdot \frac{z^3}{(z^2 + a^2)^2} \quad (8b)$$

$$\tau_{xz} = 0 \quad (8c)$$

The aim is to eliminate σ_x in several steps by successive mirroring. The solution for the quarter-plane is thus obtained by overlapping two half-planes.

By integrating Eq. (8a) with respect to z , we obtain the total lateral force bounded at z (Guilbault, 2011) and which we will denote by R_1 .

$$\int_0^\infty \sigma_x dz = -\frac{2P}{\pi} \quad (9)$$

Relative to the distance at which P is applied, the placement of R_1 moves away from the axis.

$$\frac{\int_0^\infty z \sigma_x dz}{\int_0^\infty \sigma_x dz} = \frac{\pi}{2} a \quad (10)$$

First, consider a half-space $x > 0$ loaded along the boundary, on the z -axis, and aim to remove the load R_1 , located on x at $(\pi/2)a$ from z , by adding a mirror force $-R_1$, but which in turn will generate an additional normal load R_2 , on the initial boundary $z = 0$ at distance $(\pi/2)^2 a$ from the x -axis. The normal load $R_2 = (-2/\pi)^2 P$ must be eliminated by adding a mirror force $-R_2$, which produces a new reaction on the origin axis $x = 0$ of amplitude $R_3 = (-2/\pi)^3 P$, at distance $(\pi/2)^3 a$ from the vertical.

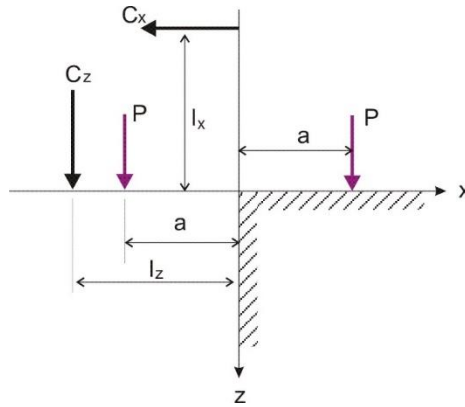


Fig. 7 – Quarter plan conditions (Guilbault, 2011).

As can easily be seen the amplitude of the stress decreases, so that in a number of steps, $n = \infty$ the amplitude $R_\infty = (-2/\pi)^\infty P$, i.e., very close to zero as the position of the load application moves away from the position of the original load $(\pi/2)^\infty$. The method can be regarded, as Hetényi himself puts it, as a "systematic relaxation of constraints" (Hetényi, 1960) and leads to convergent geometric series.

In the same paper Hetényi also determines the total normal load acting on the boundary of the initial half-plane, i.e. $[\sigma_z]_{z=0}$ and $[\sigma_x]_{x=0}$ representing the distribution of cumulative stresses generated during successive corrections.

$$C_z = \int_0^\infty [\sigma_z]_{z=0} dx = \frac{P}{a} \left(\frac{4}{\pi^2 - 4} \right) \quad (11a)$$

$$C_x = \int_0^\infty [\sigma_x]_{x=0} dz = \frac{P}{a} \left(\frac{2\pi}{\pi^2 - 4} \right) \quad (11b)$$

In 1970, Hetényi, in *A General Solution for the Elastic Quarter Space*, by a method like that used in 1960 for the elastic quarter plane, obtains generalized solutions for the quarter space. In this analysis the author starts from the Cartesian stress components defined by the Boussinesq solutions for the elastic half-space.

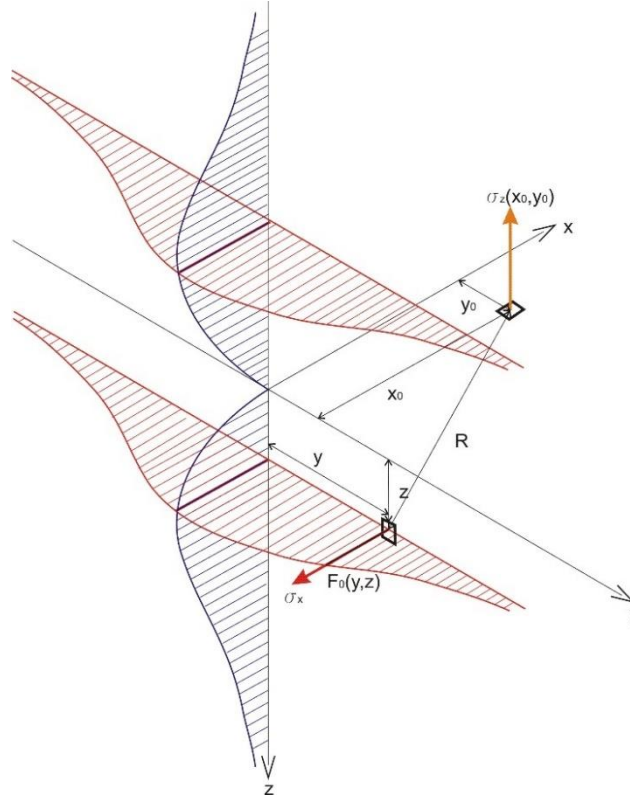


Fig. 8 – First mirror image of border loads (Hetenyi, 1970).

$$\sigma_x = -\frac{3P}{2\pi} \cdot \frac{x^2 z}{R^5} + \frac{(1-2\mu)P}{2\pi} \cdot \left\{ \frac{z}{R^3} - \frac{1}{R(R+z)} \left[1 - \frac{x^2}{R(R+z)} - \frac{x^2}{R^2} \right] \right\} \quad (12a)$$

$$\sigma_y = -\frac{3P}{2\pi} \cdot \frac{y^2 z}{R^5} + \frac{(1-2\mu)P}{2\pi} \cdot \left\{ \frac{z}{R^3} - \frac{1}{R(R+z)} \left[1 - \frac{y^2}{R(R+z)} - \frac{y^2}{R^2} \right] \right\} \quad (12b)$$

$$\sigma_z = -\frac{3P}{2\pi} \cdot \frac{z^3}{R^5} \quad (12c)$$

$$\tau_{xy} = -\frac{3P}{2\pi} \cdot \frac{xyz}{R^5} + \frac{(1-2\mu)P}{2\pi} \cdot \frac{xy}{R^2(R+z)} \left[\frac{1}{R+z} + \frac{1}{R} \right] \quad (12d)$$

$$\tau_{xz} = -\frac{3P}{2\pi} \cdot \frac{xz^2}{R^5} \quad (12e)$$

$$\tau_{yz} = -\frac{3P}{2\pi} \cdot \frac{yz^2}{R^5} \quad (12f)$$

where $R^2 = x^2 + y^2 + z^2$ and μ is the Poisson coefficient.

Due to the symmetry of the stress components, τ_{xy} and τ_{xz} will cancel at $x = 0$, resulting in a quarter of elastic space subject to a concentrated load P , applied at distance a from the origin and simultaneously loaded in the lateral plane yz by the distribution of a compressive load. Eliminating this load distribution leads to the solution for the quarter space. By integrating Eq. (12a) between the limits 0 and ∞ for z and $(-\infty, +\infty)$ for y , thus for σ_x , we obtain:

$$-\frac{3P}{\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{a^2 z}{R^5} dy dz + I_1 = -\frac{2P}{\pi} \quad (13)$$

$$\text{where } I_1 = \frac{(1-2\nu)P}{\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \left\{ \frac{z}{R^3} - \frac{1}{R(R+z)} \left[1 - \frac{a^2}{R(R+z)} - \frac{a^2}{R^2} \right] \right\} dy dz = 0.$$

The result obtained by integrating σ_x is important in proving the convergence of the solution. At the same time, $I_1 = 0$, leads to a load line parallel to the y -axis, equivalent to the case of an elastic half-plane, where the stress components are known to be independent of the Poisson coefficient, as can be seen from Eq. (13).

To simplify the calculations Hetényi performs the whole proof for $\nu = 1/2$ and introduces the dimensionless coordinates $\bar{x} = x/a$, $\bar{y} = y/a$, $\bar{z} = z/a$ and $R^2 = 1 + \bar{y}^2 + \bar{z}^2$. With these simplifications, the lateral compressive load distributed on the lateral yz plane of the elastic half-space $z = 0$ becomes:

$$\sigma_x = -\frac{P}{a^2} F_0(\bar{y}, \bar{z}) \quad (14)$$

where,

$$F_0(\bar{y}, \bar{z}) = \frac{3}{\pi} \frac{\bar{z}}{(1 + \bar{y}^2 + \bar{z}^2)^{5/2}} \quad (15)$$

By mirroring, the aim is to cancel these charges. The mirroring shown in Fig. 8 causes a drag σ_z to appear in the xy -plane at any point (x_0, y_0) .

$$\begin{aligned}\sigma_z &= \frac{P}{a^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_0(\bar{y}, \bar{z}) \cdot \frac{3}{\pi} \left\{ \frac{\bar{x}_0 \bar{z}^2}{[\bar{x}_0^2 + (\bar{y} - \bar{y}_0)^2 + \bar{z}^2]^{5/2}} \right\} d\bar{y} d\bar{z} = \\ &= \frac{P}{a^2} F_1(\bar{x}_0, \bar{y}_0)\end{aligned}\quad (16)$$

The load generated on the horizontal surface of the half-space $z \geq 0$, produces a compression on the yx plane, which requires, to be cancel, a new mirroring. A repetitive sequence is defined, alternatively for σ_x and σ_z .

$$\begin{aligned}F_0(\bar{y}, \bar{z}) &= \frac{3}{\pi} \frac{\bar{z}}{(1 + \bar{y}^2 + \bar{z}^2)^{5/2}} \\ F_1(\bar{x}_0, \bar{y}_0) &= \int_0^{\infty} \int_0^{\infty} F_0(\bar{y}, \bar{z}) \cdot \frac{3}{\pi} \left\{ \frac{\bar{x}_0 \bar{z}^2}{[\bar{x}_0^2 + (\bar{y} - \bar{y}_0)^2 + \bar{z}^2]^{5/2}} \right\} d\bar{y} d\bar{z} \\ F_2(\bar{y}, \bar{z}) &= \int_0^{\infty} \int_0^{\infty} F_1(\bar{x}_0, \bar{y}_0) \cdot \frac{3}{\pi} \left\{ \frac{\bar{x}_0^2 \bar{z}}{[\bar{x}_0^2 + (\bar{y} - \bar{y}_0)^2 + \bar{z}^2]^{5/2}} \right\} d\bar{x}_0 d\bar{y}_0 \\ F_3(\bar{x}_0, \bar{y}_0) &= \int_0^{\infty} \int_0^{\infty} F_2(\bar{y}, \bar{z}) \cdot \frac{3}{\pi} \left\{ \frac{\bar{x}_0 \bar{z}^2}{[\bar{x}_0^2 + (\bar{y} - \bar{y}_0)^2 + \bar{z}^2]^{5/2}} \right\} d\bar{y} d\bar{z} \\ F_4(\bar{y}, \bar{z}) &= \int_0^{\infty} \int_0^{\infty} F_3(\bar{x}_0, \bar{y}_0) \cdot \frac{3}{\pi} \left\{ \frac{\bar{x}_0^2 \bar{z}}{[\bar{x}_0^2 + (\bar{y} - \bar{y}_0)^2 + \bar{z}^2]^{5/2}} \right\} d\bar{x}_0 d\bar{y}_0 \\ &\vdots \\ F_n(\bar{y}, \bar{z}) &= \int_0^{\infty} \int_0^{\infty} F_{n-1}(\bar{x}_0, \bar{y}_0) \cdot \frac{3}{\pi} \left\{ \frac{\bar{x}_0^2 \bar{z}}{[\bar{x}_0^2 + (\bar{y} - \bar{y}_0)^2 + \bar{z}^2]^{5/2}} \right\} d\bar{x}_0 d\bar{y}_0\end{aligned}\quad (17)$$

The convergence of the method is evidenced by the results obtained by integrating the functions (17). The solution method consists of the superposition of two elastic half spaces, one with horizontal boundary surface, $z \geq 0$, and one with vertical boundary surface, $x \geq 0$. The horizontal surface is loaded compressively with $\sum F_{impar}$, and the vertical surface is loaded tensile $\sum F_{par}$. The result can be expressed by two integral equations:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\sum F_{par}(\bar{y}, \bar{z})] \cdot \left\{ \frac{3}{\pi} \cdot \frac{\bar{x}_0^2 \bar{z}^2}{[\bar{x}_0^2 + (\bar{y} - \bar{y}_0)^2 + \bar{z}^2]^{5/2}} \right\} d\bar{y} d\bar{z} - \sum F_{impar}(\bar{x}_0, \bar{y}_0) = 0 \quad (18)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\sum F_{impar}(\bar{x}_0, \bar{y}_0)] \cdot \left\{ \frac{3}{\pi} \cdot \frac{\bar{x}_0^2 \bar{z}}{[\bar{x}_0^2 + (\bar{y} - \bar{y}_0)^2 + \bar{z}^2]^{5/2}} \right\} d\bar{x}_0 d\bar{y}_0 + \sum F_{par}(\bar{y}, \bar{z}) = F_0(\bar{y}, \bar{z}) \quad (19)$$

The method proposed by Hetényi for solving these integrals is based on the Gauss method whereby a definite integral can be replaced by a finite sum of a product of $g(x_i)$, the value of the integrand at the chosen point x_i , also called the "knot", and the coefficient A_i representing the "weight" of each value: $\int_a^b f(x) dx = \sum_{i=1}^n A_i g(x_i)$.

2.2. The Keer approach

Keer proposes a different solution by analysing tangential and normal point charges on the quarter-space using the representation of solutions by potential functions, developed by Boussinesq. The definition of the problem has been carried out using Hetényi's idea of overlapping half-spaces: if two half-spaces (half-space A: $x > 0$, $-\infty < y < \infty$ and $-\infty < z < \infty$ and half-space B: $z > 0$, $-\infty < x < \infty$ and $-\infty < y < \infty$) overlap, then the resulting common region is a quarter-space where $x > 0$, $z > 0$ and $-\infty < y < \infty$. Keer reduces the problem to solving two uncoupled integral equations, where the unknowns are the stress distributions on the two overlapping half-spaces. The solution of the integrals is done in the Fourier transform domain (Keer *et al.*, 1983). Another method developed by Keer together with Hanson (Hanson and Keer, 1990) formulates the problem as two coupled two-dimensional integral equations. The equations contain a logarithmic singularity that varies along the length of the quarter-space edge.

Using the Boussinesq normal point loading solution for the hemisphere, the total voltage for each face of the hemisphere is written:

$$\sigma_{xx}^{total}(0, y, z) = q(y, z) + \int_0^{\infty} \int_{-\infty}^{\infty} p(\alpha, \eta) K(\alpha, y - \eta, z) d\alpha d\eta \quad (20)$$

$z > 0$ și $-\infty < y < \infty$

$$\sigma_{zz}^{total}(x, y, 0) = p(x, y) + \int_0^{\infty} \int_{-\infty}^{\infty} q(\eta, \beta) K(\beta, y - \eta, x) d\beta d\eta \quad (21)$$

$x > 0$, $-\infty < y < \infty$

where $K(\alpha, y - \eta, z)$ is the normal stress on the plane of symmetry $x = 0$ for half space B, determined by two-point normal loads symmetrically arranged at $x = \pm \alpha$, $y = \eta$. It is calculated with the relation:

$$K(\alpha, y - \eta, z) = \frac{3z\alpha^2}{\pi R^5} - \frac{(1-2\nu)}{\pi} \left\{ \frac{z}{R^3} - \frac{1}{R(R+z)} \left[1 - \frac{\alpha^2}{R(R+z)} - \frac{\alpha^2}{R^2} \right] \right\} \quad (22)$$

$$R^2 = \alpha^2 + (y - \eta)^2 + z^2$$

The total normal stress on each boundary plane of the quarter space must be equal to the applied load. This yields the two coupled two-dimensional integral equations from which both above methods are derived.

$$q(y, z) + \int_0^\infty \int_{-\infty}^\infty p(\alpha, \eta) K(\alpha, y - \eta, z) d\alpha d\eta = \sigma_{xx}^a(y, z) \quad (23)$$

$z > 0$ și $-\infty < y < \infty$

$$p(x, y) + \int_0^\infty \int_{-\infty}^\infty q(\eta, \beta) K(\beta, y - \eta, x) d\beta d\eta = \sigma_{zz}^a(x, y) \quad (24)$$

$x > 0$, $-\infty < y < \infty$

Equations (23) and (24) were solved by Keer in the Fourier transform domain (Keer *et al.*, 1983) by applying a series of transformations. They obtained $\bar{p}$ and $\bar{q}$, then substituted x with z and β with α to obtain two decoupled equations:

$$\bar{\phi}_1(z, t) - \int_0^\infty \bar{\phi}_1(\alpha, t) \bar{K}(\alpha, t, z) d\alpha = \bar{\sigma}_{zz}^a(z, t) - \bar{\sigma}_{xx}^a(t, z) \quad (25)$$

$$\bar{\phi}_2(z, t) + \int_0^\infty \bar{\phi}_2(\alpha, t) \bar{K}(\alpha, t, z) d\alpha = \bar{\sigma}_{zz}^a(z, t) + \bar{\sigma}_{xx}^a(t, z) \quad (26)$$

with

$$\begin{aligned} \bar{K}(\alpha, t, z) = & \frac{2z\alpha^2 t^2}{\pi(\alpha^2 + z^2)} K_2(|t|(\alpha^2 + z^2)^{1/2}) \\ & - (1 - 2\nu) \left\{ |t| e^{-\alpha|t|} \right. \\ & - \frac{2t^2}{\pi} \int_0^z K_0(|t|(\alpha^2 + s^2)^{1/2}) ds \left. \right\} \end{aligned} \quad (27)$$

$$\overline{\phi_1}(z, t) = \bar{p}(z, t) - \bar{q}(t, z)$$

$$\overline{\phi_2}(z, t) = \bar{p}(z, t) + \bar{q}(t, z)$$

$K_2(s)$ and $K_0(s)$ are modified Bessel functions of type two.

Using the Gausse - Legendre integration scheme, with the transformation variable t as a parameter, Eqs. (25) and (26) have been solved. To obtain an accurate solution, the equations must be solved repeatedly for different values of t . Further numerical integration is required by calculating the inverse Fourier transform in order to substitute $\bar{p}$ and $\bar{q}$ in the initial integrals. A considerable computational effort is required to find the unknowns of the problem.

Starting from Eqs. (23), (24), Hanson and Keer propose a direct solution in 1990. In order to solve the two integral equations, a higher order approximation, a piecewise constant approximation of the functions $p(x, y)$ and $q(y, z)$, is proposed. The internal stress and displacements are calculated using Love's solutions (Love, 1929) applied to a rectangular, uniformly loaded surface. A coarse "piecewise" approximation of the p and q functions will yield poor results near the quarter-space boundary due to the logarithmic singularity present in these distributions. An example of this singularity is given by the following substitution in Eqs. (23) and (24):

$$q(y, z) = A(y) \ln z + q_0(y) \quad (28)$$

$$p(x, y) = B(y) \ln x + p_0(y) \quad (29)$$

where p_0, q_0 – limited, x and $z \rightarrow 0$ and $\nu = 1/2$.

Using the limits

$$\lim_{z \rightarrow 0} \int_{-\infty}^{\infty} B(\eta) \int_0^{\infty} \frac{3z\alpha^2 \ln \alpha}{\pi R^5} d\alpha d\eta = B(y)(1 + \ln z) \quad (30)$$

$$\lim_{z \rightarrow 0} \int_{-\infty}^{\infty} p_0(\eta) \int_0^{\infty} \frac{3z\alpha^2}{\pi R^5} d\alpha d\eta = p_0(y) \quad (31)$$

the following equations are obtained:

$$A(y) \ln z + q_0(y) + B(y)(1 + \ln z) + p_0(y) = \sigma_{xx}^a(y, 0) \quad (32)$$

$$B(y) \ln x + p_0(y) + A(y)(1 + \ln x) + q_0(y) = \sigma_{zz}^a(y, 0) \quad (33)$$

By subtracting (33) from (32) and cancelling the singularity in each equation, we obtain:

$$A(y) = \frac{1}{2}(\sigma_{zz}^a(0, y) - \sigma_{xx}^a(y, 0)) \quad (34a)$$

$$B(y) = -\frac{1}{2}(\sigma_{zz}^a(0, y) - \sigma_{xx}^a(y, 0)) \quad (34b)$$

For arbitrary values of the Poisson coefficient, the determination of the logarithmic singularity coefficients must be incorporated directly into the numerical solution.

On each hemisphere isolating the singularity, an additional singular symmetric normal voltage distribution is developed by finding a Green's function. Then, this fundamental solution is distributed, with unknown density, along the y-axis which represents the edge of the quarter-space.

2.3. The de Mul approach

To model the quarter-space and find solutions for the tangential stresses acting on the ends of bodies in contact, de Mul starts from Boussineqs results for the half-space loaded with a concentrated force, which he puts into Cartesian coordinates, to obtain the normal displacement, w , of a point (x, y) due to contact pressure $p(x', y')$ (de Mul *et al.*, 1986).

$$w(x, y) = \frac{1 - \nu}{2\pi G} \iint_A \frac{p(x', y')}{S} dx' dy' \quad (35)$$

where A is the contact area and S is the distance between (x, y) and (x', y') calculated as follows:

$$S = \sqrt{(x - x')^2 + (y - y')^2}$$

To determine the solutions, the problem is dissected by dividing the contact area into rectangles on which the pressure is constant. Thus on any rectangle i , belonging to the contact area, $p = p_i$.

The displacement due to pressure on a rectangle, i , is calculated as follows:

$$w_i(x) = \frac{1 - \nu}{\pi G} \iint_{A_i} \frac{dx' dy'}{|x - x'|} \quad (36)$$

Each rectangle contributes to the displacement w :

$$w(x) = \sum_{i=1}^n p_i w_i(x) \quad (37)$$

To increase computational speed, de Mul proposes a series of approximations, the solutions being expressed for $p \geq 0$, as follows:

$$A_i d_i = \frac{\partial C}{\partial p_i} = A_i B_i + \sum_{j=1}^n T_{ij} p_j = 0 \quad \text{dacă} \quad p_i > 0 \quad (38a)$$

$$A_i d_i = \frac{\partial C}{\partial p_i} = A_i B_i + \sum_{j=1}^n T_{ij} p_j \geq 0 \quad \text{dacă} \quad p_i = 0 \quad (38b)$$

where

$$T_{ij} = \iint_{A_j} w_i(x) dx dy \quad (39)$$

and with d_i represents the discretized analogue of the deformed gap D ($D = 0$ for $p > 0$, inside the contact area A ; $D \geq 0$ for $p = 0$, outside the contact area).

The method proposed by de Mul does not consider the normal stresses, which are considered very small, only cancelling the tangential stresses on the contact end surfaces by obliterating the pressures.

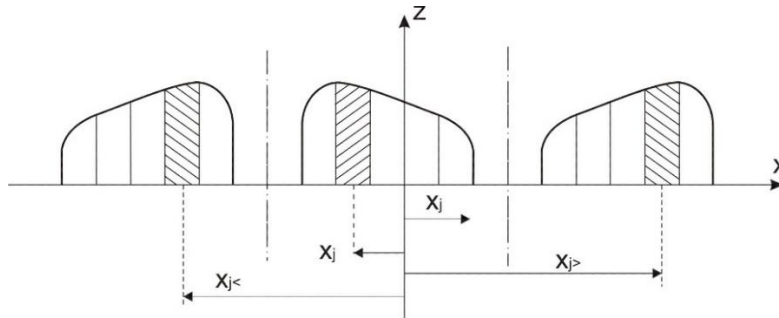


Fig. 9 – Pressure mirroring to cancel tangential stresses (de Mul *et al.*, 1986).

In the calculation of solutions this mirroring appears as follows:

– for bodies of the same length

$$A_i d_i = A_i B_i + \sum_{j=1}^n (T_{ij<} + T_{ij} + T_{ij>}) p_j \quad (40)$$

– for bodies of different lengths

$$A_i d_i = A_i B_i + \sum_{j=1}^n (T_{ij<}^+ + T_{ij>}^+ + T_{ij} + T_{ij<}^- + T_{ij>}^-) p_j \quad (41)$$

2.4. The Gouilbault approach

Starting from the total normal load acting on the boundary of the half-spaces $z = 0$, C_z , and on the boundary $x = 0$, C_x , expressed by relations (11a), (11b) and determined by Hetényi, Gouilbault establishes the influence of these correction forces on the displacement of the application point P belonging to the half-plane (Gouilbault, 2011). The representation of these correction forces is shown in Fig. 9.

$$u_z(C_x) = \delta(C_x) = \frac{C_x(1-v)}{2\pi E} \left[\frac{a \cdot l_x}{\rho^3} - (1-2v) \frac{l_x}{\rho(\rho+a)} \right] \quad (42a)$$

$$u_z(C_z) = \delta(C_z) = \frac{C_z(1+v)}{2\pi E} \cdot \frac{2(1-v)}{l_z+a} \quad (42b)$$

$$u_z(P) = \delta(P) = \frac{P(1+v)}{2\pi E} \cdot \frac{2(1-v)}{2a} \quad (\text{mirrored force}) \quad (42c)$$

The correction factor is determined as the ratio of the displacement produced by the mirrored correction pressure P^* to the displacement of the point where P is applied. The overcorrection P^* simulates the cumulative influence of the force mirroring correction.

$$\delta(P^*) = \delta(P) + \delta(C_x) + \delta(C_z) \quad (43)$$

The correction corresponding to ψ , is calculated from Eq. (43) as a convergent series.

$$\begin{aligned} \psi = \frac{P^*}{P} = & \\ = & \left\{ 1 + \sum_{n=1}^{\infty} \frac{2^{4n+1}}{\pi^{2n}(\pi^{2n} + 2^{2n})} + \right. \\ & + \frac{1}{(1-v)} \left[\sum_{n=1}^{\infty} \frac{1}{\left(\left(\frac{\pi}{2} \right)^{2(2n-1)} + 1 \right)^{3/2}} - \right. \\ & \left. \left. - (1-2v) \sum_{n=1}^{\infty} \frac{1}{\left(\left(\frac{\pi}{2} \right)^{2(2n-1)} + 1 \right) \left(\left(\left(\frac{\pi}{2} \right)^{2(2n-1)} + 1 \right)^{1/2} + 1 \right)} \right] \right\} \quad (44) \end{aligned}$$

In an acceptable approximation, the correction factor ψ , has (Guilbault, 2011) the following expression:

$$\psi = \frac{P^*}{P} = 1,29 - \frac{1}{(1 - \nu)} [0,08 - 0,5\nu] \quad (45)$$

The overcorrection proposed by Guilbault considers the normal stress in the mirroring process for shear stress correction. To validate the proposed method Guilbault resorts to FEM validation (Yalpanian and Guilbault, 2020). The comparison with FEM showed that the introduction of the ψ factor better simulates the elimination of normal and shear stresses on the displacement without affecting the computational times of the mirroring.

3. Conclusions

To solve the problem of concentrated contacts, the mathematical difficulties have so far been overcome in few cases. The best known are the relations obtained by Hertz in 1880 for point contact and linear contact.

Recent research has shown that for linear contacts of finite length, the pressure distribution is non-uniform, with maximum values at the end regions. This pressure concentration effect depends, crucially, on the geometry of the edge of the bodies and on the relationship between the contact lengths of the two bodies. The contact between the two bodies can be coincident or non-coincident. On the edges, the concentration of pressures can lead to the material flow limit being exceeded, creating plastic deformations, and introducing residual stresses. This phenomenon occurs in the first few cycles of operation.

To improve the durability of cylindrical roller bearings, the geometry of the contact area between the roller and the raceway can be determined by using specific analytical equations for the problem, with consideration of the space quarter.

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ABORDAREA CONTACTULUI CONCENTRAT LINIAR CA O PROBLEMĂ DE SFERT DE SPAȚIU

(Rezumat)

Contactul concentrat apare frecvent între organele de mașini, ceea ce duce la apariția diferitelor forme de deteriorare: uzura abrazivă, uzura adezivă și griparea, precum și distrugerea suprafeței prin oboseala de contact. Oboseala de contact este un mod tipic de deteriorare a organelor de mașini (rulmenți, angrenaje, contactul roată-șină, dar și came și supape) supuse unor presiuni de contact hertziene puternice și repetate.

Cele mai multe probleme de contact concentrat, în care corpurile se ating într-un punct sau pe o linie, sunt bine descrise și rezolvate cu ajutorul ipotezelor și ecuațiilor teoriei lui Hertz, care asociază corpurile cu semispații elastice. În cazul corpurilor cilindrice de lungime finită, ecuațiile lui Hertz nu descriu corect starea de tensiuni care apare la limită. La contact linear concentrate necoincident, aceste tensiuni necompensate determină apariția așa numitului „efect de capăt”: presiunea crește în apropierea marginilor cilindrului de lungime mai mică. Aceste tensiuni necompensate trebuie să fie luate în considerare în proiectarea geometrică și funcțională a organelor de mașini.

În această lucrare, sunt prezentate principalele metode de soluționare a tensiunilor ce apar la contactul linear concentrate cu lungime finite, în abordarea problemei ca o problemă de sfert de spațiu.