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Power Unit Garima Distribution: Properties, Comparison of Estimation Procedures and Diverse Applications

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SUMMARY

This paper derives the power unit Garima distribution (PUGD), which is an extension of the unit Garima distribution (UGD) obtained through a power transformation. Properties of this new model, among which are mode, moments, moment generating function, entropies, extropy, and Bonferroni and Lorenz curves, are studied in detail. The relationship between the PUGD and the Garima distribution is established mathematically. Additionally, several estimation techniques for the parameters of the PUGD are examined through a Monte Carlo simulation procedure, including maximum likelihood, ordinary least squares, weighted least squares, and the Anderson–Darling and Cramer–von Mises methods. The simulation results obtained are favorable to the maximum likelihood approach. Both the observed and expected information matrices are discussed. The goodness of fit of the distribution to three data sets relating to various fields is determined relative to that of the unit power Lindley, UGD, Kumaraswamy and beta distributions. Among the distributions considered, the PUGD proved to be the best in modeling the data.

Key words: estimation methods, exponential integral, goodness of fit, Lambert W function, power transformation, Tricomi confluent hypergeometric function, unit Garima distribution

1. Introduction

It is an undeniable fact that probability distributions are fundamental to the modeling of stochastic phenomena. New distributions may be required when it is necessary to reflect certain distributional properties of given data. Features of data, including shape characteristics, supports (data range), and discrete or continuous nature, largely influence the choice of a distribution for the data. For

instance, a right-skewed distribution is an appropriate choice for data that are skewed to the right.

The literature on distribution theory is to some extent predicated on distributions that have the support (0, 1), which are called unit distributions. Unit distributions are good at modeling proportions (Flies et al., 1993; Seber, 2013). The standard beta distribution and Kumaraswamy distribution are unit distributions. Despite the wide applicability of these distributions, they are found wanting in quite a number of situations requiring the fitting of distributions to data lying in the interval (0, 1). As a consequence, new unit distributions have been developed. The unit Garima distribution is one of the recently proposed unit distributions (Ayuyuen and Bodhisuwan, 2024). These authors paid attention to just a few properties of the distribution, including the probability density function (pdf), cumulative distribution function (cdf) and quantile function of the distribution. They eventually proposed the unit Garima family of distributions.

The unit Garima distribution, just like other one-parameter unit distributions, often requires some modification to make it flexible enough for certain data. Power transformation has gained attention in the generalization of unit distributions (Alsadat et al., 2023; Yıldırım, 2023; Gemeay et al., 2024). The essence of this paper is to proffer a new extension of the unit Garima distribution using the power transformation approach. We will later refer to the new distribution as the power unit Garima distribution (PUGD). The remaining portion of this paper is structured in the following way. Section 2 deals with the derivation of the pdf of the PUGD alongside other properties of the distribution. The estimation procedures employed in this study are dealt with in Section 3. Simulation results are presented in Section 4. In Section 5, the fits of the distribution to real-world data are compared with those of competing distributions. The work is concluded in Section 6.

2. Properties of the PUGD

Several properties of the PUGD are derived in this section. In particular, due

consideration is given to the pdf, cdf, ordinary moments, incomplete moment, mean residual life function, stochastic ordering, and order statistics of the distribution, among other properties.

2.1. Derivation of the pdf of PUGD

The derivation of the PUGD requires the definition of the UGD. For this purpose, Definition 2.1 is provided.

Definition 2.1. A random variable Y is statistically said to follow the UGD if its pdf and cdf are respectively given by

$$g(y)=\frac{\lambda(\lambda+y)}{(2+\lambda)y^3}\exp\left(-\lambda\left(\frac{1}{y}-1\right)\right),0<y<1,\lambda>0. \tag{2.1}$$

and

$$G(y)=\left(1+\frac{\lambda}{2+\lambda}\left(\frac{1}{y}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{y}-1\right)\right),0<y<1,\lambda>0. \tag{2.2}$$

Proposition 2.1. Suppose Y follows the UGD with pdf (2.1) and cdf (2.2). Then the random variable $X=Y^{\frac{1}{\beta}}$ follows the PUGD with parameters β and λ while its pdf is

$$f(x)=\frac{\beta\lambda\left(x^{\beta}+\lambda\right)}{(\lambda+2)x^{2\beta+1}}\exp\left(-\lambda\left(\frac{1}{x^{\beta}}-1\right)\right),0<x<1,\beta,\lambda>0. \tag{2.3}$$

Proof: To prove Proposition 2.1, we let $x=y^{\frac{1}{\beta}}$. Now, $\frac{dy}{dx}=\beta x^{\beta-1}$. Consequently, the pdf of X is

$$f(x)=g(x^{\beta})\left|\frac{dy}{dx}\right|=\beta x^{\beta-1}g(x^{\beta})$$

$$=\frac{\beta\lambda\left(\lambda+x^{\beta}\right)}{(2+\lambda)x^{2\beta+1}}\exp\left(-\lambda\left(\frac{1}{x^{\beta}}-1\right)\right),0<x<1,\beta,\lambda>0.$$

This completes the proof. □

Figure 2.1 constitutes a graphical representation of the pdf of the PUGD. It is obvious that the distribution can be positively or negatively skewed and unimodal.

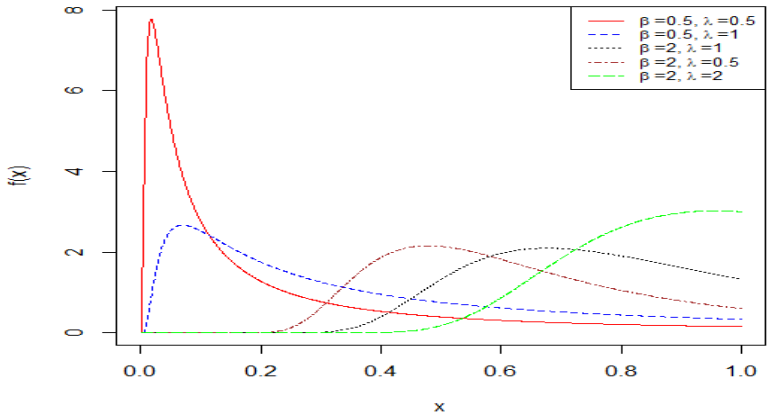


Figure 2.1: Plots of the pdf of PUGD based on various parameter values

Integrating (2.3) with respect to x gives the cdf of the PUGD, defined by

$$F(x)=\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x^{\beta}}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{x^{\beta}}-1\right)\right),0<x<1,\beta,\lambda>0. \tag{2.4}$$

Survival function and hazard rate function are two important concepts in reliability and survival analysis. For the PUGD, let $S(x)$ and $h(x)$ be the survival function and hazard rate function respectively.

Mathematically,

$$S(x)=1-\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x^{\beta}}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{x^{\beta}}-1\right)\right),0<x<1,\beta,\lambda>0. \tag{2.5}$$

Also,

$$h(x)=\frac{\beta\lambda\left(\lambda+x^{\beta}\right)\exp\left(-\lambda\left(\frac{1}{x^{\beta}}-1\right)\right)}{\left[1-\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x^{\beta}}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{x^{\beta}}-1\right)\right)\right](2+\lambda)x^{2\beta+1}},0<x<1,\beta,\lambda>0. \tag{2.6}$$

We present Figure 2.2. in order to illustrate the possible shape of (2.6). Clearly, it indicates that the hazard rate function of PUGD can be a nondecreasing function of x.

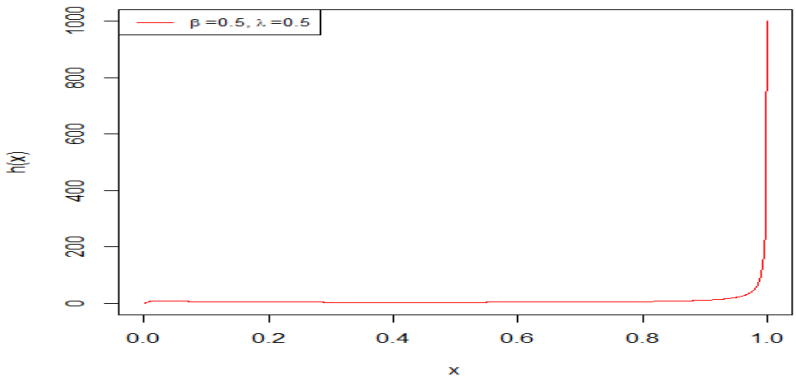


Figure 2.2. Plot of the hazard rate function of the PUGD

In what follows, if a random variable follows the PUGD with parameters β and λ, we write $X \sim \text{PUGD}(\beta, \lambda)$.

2.2. Mode

The mode is one of the measures of location in statistics. It is well known that the maximum value of the pdf of a continuous distribution is associated with the mode of the distribution.

The task of finding the mode of the PUGD requires obtaining the first derivative of $\ln f(x)$, equating it to 0 and solving the resulting equation. Based on (2.3),

$$\ln f(x) = \ln(\beta\lambda) - \ln(2+\lambda) - (2\beta+1)\ln x + \ln(\lambda + x^{\beta}) - \lambda\left(\frac{1}{x^{\beta}} - 1\right).$$

Consequently, $\frac{\mathrm{d}}{\mathrm{d}x}\ln f(x) = 0$ implies that

$$-\frac{2\beta+1}{x} + \frac{\beta x^{\beta-1}}{x^{\beta}+\lambda} + \frac{\beta\lambda}{x^{\beta+1}} = 0. \qquad (2.7)$$

Multiplying through by $(\lambda + x^{\beta})x$ and denoting x^{β} by f leads to the equation

$$f^2+\lambda f-\left(\frac{\beta}{\beta+1}\right)\lambda^2=0.$$

Solving for f, we have

$$f=\frac{\lambda}{2}\left[-1+\sqrt{1+\frac{4\beta}{(\beta+1)}}\right].$$

Let the value of x that satisfies (2.7) be x_0.Then

$$x_0=\left(\frac{\lambda}{2}\left[-1+\sqrt{1+\frac{4\beta}{(\beta+1)}}\right]\right)^{\frac{1}{\beta}},\beta>0,\lambda>0.$$

Consider the second derivative

$$\frac{d^2\ln f(x)}{dx^2}=x^{-2}\left[\frac{\beta x^{\beta}\left((\beta-1)\lambda-x^{\beta}\right)}{\left(\lambda+x^{\beta}\right)^2}+2\beta+1-\beta\lambda(\beta+1)x^{-\beta}\right].$$

This implies that

$$\left.\frac{d^2\ln f(x)}{dx^2}\right|_{x=x_0}=x_0^{-2}\left[\frac{\beta f((\beta-1)\lambda-f)}{\left(\lambda+f\right)^2}+2(\beta+1)-\beta\lambda(\beta+1)f^{-1}\right]. \tag{2.8}$$

At the mode, we have $\frac{d\ln f(x)}{dx}=0$, leading to the relationship

$$2(\beta+1)=\frac{\beta f}{\left(\lambda+f\right)}+\beta\lambda f^{-1} \tag{2.9}$$

.

Applying (2.9) in (2.8) and simplifying the resulting expression leads to

$$\left.\frac{d^2\ln f(x)}{dx^2}\right|_{x=x_0}=-\beta^2\lambda^2x_0^{-2}\left[\frac{\lambda+2f}{f\left(\lambda+f\right)^2}\right].$$

Since $\left.\frac{d^2\ln f(x)}{dx^2}\right|_{x=x_0}<0$, x_0 is the mode of the PUGD.

2.3. Raw moments

The raw moments of a random variable with known distribution play a crucial role in the study of other properties such as the shape characteristics and

dispersion index of the distribution. In particular, the *r*th raw moment of the PUGD is given by

$$E\left(X^r\right)=\mu_r{}'=\frac{\beta\lambda\exp(\lambda)}{\left(\lambda+2\right)}\int_0^1 x^r\,\frac{\left(\lambda+x^\beta\right)}{x^{2\beta+1}}\exp\left(-\frac{\lambda}{x^\beta}\right)dx.$$

Let $u=\dfrac{\lambda}{x^\beta}$. This implies that $x=\lambda^{\frac{1}{\beta}}u^{-\frac{1}{\beta}}$ and $dx=\dfrac{1}{\beta}\lambda^{\frac{1}{\beta}}u^{-\frac{1}{\beta}-1}du$. Hence,

$$E\left(X^r\right)=\mu_r{}'=\frac{\lambda^{\frac{r}{\beta}}\exp(\lambda)}{\left(\lambda+2\right)}\int_\lambda^\infty u^{1-\frac{r}{\beta}}\left(1+u^{-1}\right)\exp\left(-u\right)du.$$

$$=\frac{\lambda^{\frac{r}{\beta}}\exp(\lambda)}{\left(\lambda+2\right)}\left[\Gamma\left(2-\frac{r}{\beta},\lambda\right)+\Gamma\left(1-\frac{r}{\beta},\lambda\right)\right], \tag{2.10}$$

where $\Gamma\left(a,z\right)=\int_z^\infty m^{a-1}\exp\left(-m\right)dm$ refers to the complementary gamma function. Details of this function are available in Nadarajah and Chan (2019), among other relevant literature.

Using (2.10), we obtain the following first four raw moments:

$$E\left(X\right)=\mu=\frac{\lambda^{\frac{1}{\beta}}\exp(\lambda)}{\left(\lambda+2\right)}\left[\Gamma\left(2-\frac{1}{\beta},\lambda\right)+\Gamma\left(1-\frac{1}{\beta},\lambda\right)\right];\beta>\frac{1}{2}$$

$$E\left(X^2\right)=\mu_2{}'=\frac{\lambda^{\frac{2}{\beta}}\exp(\lambda)}{\left(\lambda+2\right)}\left[\Gamma\left(2-\frac{2}{\beta},\lambda\right)+\Gamma\left(1-\frac{2}{\beta},\lambda\right)\right];$$

$$E\left(X^3\right)=\mu_3{}'=\frac{\lambda^{\frac{3}{\beta}}\exp(\lambda)}{\left(\lambda+2\right)}\left[\Gamma\left(2-\frac{3}{\beta},\lambda\right)+\Gamma\left(1-\frac{3}{\beta},\lambda\right)\right];$$

$$E\left(X^4\right)=\mu_4{}'=\frac{\lambda^{\frac{4}{\beta}}\exp(\lambda)}{\left(\lambda+2\right)}\left[\Gamma\left(2-\frac{4}{\beta},\lambda\right)+\Gamma\left(1-\frac{4}{\beta},\lambda\right)\right].$$

From the foregoing, we have

$$\text{Variance} = \text{Var}(X) = \sigma^2 = \mu_2' - \mu^2$$

$$= \frac{\lambda^{\frac{2}{\beta}} \exp(\lambda)}{(\lambda+2)}\left[\Gamma\left(2-\frac{2}{\beta},\lambda\right)+\Gamma\left(1-\frac{2}{\beta},\lambda\right)\right]-\left(\frac{\lambda^{\frac{1}{\beta}} \exp(\lambda)}{(\lambda+2)}\left[\Gamma\left(2-\frac{1}{\beta},\lambda\right)+\Gamma\left(1-\frac{1}{\beta},\lambda\right)\right]\right)^2.$$

The moment based coefficients of skewness and kurtosis represented by S and K, respectively, are mathematically defined as follows

$$S = \frac{\mathrm{E}\left(X-\mu\right)^3}{\sigma^3} = \frac{\mathrm{E}\left(X^3\right)-3\mu\mathrm{E}\left(X^2\right)+2\mu^3}{\sigma^3},$$

$$K = \frac{\mathrm{E}\left(X-\mu\right)^4}{\sigma^4} = \frac{\mathrm{E}\left(X^4\right)-4\mu\mathrm{E}\left(X^3\right)+6\mu^2\mathrm{E}\left(X^2\right)-3\mu^4}{\sigma^4}.$$

Table 2.1 contains numerical values of μ, μ_2, S and K. In the table, it is evident that, for fixed β, the mean of the distribution increases as λ increases. An increase in λ results in a decrease in the coefficient of skewness when β is constant. There is no monotonic relationship between either of the variance and kurtosis of the distribution and β when λ is constant. Notably, the distribution

Table 2.1: Descriptive statistics for the PUGD

β	λ	μ	μ_2	S	K
0.5	0.5	0.2	0.0510	1.6191	4.9196
0.5	1	0.3333	0.0663	0.8466	2.6799
0.5	2	0.5	0.0651	0.2038	1.9398
0.5	5	0.7143	0.0381	-0.5242	2.4200
0.5	10	0.8333	0.0176	-0.9915	3.5650
1	0.5	0.3846	0.0521	0.8359	2.8099
1	1	0.5321	0.0502	0.2811	2.0694
1	2	0.6807	0.0367	-0.2299	2.1135
1	5	0.8360	0.0154	-0.8462	3.1726
1	10	0.9096	0.0059	-1.2397	4.5094
2	0.5	0.5934	0.0324	0.3659	2.2482
2	1	0.7123	0.0248	-0.0714	2.1028
2	2	0.8160	0.0148	-0.5049	2.4983
2	5	0.9116	0.0050	-1.0457	3.8130
2	10	0.9528	0.0017	-1.3866	5.1810

can be positively or negatively skewed. The distribution is also capable of being either platykurtic or leptokurtic.

2.4. Inverse raw moments

The *r*th inverse raw moment of the PUGD, denoted by μ_{-r}', is derived as follows:

$$E\left(X^{-r}\right)=\mu_{-r}'=\int_0^1 x^{-r} f(x)dx=\frac{\beta\lambda\exp(\lambda)}{(\lambda+2)}\int_0^1 x^{-r}\frac{\left(\lambda+x^{\beta}\right)}{x^{2\beta+1}}\exp\left(-\frac{\lambda}{x^{\beta}}\right)dx.$$

$$=\frac{\lambda^{-\frac{r}{\beta}}\exp(\lambda)}{(\lambda+2)}\int_{\lambda}^{\infty}u^{\frac{r}{\beta}+1}\left(1+u^{-1}\right)\exp(-u)\,du$$

$$=\frac{\lambda^{-\frac{r}{\beta}}\exp(\lambda)}{(\lambda+2)}\left[\Gamma\left(\frac{r}{\beta}+2,\lambda\right)+\Gamma\left(\frac{r}{\beta}+1,\lambda\right)\right]$$

It may be noted that the harmonic mean of the PUGD is

$$\mu_{-1}'=\frac{\lambda^{-\frac{1}{\beta}}\exp(\lambda)}{(\lambda+2)}\left[\Gamma\left(\frac{1}{\beta}+2,\lambda\right)+\Gamma\left(\frac{1}{\beta}+1,\lambda\right)\right].$$

2.5. Incomplete moments and measures of dispersion

Incomplete moments are arguably an essential property of a continuous random variable, as they can be used to derive other notable properties, such as the mean deviations and income inequality measures. The *r*th incomplete moment for the PUGD is

$$J_r(z)=\int_0^z x^r f(x)dx \quad =\frac{\beta\lambda}{(\lambda+2)}\exp(\lambda)\int_0^z x^r\left(\frac{\lambda+x^{\beta}}{x^{2\beta+1}}\right)\exp(-\lambda x^{-\beta})dx$$

$$=\frac{\lambda^{\frac{r}{\beta}}}{(\lambda+2)}\exp(\lambda)\int_{\lambda z^{-\beta}}^{\infty}u^{-\frac{r}{\beta}+1}\left(1+u^{-1}\right)\exp(-u)du$$

$$=\frac{\lambda^{\frac{r}{\beta}}\exp(\lambda)}{(\lambda+2)}\left[\Gamma\left(2-\frac{r}{\beta},\frac{\lambda}{z^{\beta}}\right)+\Gamma\left(1-\frac{r}{\beta},\frac{\lambda}{z^{\beta}}\right)\right],\beta\geq\frac{r}{2}. \qquad (2.11)$$

The mean deviation about the mean μ and mean deviation about the median M, denoted by $\delta_1(x)$ and $\delta_2(x)$, respectively, are measures of dispersion.

Mathematically,

$$\delta_1(x) = \int_0^1 |x-\mu| f(x)dx = 2\mu F(\mu) - 2J_1(\mu). \tag{2.12}$$

Applying (2.4) and (2.11) in (2.12), the mean deviation about the mean of the PUGD becomes

$$\delta_1(x) = 2\mu\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{\mu^\beta}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{\mu^\beta}-1\right)\right)$$
$$-\frac{2\lambda^{\frac{1}{\beta}}\exp(\lambda)}{(\lambda+2)}\left[\Gamma\left(2-\frac{1}{\beta},\frac{\lambda}{\mu^\beta}\right)+\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{\mu^\beta}\right)\right].$$

Similarly,

$$\delta_2(x) = \int_0^1 |x-M| f(x)dx = \mu - 2J_1(\mu)$$

$$= \mu - \frac{2\lambda^{\frac{1}{\beta}}\exp(\lambda)}{(\lambda+2)}\left[\Gamma\left(2-\frac{1}{\beta},\frac{\lambda}{\mu^\beta}\right)+\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{\mu^\beta}\right)\right].$$

2.6. Relationship with Garima distribution and expectations of functions of a power unit Garima distributed random variable

Here, we commence by establishing the relationship between the PUGD and Garima distribution. This relationship may help in deriving other properties of the distribution. It can also serve as an alternative approach to obtaining simulation results corresponding to the PUGD.

Proposition 2.1. Let W be a standard Garima distributed random variable. That is, W is a continuous random variable having a mixture of exponential distribution with parameter 1 and gamma distribution with shape parameter and scale

parameter given by 2 and 1, respectively. If the mixing proportions are $\frac{\lambda+1}{\lambda+2}$ and $\frac{1}{\lambda+2}$, respectively, then for $\beta>0,\lambda>0$, the random variable

$$X=\left(\frac{\lambda}{\lambda+W}\right)^{\frac{1}{\beta}} \sim PUGD(\beta,\lambda).$$

Proof: The pdf of W is

$$g(w)=\frac{\lambda+1}{\lambda+2}e^{-w}+\frac{1}{\lambda+2}we^{-w}, w>0.$$

From the transformation $x=\left(\frac{\lambda}{\lambda+w}\right)^{\frac{1}{\beta}}$, we have $w=\lambda\left(x^{-\beta}-1\right)$ and $dw=-\beta\lambda x^{-\beta-1}dx$.

Hence, the pdf of X is

$$f(x)=g\left[\lambda\left(x^{-\beta}-1\right)\right]\left|\frac{dw}{dx}\right| \quad \frac{\lambda+1}{\lambda+2}e^{-\lambda\left(x^{-\beta}-1\right)}+\frac{1}{\lambda+2}\lambda\left(x^{-\beta}-1\right)e^{-\lambda\left(x^{-\beta}-1\right)}\left|-\beta\lambda x^{-\beta}\right|$$

$$=\frac{\beta\lambda\left(\lambda+x^{\beta}\right)}{\lambda+2}x^{-(2\beta+1)}e^{-\lambda\left(x^{-\beta}-1\right)}, \beta>0,\lambda>0,0<x<1.$$

This completes the proof. □

It is easy to deduce that $M=\frac{W}{\lambda}=X^{-\beta}-1$ follows a Garima distribution with parameter λ. Expectations of certain useful functions of the power unit Garima distributed variable *X* are derived below.

$$\mathrm{E}\left(\ln \mathrm{X}\right)=\frac{1}{\beta}\left[\ln\lambda-\mathrm{E}\left(\ln(\lambda+W)\right)\right]. \tag{2.13}$$

Let $W_1 \sim \mathrm{EXp}(1)$ and $W_2 \sim \mathrm{Gamma}(2,1)$. Then,

$$\mathrm{E}\left(\ln(\lambda+W)\right)=\frac{\lambda+1}{\lambda+2}\mathrm{E}\left(\ln(\lambda+W_1)\right)+\frac{1}{\lambda+2}\mathrm{E}\left(\ln(\lambda+W_2)\right)$$

$$=\frac{\lambda+1}{\lambda+2}\int_0^1 \ln(\lambda+w_1)\mathrm{e}^{-w_1}dw_1+\frac{1}{\lambda+2}\int_0^1 \ln(\lambda+w_2)\mathrm{e}^{-w_2}dw_2$$

$$=\frac{\lambda+1}{\lambda+2}\mathrm{e}^{\lambda}\left[\mathrm{e}^{-\lambda}\ln\lambda+\mathrm{E}_1(\lambda)\right]+\frac{1}{\lambda+2}\left[\ln\lambda+1-(\lambda+1)\mathrm{e}^{\lambda}\mathrm{E}_1(\lambda)\right]$$

$$= \ln\lambda + \frac{1}{\lambda+2}. \tag{2.14}$$

(2.14)

$$\mathrm{E}\left(\mathrm{X}^{-\beta}\right) = 1 + \frac{\mathrm{E}(W)}{\lambda} = 1 + \frac{1}{\lambda}\left[\frac{\lambda+1}{\lambda+2}\mathrm{E}(W_1) + \frac{1}{\lambda+2}\mathrm{E}(W_2)\right]$$

$$= 1 + \frac{1}{\lambda}\left[\frac{+1}{\lambda+2} + \frac{2}{\lambda+2}\right] = \frac{\lambda^2+3\lambda+3}{\lambda(\lambda+2)}. \tag{2.15}$$

Similarly,

$$\mathrm{E}(\ln \mathrm{X}) = \frac{1}{\beta}\left[\ln\lambda - \ln(\lambda+W)\right] = -\frac{1}{\beta(\lambda+2)}, \tag{2.16}$$

$$\mathrm{E}\left(\ln\left(\lambda + X^{\beta}\right)\right) = \ln\lambda + \mathrm{E}(\lambda+1+W) - \mathrm{E}(\lambda+W)$$

$$= \ln(\lambda+1) + \frac{1}{\lambda+3} - \frac{1}{\lambda+2}, \tag{2.17}$$

$$E\left(\frac{1}{\left(\lambda+X^{\beta}\right)^2}\right) = \frac{1}{\lambda^2}E\left(\frac{(\lambda+W)^2}{(\lambda+1+W)^2}\right)$$

$$= \frac{\lambda + \exp(\lambda+1)\mathrm{E}_1(\lambda+1)}{\lambda^2(\lambda+2)}, \tag{2.18}$$

$$E\left(X^{-\beta}\ln X\right) = \frac{1}{\beta\lambda}E\left((\lambda+W)\left(\ln\lambda - \ln(\lambda+W)\right)\right)$$

$$== \frac{1}{\beta\lambda}\left[1 - \frac{\exp(\lambda)E_1(\lambda)}{\lambda+2}\right], \tag{2.19}$$

$$E\left(X^{-\beta}(\ln X)^2\right) = \frac{2}{\beta^2(\lambda+2)^2} + \frac{2\exp(\lambda)\mathrm{E}_1(\lambda)}{\beta^2(\lambda+2)} - \frac{2\exp(\lambda)\mathrm{E}_1(\lambda)}{\beta^2(\lambda+2)^2}, \tag{2.20}$$

$$E\left(\frac{X^{\beta}\ln X}{\left(\lambda+X^{\beta}\right)^2}\right) = \frac{1}{\beta\lambda}E\left(\frac{(\lambda+W)\left(\ln\lambda - \ln(\lambda+W)\right)}{\lambda+W+1}\right)$$

$$== \frac{1}{\beta\lambda}\left[\frac{\exp(\lambda+1)E_1(\lambda+1) - 1}{\lambda+2}\right], \tag{2.21}$$

$$E\left(\frac{X^{\beta}\left(\ln X\right)^{2}}{\left(\lambda+X^{\beta}\right)^{2}}\right)=\frac{2}{\beta^{2}\lambda\left(\lambda+2\right)}\left(\exp\left(\lambda+1\right)\mathrm{E}_{1}\left(\lambda+1\right)-1\right). \tag{2.22}$$

2.7. Stochastic ordering

This concept plays a central role in the evaluation of the comparative behavior of positive continuous random variables. Let $X_l \sim P\,UGD(\beta_l, \lambda_l), l = 1,2$ with pdf and cdf given by $f_l(x)$ and $F_l(x)$, respectively. Then X_1 is smaller than X_2 in the

- stochastic order $\left(X_1 \leq_{\mathrm{st}} X_2\right)$ if $F_1\left(x\right) \geq F_2\left(x\right)$ for all x;
- hazard rate order $\left(X_1 \leq_{\mathrm{hr}} X_2\right)$ if $h_1\left(x\right) \geq h_2\left(x\right)$ for all x;
- likelihood ratio order $\left(X_1 \leq_{\mathrm{lr}} X_2\right)$ if $\frac{f_1\left(x\right)}{f_2\left(x\right)}$ decreases with increasing x.

It is known that $X_1 \leq_{\mathrm{lr}} X_2 \Rightarrow X_1 \leq_{\mathrm{hr}} X_2 \Rightarrow X_1 \leq_{\mathrm{st}} X_2$ (Shaked and Shanthikumar, 1994). The following theorem establishes that the PUGD is ordered with respect to the likelihood ratio ordering.

Theorem: Let $X_1 \sim PUGD(\beta_1, \lambda_1)$ and $X_2 \sim PUGD(\beta_2, \lambda_2)$. If $\beta_1 = \beta_2 = \beta$ and $\lambda_1 < \lambda_2$, then $(X_1 \leq_{lr} X_2)$.

Proof:

$$\frac{f_1\left(x\right)}{f_2\left(x\right)}=\frac{\beta_1\lambda_1\left(x^{\beta_1}+\lambda_1\right)\left(\lambda_2+2\right)x^{2\beta_2+1}\exp\left(-\lambda_1\left(x^{-\beta_1}-1\right)\right)}{\beta_2\lambda_2\left(\lambda_1+2\right)\left(x^{\beta_2}+\lambda_2\right)x^{2\beta_1+1}\exp\left(-\lambda_2\left(x^{-\beta_2}-1\right)\right)}.$$

The natural logarithm of the ratio of the pdfs is

$$\ln\left[\frac{f_1\left(x\right)}{f_2\left(x\right)}\right]=L=\ln\left(\beta_1\right)+\ln\left(\lambda_1\right)+\ln\left(x^{\beta_1}+\lambda_1\right)-\lambda_1x^{-\beta_1}+\lambda_1+\ln\left(\lambda_2+2\right)+\left(2\beta_2+1\right)\ln x$$

$$-\ln\left(\lambda_1+2\right)-\left(2\beta_1+1\right)\ln x-\ln\left(\beta_2\right)-\ln\left(\lambda_2\right)-\ln\left(x^{\beta_2}+\lambda_2\right)+\lambda_2x^{-\beta_2}-\lambda_2$$

Differentiating the natural log of the ratio with respect to x yields

$$\frac{dL}{dx}=\frac{\beta_1x^{\beta_1-1}}{\lambda_1+x^{\beta_1}}+\beta_1\lambda_1x^{-\beta_1-1}+\frac{2\beta_2+1}{x}-\frac{2\beta_1+1}{x}-\frac{\beta_2x^{\beta_2-1}}{\lambda_2+x^{\beta_2}}-\beta_2\lambda_2x^{\beta_2-1}.$$

If $\beta_1 = \beta_2 = \beta$,

$$\frac{dL}{dx} = \beta\left(\lambda_2 - \lambda_1\right)\left[\frac{x^{\beta-1}}{\left(\lambda_1 + x^{\beta}\right)\left(\lambda_2 + x^{\beta}\right)} - x^{-\beta-1}\right]$$

$$= -\frac{\beta\left(\lambda_2 - \lambda_1\right)x^{-\beta-1}\left(\lambda_1\lambda_2 + \lambda_1 x^{\beta} + \lambda_2 x^{\beta}\right)}{\left(\lambda_1 + x^{\beta}\right)\left(\lambda_2 + x^{\beta}\right)},$$

which is less than 0 if $\lambda_1 < \lambda_2$. The possession of the likelihood ratio ordering property by the PUGD is an added advantage for the distribution, as several distributions on the unit interval lack the property, in particular, beta, kumaraswamy (Kum), unit gamma and arcsine distributions, among others, lack the property. This theoretical property makes the PUGD suitable for applications in reliability engineering, economics and quality control, where monotonic comparisons are needed.

2.8. Conditional moment, mean residual life function and mean past lifetime function

Conditional moment plays a central role in probability theory and economics. Conditional moment representations for dependent random variables were studied by Bryc (1996). In turn, DomÃŋnguez and Lobato (2004) suggested a simple and consistent estimation procedure for economic models using conditional moments. The conditional moment for PUGD(β, λ) is provided in Lemma 2.1.

Lemma 2.1: Suppose that $X \sim \text{PUGD}(\beta, \lambda)$. The nth conditional moment of X is

$$E\left(X^r \middle| X > t\right) = \frac{\lambda^{\frac{r}{\beta}}\exp(\lambda)}{(\lambda+2)S(x)}\left[\Gamma\left(2-\frac{r}{\beta},\lambda\right) - \Gamma\left(2-\frac{r}{\beta},\frac{\lambda}{t^{\beta}}\right) + \Gamma\left(1-\frac{r}{\beta},\lambda\right) - \Gamma\left(1-\frac{r}{\beta},\frac{\lambda}{t^{\beta}}\right)\right].$$

Proof: On the basis of PUGD, we have

$$E\left(X^r \middle| X > t\right) = I_r(x) = \frac{1}{S(x)}\int_t^1 x^r f(x)\mathrm{d}x = \frac{\lambda^{\frac{r}{\beta}}\exp(\lambda)}{(\lambda+2)S(x)}\int_0^{\frac{\lambda}{t^{\beta}}} u^{1-\frac{r}{\beta}}\left(1+u^{-1}\right)\exp(-u)\,du$$

$$= \frac{\lambda^{\frac{r}{\beta}} \exp(\lambda)}{(\lambda+2)S(x)}\left[\Gamma\left(2-\frac{r}{\beta},\lambda\right)-\Gamma\left(2-\frac{r}{\beta},\frac{\lambda}{t^{\beta}}\right)+\Gamma\left(1-\frac{r}{\beta},\lambda\right)-\Gamma\left(1-\frac{r}{\beta},\frac{\lambda}{t^{\beta}}\right)\right],$$

which concludes the proof of Lemma 2.1. □

Mean residual life function (MRLF) is among the functions that are of interest in both parametric and nonparametric statistics. It is used to set rates and benefits for life insurance and to analyze survivorship in the biomedical sciences (Guess and Proschan, 1988). It refers to the expected additional time given that an item has survived up to time *x*. Adopting the definition in Dey et al. (2017), the MRLF for PUGD is defined as

$$m(x) = \frac{I_1(x)}{S(x)} - x$$

$$= \frac{\frac{\lambda^{\frac{1}{\beta}} \exp(\lambda)}{(\lambda+2)}\left[\Gamma\left(2-\frac{1}{\beta},\lambda\right)-\Gamma\left(2-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)+\Gamma\left(1-\frac{1}{\beta},\lambda\right)-\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{x^{\beta}}\right)\right]}{1-\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x^{\beta}}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{x^{\beta}}-1\right)\right)} - x.$$

Let *X* represent the lifetime of a component that has failed at or some time before x. The mean past lifetime function corresponding to PUGD is

$$\tau(x) = x - \frac{J_1(x)}{F(x)} = x - \frac{\lambda^{\frac{r}{\beta}} \exp(\lambda)\left[\Gamma\left(2-\frac{r}{\beta},\frac{\lambda}{x^{\beta}}\right)+\Gamma\left(1-\frac{r}{\beta},\frac{\lambda}{x^{\beta}}\right)\right]}{(\lambda+2)\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x^{\beta}}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{x^{\beta}}-1\right)\right)}.$$

2.9. Quantile function

The quantile function is a useful property of a statistical distribution. It can be used to generate random numbers from a distribution. It also plays a key role in interval estimation and hypothesis testing. Given the PUGD and a real number p satisfying the condition $p \in (0,1)$, the quantile function x_p, satisfies the equation

$$F\left(x_p\right) = p.$$

That is

$$\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x_p{}^{\beta}}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{x_p{}^{\beta}}-1\right)\right)=p. \tag{2.23}$$

Multiplying both sides of (2.23) by $-(\lambda+2)$ yields

$$-\left(\lambda+2+\lambda\left(\frac{1}{x_p{}^{\beta}}-1\right)\right)\exp\left(-(\lambda+2)-\lambda\left(\frac{1}{x_p{}^{\beta}}-1\right)\right)=-(\lambda+2)\,p\exp\left(-(\lambda+2)\right).$$

Applying the concept of the Lambert W function and solving for x_p yields the quantile function

$$x_p=\left[\frac{\lambda}{-2-W_{-1}\left(-p(\lambda+2)\exp(-(\lambda+2))\right)}\right]^{\frac{1}{\beta}},$$

where $W_{-1}(.)$ is the negative branch of the Lambert W function.

2.10. Bonferroni and Lorenz curves

These two curves are useful in studying income inequality in economics. They have also found applications in reliability, demography, insurance and medicine (Pararai et al., 2016). Let the Bonferroni curve and Lorenz curve for PUGD be $B(q)$ and $L(q)$ respectively. Then

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)\mathrm{d}x=\frac{\lambda^{\frac{1}{\beta}}\exp(\lambda)}{p\mu(\lambda+2)}\left[\Gamma\left(2-\frac{1}{\beta},\frac{\lambda}{q^{\beta}}\right)+\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{q^{\beta}}\right)\right]$$

and

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)\mathrm{d}x=\frac{\lambda^{\frac{1}{\beta}}\exp(\lambda)}{\mu(\lambda+2)}\left[\Gamma\left(2-\frac{1}{\beta},\frac{\lambda}{q^{\beta}}\right)+\Gamma\left(1-\frac{1}{\beta},\frac{\lambda}{q^{\beta}}\right)\right],$$

where $q=F^{-1}(p)$.

2.11. Stress-strength reliability

The concept of stress–strength reliability has found applications in fields such as engineering, medical statistics, quality control and econometrics (Biswas and Chakraborty, 2021; Pak et al., 2022). Let X be the strength of a system or

component that is subjected to stress Y. Suppose that $X \sim \text{PUGD}(\beta_1, \lambda_1)$ and $Y \sim \text{PUGD}(\beta_2, \lambda_2)$ are independent PUG distributed random variables with pdf and cdf $f_X(x)$ and $F_Y(y)$ respectively. Then

$$f_X(x) = \frac{\beta_1 \lambda_1 (\lambda_1 + x^{\beta_1})}{(\lambda_1 + 2)} x^{-(2\beta_1+1)} \exp\left(-\lambda_1 (x^{-\beta_1} - 1)\right)$$

and

$$F_Y(x) = \left[\frac{\lambda_2 x^{-\beta_2} + 2}{\lambda_2 + 2}\right] \exp\left(-\lambda_2 (x^{-\beta_2} - 1)\right)$$

Thus, the stress-strength reliability parameter for PUGD is

$$R = P(Y < X) = \int_0^1 F_Y(x) f_X(x)\, dx$$

$$= \frac{\beta_1 \lambda_1 \exp(\lambda_1 + \lambda_2)}{(\lambda_1 + 2)(\lambda_2 + 2)} \int_0^1 (\lambda_1 + x^{\beta_1})\left(\lambda_2 x^{-\beta_2} + 2\right) x^{-(2\beta_1+1)} \exp\left(-(\lambda_1 x^{-\beta_1} + \lambda_2 x^{-\beta_2})\right) dx.$$

If $\beta_1 = \beta_2 = \beta$,

$$R = \frac{\beta \lambda_1 \exp(\lambda_1 + \lambda_2)}{(\lambda_1 + 2)(\lambda_2 + 2)} \int_0^1 (\lambda_1 + x^{\beta})(\lambda_2 x^{-\beta} + 2) x^{-(2\beta+1)} \exp\left(-(\lambda_1 + \lambda_2) x^{-\beta}\right) dx \qquad (2.24)$$

That is, the value of R corresponding to values of its parameters is obtained by evaluating the integral (2.24). If we perform the substitution $x = (\lambda_1 + \lambda_2)^{\frac{1}{\beta}} s^{-\frac{1}{\beta}}$ in (2.24), we obtain

$$R = \frac{\lambda_1 \exp(\lambda_1 + \lambda_2)}{(\lambda_1 + 2)(\lambda_2 + 2)(\lambda_1 + \lambda_2)^2} \left[\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} \int_{\lambda_1+\lambda_2}^{\infty} s^2 \exp(-s) ds + (2\lambda_1 + \lambda_2) \int_{\lambda_1+\lambda_2}^{\infty} s \exp(-s) ds + 2(\lambda_1 + \lambda_2) \int_{\lambda_1+\lambda_2}^{\infty} \exp(-s) ds\right]$$

$$= \frac{\lambda_1 \exp(\lambda_1 + \lambda_2 - \lambda_1 - \lambda_2)}{(\lambda_1 + 2)(\lambda_2 + 2)(\lambda_1 + \lambda_2)^2} \left[\frac{\lambda_1 \lambda_2 \left[(\lambda_1 + \lambda_2)^2 + 2(\lambda_1 + \lambda_2) + 2\right]}{\lambda_1 + \lambda_2} + (2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + 1) + 2(\lambda_1 + \lambda_2)\right]$$

$$R = \frac{\lambda_1 \left[\lambda_1 \lambda_2 \left[(\lambda_1 + \lambda_2)^2 + 2(\lambda_1 + \lambda_2) + 2\right] + (\lambda_1 + \lambda_2)\left((2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + 1) + 2(\lambda_1 + \lambda_2)\right)\right]}{(\lambda_1 + 2)(\lambda_2 + 2)(\lambda_1 + \lambda_2)^3}. \qquad (2.25)$$

It is easily seen from (2.25) that for $\beta_1 = \beta_2 = \beta$, R is independent of β. If the additional condition that $\lambda_1 = \lambda_2 = \lambda$ is imposed, then $R = 1/2$, which is a constant that does not depend on any parameter. Hence, when both the stress and strength random variables are independent and identically distributed power unit

Garima random variables, R is constant. Furthermore, procedures such as the maximum likelihood method and Bayesian technique for estimation of R have been discussed in the literature (Biswas and Chakraborty, 2021).

2.12. Uncertainty measures

Entropy is a measure of the uncertainty of a random variable. The larger the entropy measure, the greater the uncertainty in the data. It is of utmost importance in various fields, among which are communication theory, economics, physics, probability and statistics. Quite a number of entropy measures have been suggested in the literature. In this segment of the paper, we focus on the derivation of the Rényi entropy measure based on the PUGD. Here, the entropy measure is mathematically defined by

$$RE = \frac{1}{\delta - 1} \ln \left[\int_0^1 f^{\delta}(x) dx \right], \delta \neq 1, \delta > 0.$$

$$= \frac{1}{\delta - 1} \ln \left[\left(\frac{\beta \lambda}{\lambda + 2} \right)^{\delta} \int_0^1 \left(1 + \lambda x^{\beta}\right)^{\delta} x^{-\delta(2\beta+1)} \exp\left(-\delta\lambda\left(x^{-\beta} - 1\right)\right) dx \right]$$

Let $t = x^{-\beta} - 1$. Then $x^{-\beta} = t + 1, x^{\beta} = \frac{1}{t+1}, x = (t+1)^{-\frac{1}{\beta}}$ and $dx = -\frac{1}{\beta}(t+1)^{-\frac{1}{\beta}-1} dt$

Therefore,

$$RE = \frac{1}{\delta - 1} \ln \left[\frac{1}{\beta} \left(\frac{\beta \lambda}{\lambda + 2} \right)^{\delta} \int_0^{\infty} \left(\lambda(t+1) + 1\right)^{\delta} (t+1)^{\delta + \frac{\delta}{\beta} - \frac{1}{\beta} - 1} \exp(-\delta\lambda t) dt \right].$$

Notably,

$$\left(\lambda(t+1) + 1\right)^{\delta} = \sum_{j=0}^{\infty} \binom{\delta}{j} (\lambda + 1)^{\delta - j} \lambda^j t^j.$$

As a consequence,

$$RE = \frac{1}{\delta - 1} \ln \left[\frac{1}{\beta} \left(\frac{\beta \lambda}{\lambda + 2} \right)^{\delta} \sum_{j=0}^{\infty} \binom{\delta}{j} (\lambda + 1)^{\delta - j} \lambda^j \int_0^{\infty} t^j (t+1)^{(\delta - 1)(1 + \frac{1}{\beta})} \exp(-\delta\lambda t) dt \right]$$

$$=\frac{1}{\delta-1}\ln\left[\frac{1}{\beta}\left(\frac{\beta\lambda}{\lambda+2}\right)^{\delta}\sum_{j=0}^{\infty}\binom{\delta}{j}(\lambda+1)^{\delta-j}\lambda^{j}\Gamma(j+1)U\left(j+1,(\delta-1)\left(1+\frac{1}{\beta}\right)+j+2,\delta\lambda\right)\right],$$

where, for $a>0$,

$$U(a,b,c)=\frac{1}{\Gamma(a)}\int_{0}^{\infty}t^{a-1}(t+1)^{b-a-1}e^{-zt}dt$$

is Tricomi's confluent hypergeometric function (Slater, 1960; Fitzgerald, 2000). An alternative entropy measure for the PUGD is the Shannon entropy (S_E) defined by

$$S_E=-E(\ln f(X))$$

$$=-\ln\left(\frac{\beta\lambda}{\lambda+2}\right)-\lambda+\lambda E\left(X^{-\beta}\right)-E\left(\ln(X^{\beta}+\lambda)\right)+(2\beta+1)E(\ln(X))$$

$$=-\ln\left(\frac{\beta\lambda}{\lambda+2}\right)-\lambda+\frac{\lambda\left(\lambda^{2}+3\lambda+3\right)}{\lambda(\lambda+2)}-\ln(\lambda+1)-\frac{1}{\lambda+3}+\frac{1}{\lambda+2}-\frac{2}{\lambda+2}-\frac{1}{\beta(\lambda+2)}$$

$$=1-\ln\left(\frac{\beta\lambda}{\lambda+2}\right)-\ln(\lambda+1)-\frac{1}{\lambda+3}-\frac{1}{\beta(\lambda+2)}.$$

Lad et al. (2015) proposed a measure of uncertainty called extropy. It is of immense applicability in the assessment of the accuracy of predicting distributions based on the total log scoring approach (Gemeay et al., 2024). Based on $\text{PUGD}(\beta,\lambda)$, extropy is given by

$$E=-\frac{1}{2}\int_{0}^{1}f^{2}(x)dx$$

$$=-\frac{\beta^{2}\lambda^{2}}{2(\lambda+2)^{2}}\int_{0}^{1}x^{-2(2\beta+1)}\left(x^{\beta}+\lambda\right)^{2}\exp\left(-2\lambda\left(\frac{1}{x^{\beta}}-1\right)\right)dx$$

Let $z=2\lambda x^{-\beta}$. Then

$$E=-\frac{2^{-5-\frac{1}{\beta}}\beta\lambda^{-\frac{1}{\beta}}}{(\lambda+2)^{2}}\int_{2\lambda}^{\infty}z^{3+\frac{1}{\beta}}\left(2z^{-1}+1\right)^{2}\exp(-z)dz$$

$$=-\frac{2^{-5-\frac{1}{\beta}}\beta\lambda^{-\frac{1}{\beta}}}{(\lambda+2)^{2}}\left[4\Gamma\left(2+\frac{1}{\beta},2\lambda\right)+4\Gamma\left(3+\frac{1}{\beta},2\lambda\right)+\Gamma\left(4+\frac{1}{\beta},2\lambda\right)\right].$$

2.13. Order statistics

Consider a random sample $X_1, X_2, ..., X_n$ of size n from PUGD(β, λ). If $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are the associated order statistics, the pdf $f_{X_{(k)}}(x)$ of the kth order statistic based on the distribution is

$$f_{X_k}(x) = \frac{n!}{(k-1)!(n-k)!}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j} f(x)F^{j+k-1}(x)$$

$$= \frac{\beta\lambda n!}{(\lambda+2)(k-1)!(n-k)!}\sum_{j=0}^{n-k}(-1)^j\binom{n-k}{j}\left(\frac{\lambda+x^\beta}{x^{2\beta+1}}\right)\left[1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x^\beta}-1\right)\right]^{j+k-1}\exp\left[-\lambda(j+k)\left(\frac{1}{x^\beta}-1\right)\right]$$

$$= \frac{\beta n!}{(k-1)!(n-k)!}\sum_{m=0}^{l}\sum_{l=0}^{j+k-1}\sum_{j=0}^{n-k}\left[\begin{array}{l}(-1)^{j+l+m}\binom{l}{m}\binom{j+k-1}{l}\binom{n-k}{j}\left(\frac{\lambda+x^\beta}{x^{2\beta+m\beta+1}}\right)\left(\frac{\lambda}{\lambda+2}\right)^l \\ \times\exp\left[-\lambda(j+k)\left(\frac{1}{x^\beta}-1\right)\right]\end{array}\right].$$

The rth moment of the kth order statistic for the PUGD is

$$E\left(X_{(k)}{}^r\right) = \frac{\beta n!}{(k-1)!(n-k)!}\sum_{m=0}^{l}\sum_{l=0}^{j+k-1}\sum_{j=0}^{n-k}\left[\begin{array}{l}(-1)^{j+l+m}\binom{l}{m}\binom{j+k-1}{l}\binom{n-k}{j}\left(\frac{\lambda}{\lambda+2}\right)^l\exp\left[\lambda(j+k)\right] \\ \times\int_0^1 x^{r-2\beta-m\beta-1}\left(\lambda+x^\beta\right)\exp\left[-\frac{\lambda(j+k)}{x^\beta}\right]dx\end{array}\right].$$

Let $v = \frac{\lambda(j+k)}{x^\beta}$. Then

$$E\left(X_{(k)}{}^r\right) = \frac{n!}{(k-1)!(n-k)!}\sum_{m=0}^{l}\sum_{l=0}^{j+k-1}\sum_{j=0}^{n-k}\left[\begin{array}{l}(-1)^{j+l+m}\binom{l}{m}\binom{j+k-1}{l}\binom{n-k}{j}\lambda^{l+\frac{r}{2}-m}(j+k)^{\frac{r}{\beta}-m-2}\left(\frac{1}{\lambda+2}\right)^l \\ \times\exp\left[\lambda(j+k)\right] \\ \times\left[\Gamma\left(-\frac{r}{\beta}+m+2,\lambda(j+k)\right)+(j+k)\Gamma\left(-\frac{r}{\beta}+m+1,\lambda(j+k)\right)\right]\end{array}\right].$$

3. Estimation

In this part of the article, five estimation methods for estimating parameters of the PUGD are considered: maximum likelihood estimation (ML), least squares estimation (LS), weighted least squares estimation (WL), Anderson–Darling estimation (AD) and Cramer–Von Mises estimation (CVM). An explicit discussion of each of the procedures is presented below.

3.1. ML method

Let $x_1, x_2, ..., x_n$ be observations corresponding to a random sample $X_1, X_2, ..., X_n$ of size *n* from PUGD(β, λ). Given that $\boldsymbol{\Psi} = (\beta, \lambda)'$ is the vector of the concerned model parameters, the log-likelihood function may be written as

$$\ell(\boldsymbol{\Psi}) = n \ln \beta + n \ln \lambda + \sum_{i=1}^{n} \ln\left(x_i^{\beta} + \lambda\right) - n \ln(\lambda + 2) - (2\beta + 1) \sum_{i=1}^{n} \ln x_i - \lambda \sum_{i=1}^{n} x_i^{-\beta} + n\lambda.$$

The maximum likelihood estimates (MLEs) $\hat{\beta}_{MLE}$ and $\hat{\lambda}_{MLE}$ of β and λ respectively are determined by solving the simultaneous equations $\frac{\partial \ell(\boldsymbol{\Psi})}{\partial \beta} = 0$ and $\frac{\partial \ell(\boldsymbol{\Psi})}{\partial \lambda} = 0$. Accordingly, the equations are

$$\frac{\partial \ell(\boldsymbol{\Psi})}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{x_i^{\beta} \ln x_i}{x_i^{\beta} + \lambda} - (2\beta + 1) 2 \sum_{i=1}^{n} \ln x_i - \lambda \sum_{i=1}^{n} x_i^{-\beta} + n\lambda = 0$$

$$\frac{\partial \ell(\boldsymbol{\Psi})}{\partial \lambda} = \frac{n}{\beta} + \sum_{i=1}^{n} \frac{1}{x_i^{\beta} + \lambda} - \frac{n}{\lambda + 2} - \sum_{i=1}^{n} \frac{1}{x_i^{\beta}} + n = 0.$$

It is easily noticed that it is not possible to find an analytical solution of the simultaneous equations. For this reason, a numerical solution of the equations is considered, and the fitdistrplus package in R is used to obtain the MLEs associated with a given data set. The observed information matrix, denoted by $J(\hat{\beta}, \hat{\lambda})$, can easily be derived from the second derivatives of the log-likelihood function evaluated at the maximum likelihood estimates of the parameters in question. Symbolically,

$$J(\hat{\beta}, \hat{\lambda}) = -\begin{pmatrix} \frac{\partial^2 l(\boldsymbol{\Psi})}{\partial \beta^2} & \frac{\partial^2 l(\boldsymbol{\Psi})}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l(\boldsymbol{\Psi})}{\partial \beta \partial \lambda} & \frac{\partial^2 l(\boldsymbol{\Psi})}{\partial \lambda^2} \end{pmatrix} \Bigg| \beta = \hat{\beta}_{MLE}, \lambda = \hat{\lambda}_{MLE},$$

where

$$\frac{\partial^2 l(\boldsymbol{\Psi})}{\partial \beta^2} = -\frac{n}{\beta^2} + \sum_{i=1}^{n} \frac{\beta x_i^{\beta} (\ln x_i)^2}{\left(\lambda + x_i^{\beta}\right)^2} - \lambda \sum_{i=1}^{n} x_i^{-\beta} (\ln x_i)^2;$$

$$\frac{\partial^2 l(\boldsymbol{\Psi})}{\partial\beta\partial\lambda}=\frac{\partial^2 l(\boldsymbol{\Psi})}{\partial\lambda\partial\beta}=-\sum_{i=1}^{n}\frac{x_i^{\beta}\ln x_i}{\left(\lambda+x_i^{\beta}\right)^2}+\sum_{i=1}^{n}x_i^{-\beta}\ln x_i$$

and

$$\frac{\partial^2 l(\boldsymbol{\Psi})}{\partial\beta^2}=-\frac{n}{\lambda^2}+\frac{n}{(\lambda+2)^2}-\sum_{i=1}^{n}\frac{1}{\left(\lambda+x_i^{\beta}\right)^2}.$$

The inverse of the observed information matrix provides asymptotic standard errors for the maximum likelihood estimates, which are useful for finite-sample inference, particularly the construction of Wald confidence intervals and hypothesis tests.

Let l^* be the likelihood function corresponding to a single power unit Garima distributed random variable X. Then

$$l^*=\ln\beta+\ln\lambda+\ln\left(\lambda+x^{\beta}\right)-(2\beta+1)\ln x-\lambda x^{-\beta}+\lambda-\ln(\lambda+2).$$

Consequently, we have the following partial derivatives:

$$\frac{\partial^2 l^*}{\partial\beta^2}=-\frac{1}{\beta^2}+\frac{\lambda x^{\beta}(\ln x)^2}{\left(\lambda+x^{\beta}\right)^2}-\lambda(\ln x)^2 x^{-\beta};$$

$$\frac{\partial^2 l^*}{\partial\beta\partial\lambda}=\frac{\partial^2 l^*}{\partial\lambda\partial\beta}-\frac{x^{\beta}\ln x}{\left(\lambda+x^{\beta}\right)^2}+x^{-\beta}\ln x$$

and

$$\frac{\partial^2 l^*}{\partial\lambda^2}=-\frac{1}{\lambda^2}+\frac{1}{(\lambda+2)^2}-\frac{1}{\left(\lambda+x^{\beta}\right)^2}.$$

The expected information matrix I^* also plays a key role, especially in Bayesian inference based on the Jeffery prior. Mathematically,

$$I^*=-E\begin{pmatrix}\dfrac{\partial^2 l^*}{\partial\beta^2} & \dfrac{\partial^2 l^*}{\partial\beta\partial\lambda}\\ \dfrac{\partial^2 l^*}{\partial\lambda\partial\beta} & \dfrac{\partial^2 l^*}{\partial\lambda^2}\end{pmatrix}=\begin{pmatrix}I_{\beta\beta}^* & I_{\beta\lambda}^*\\ I_{\beta\lambda}^* & I_{\lambda\lambda}^*\end{pmatrix},$$

where

$$I_{\beta\beta}^{*}=-E\left(\frac{\partial^2 l^{*}}{\partial\beta^2}\right)=\frac{1}{\beta}\left[1-\frac{2\left(\exp(\lambda+1)\mathrm{E}_1(\lambda+1)-1\right)}{\lambda+2}+\frac{2\lambda}{(\lambda+2)^2}+\frac{2\lambda\exp(\lambda)\mathrm{E}_1(\lambda)}{(\lambda+2)}-\frac{2\lambda\exp(\lambda)\mathrm{E}_1(\lambda)}{(\lambda+2)^2}\right],$$

$$I_{\beta\lambda}^{*}=-E\left(\frac{\partial^2 l^{*}}{\partial\beta\partial\lambda}\right)=I_{\lambda\beta}^{*}=-E\left(\frac{\partial^2 l^{*}}{\partial\lambda\partial\beta}\right)=\frac{1}{\beta\lambda}\left[\frac{\exp(\lambda+1)\mathrm{E}_1(\lambda+1)-1}{\lambda+2}+\frac{\exp(\lambda)\mathrm{E}_1(\lambda)}{(\lambda+2)}-1\right]$$

and

$$I_{\lambda\lambda}^{*}=-E\left(\frac{\partial^2 l^{*}}{\partial\lambda^2}\right)=-\frac{1}{\lambda^2}-\frac{1}{(\lambda+2)^2}+\frac{\lambda+\exp(\lambda+1)\mathrm{E}_1(\lambda+1)}{\lambda^2(\lambda+2)}.$$

3.2. LS method

Let $x_1, x_2, ..., x_{(n)}$ be the observed values of the order statistics $X_1, X_2, ..., X_{(n)}$ from a random sample based on $\mathrm{PUGD}(\beta,\lambda)$. The LS estimates are obtained via the minimization of the following function with respect to β and λ:

$$LS(\boldsymbol{\Psi})=\sum_{i=1}^{n}\left[F\left(x_{(i)}\right)-\frac{i}{n+1}\right]^2$$

$$=\sum_{i=1}^{n}\left[\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x_{(i)}^{\beta}}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{x_{(i)}^{\beta}}-1\right)\right)-\frac{i}{n+1}\right]^2.$$

3.3. WLS method

The weighted least squares estimates of the parameters of $\mathrm{PUGD}(\beta,\lambda)$ are those values of β and λ that minimize

$$WLS(\boldsymbol{\Psi})=\sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left[F\left(x_{(i)}\right)-\frac{i}{n+1}\right]^2$$

$$=\sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left[\left(1+\frac{\lambda}{\lambda+2}\left(\frac{1}{x_{(i)}^{\beta}}-1\right)\right)\exp\left(-\lambda\left(\frac{1}{x_{(i)}^{\beta}}-1\right)\right)-\frac{i}{n+1}\right]^2.$$

3.4. AD method

The AD estimates of the parameters of $\mathrm{PUGD}(\beta,\lambda)$ are obtained by minimization, with respect to β and λ, of the quantity

$$AD(\boldsymbol{\Psi}) = -\mathrm{n} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{2\mathrm{i}-1}{\mathrm{n}} \left[\log F\left(x_{(\mathrm{i})}\right) + \log S\left(x_{(\mathrm{n}+1-\mathrm{i})}\right) \right]$$

$$= -\mathrm{n} - \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{2\mathrm{i}-1}{\mathrm{n}} \left[\begin{array}{l} \log\left[\left(1 + \dfrac{\lambda}{\lambda+2} \left(\dfrac{1}{x_{(\mathrm{i})}^{\beta}} - 1 \right) \right) \exp\left(-\lambda \left(\dfrac{1}{x_{(\mathrm{i})}^{\beta}} - 1 \right) \right) \right] \\ -\log\left[1 - \left(1 + \dfrac{\lambda}{\lambda+2} \left(\dfrac{1}{x_{(\mathrm{n}+1-\mathrm{i})}^{\beta}} - 1 \right) \right) \exp\left(-\lambda \left(\dfrac{1}{x_{(\mathrm{n}+1-\mathrm{i})}^{\beta}} - 1 \right) \right) \right] \end{array} \right]$$

3.5. CVM method

To find the CVM estimates of β and λ in PUGD(β, λ), we minimize, with respect to β and λ, the expression

$$CVM\left(\boldsymbol{\Psi}\right) = \frac{1}{12\mathrm{n}} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[F\left(x_{(\mathrm{i})}\right) - \frac{2\mathrm{i}-1}{2\mathrm{n}} \right]^2$$

$$= \frac{1}{12\mathrm{n}} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \left[\left[\left(1 + \frac{\lambda}{\lambda+2} \left(\frac{1}{x_{(\mathrm{i})}^{\beta}} - 1 \right) \right) \exp\left(-\lambda \left(\frac{1}{x_{(\mathrm{i})}^{\beta}} - 1 \right) \right) \right] - \frac{2\mathrm{i}-1}{2\mathrm{n}} \right]^2.$$

4. Simulation

This section is devoted to the effective comparison, via simulations, of the five point estimation methodologies for parameters of a PUGD. For all the sets of parameter values $(\beta, \lambda) = \{(0.5, 0.5), (2, 1)\}$, 1000 samples are generated from a PUGD based on each of the sample sizes n=20, 50, 100, 250, 500 and 1000. The average bias (AB) and mean squared error (MSE) needed for the comparison are computed for each sample size, estimation method and set of parameter values using the formulas

$$\mathrm{AB}\left(\hat{\vartheta}\right) = \frac{\sum_{\mathrm{i}=1}^{1000} \left(\hat{\vartheta}_{\mathrm{i}} - \vartheta \right)}{1000}$$

and

$$\text{MSE}\left(\hat{\vartheta}\right)=\frac{\sum_{i=1}^{1000}\left(\hat{\vartheta}_i-\vartheta\right)^2}{1000},$$

where $\hat{\vartheta}$ is either $\hat{\beta}$ or $\hat{\lambda}$.

In Tables 4.1 and 4.2, the absolute values of the average biases (ABs) and MSEs associated with the various classes of estimates decrease as the sample size increases. This is an indication that the estimates exhibit the consistency property. It can also be observed from the tables that the ML estimates correspond to smallest MSEs, making the ML method the best for estimating the parameters of a PUGD among the five classes of estimators being considered.

Table 4.1. Simulation results based on PUGD when $\beta = 0.5$and $\lambda = 0.5$

Method	n	Average bias		MSE	
	20	0.0329	0.1001	0.0281	0.1676
	50	0.0145	0.0277	0.0103	0.0494
	100	0.0076	0.0116	0.0050	0.0216
ML	250	0.0041	0.0040	0.0020	0.0088
	500	0.0026	0.0010	0.0009	0.0041
	1000	0.0006	0.0026	0.0005	0.0023
	20	-0.0494	0.4921	0.0456	1.1131
	50	-0.0160	0.1260	0.0178	0.1547
	100	-0.0076	0.0533	0.0084	0.0440
LS	250	-0.0014	0.0189	0.0036	0.0143
	500	0.0003	0.0075	0.0017	0.0066
	1000	-0.0023	0.0086	0.0009	0.0035
	20	-0.0416	0.4791	0.0395	1.2900
	50	-0.0082	0.0911	0.0145	0.0949
	100	-0.0019	0.0360	0.0067	0.0327
WLS	250	0.0011	0.0116	0.0027	0.0112
	500	0.0016	0.0040	0.0013	0.0052
	1000	-0.0009	0.0058	0.0007	0.0029
	20	-0.0223	0.3505	0.0316	0.7768
	50	-0.0065	0.0838	0.0129	0.0905
	100	-0.0023	0.0359	0.0063	0.0319
AD	250	0.0003	0.01313	0.0026	0.0113
	500	0.0011	0.0048	0.0013	0.0052
	1000	-0.0013	0.0065	0.0007	0.0029
	20	0.0229	0.2673	0.0558	0.6186
	50	0.0125	0.0685	0.0187	0.1103
	100	0.0065	0.0286	0.0086	0.0381
CVM	250	0.0041	0.0097	0.0036	0.0135
	500	0.0031	0.0030	0.0017	0.0064
	1000	-0.0009	0.0064	0.0009	0.0034

Table 4.2. Simulation results based on PUGD when $\beta = 2$and $\lambda = 1$

Method	n	Average bias		MSE	
	20	0.2372	0.2491	0.8792	1.6985
	50	0.1104	0.0609	0.2724	0.2808
	100	0.0323	0.0312	0.1111	0.0785
ML	250	-0.0004	0.0317	0.0587	0.0430
	500	-0.0042	0.0166	0.0256	0.0189
	1000	-0.0022	0.0115	0.0135	0.0093
	20	-0.2191	1.1265	1.2985	5.0559
	50	-0.0898	0.3869	0.5027	1.2763
	100	-0.0701	0.1538	0.2365	0.2407
LS	250	-0.0387	0.0814	0.1128	0.0915
	500	-0.0207	0.0370	0.0504	0.0341
	1000	-0.0152	0.0244	0.0256	0.0160
	20	-0.1829	1.8891	1.2447	30.3283
	50	-0.0375	0.3159	0.3966	2.4948
	100	-0.0318	0.1020	0.1720	0.1403
WLS	250	-0.0220	0.0571	0.0824	0.0639
	500	-0.0114	0.0256	0.0366	0.0258
	1000	-0.0090	0.0180	0.0185	0.0123
	20	-0.1072	0.9890	0.9257	7.1929
	50	-0.0339	0.2585	0.3428	0.9630
	100	-0.0363	0.1020	0.1516	0.1369
AD	250	-0.0282	0.0620	0.0810	0.0658
	500	-0.0138	0.0274	0.0361	0.0259
	1000	-0.0106	0.0192	0.0185	0.0124
	20	0.1710	0.6318	1.5857	3.0762
	50	0.0698	0.2287	0.5351	0.8919
	100	0.0112	0.0956	0.2346	0.1853
CVM	250	-0.0076	0.0591	0.1124	0.0838
	500	-0.0053	0.0265	0.0502	0.0326
	1000	-0.0075	0.0192	0.0255	0.0156

5. Applications

In this section, we illustrate the flexibility of PUGD by comparing the fit of the model with the fits of the unit power Lindley (UPL) distribution of Karakus et al. (2025) and the UGD, Kumaraswamy (Kum) and beta distributions, using three data sets. Let $f_1(x)$, $f_2(x)$ and $f_3(x)$ denote the pdfs of the UPL, Kum and beta distributions respectively. Then

$$f_1(x) = \frac{\beta\lambda^2}{\lambda+1}\left(1+\left(\frac{x}{1-x}\right)^{\beta}\right)\left(\frac{x}{1-x}\right)^{\beta-1}\left(\frac{1}{1-x}\right)^{2} exp\left[-\lambda\left(\frac{x}{1-x}\right)^{\beta}\right],$$

$$x \in (0,1), \beta > 0, \lambda > 0.$$

$$f_2(x) = \beta\lambda x^{\beta-1}\left(1-x^{\beta}\right)^{\lambda-1}, x \in (0,1), \beta > 0, \lambda > 0.$$

$$f_3(x) = \frac{\Gamma(\beta)\Gamma(\lambda)}{\Gamma(\beta+\lambda)} x^{\beta-1}(1-x)^{\lambda-1}, x \in (0,1), \beta > 0, \lambda > 0.$$

The first data set (Data I) comprises times between failures of secondary reactor pumps, which were modeled by Maya et al. (2024). The data are reported as:

0.2160, 0.0150, 0.4082, 0.0746, 0.0358, 0.0199, 0.0402, 0.0101, 0.0605, 0.0954, 0.1359, 0.0273, 0.0491, 0.3465, 0.0070, 0.6560, 0.1060, 0.0062, 0.4992, 0.0614, 0.5320, 0.0347, 0.1921.

The data in the second set (Data II) constitute death rates in Spain (from March 3 to May 7, 2020) due to COVID-19 infections, as reported by Afify et al. (2022). The data are as follows:

0.3330, 0.5000, 0.5000 ,0.5714, 0.2500, 0.3469, 0.4839, 0.2105, 0.2311, 0.3127, 0.4800, 0.2749, 0.3625, 0.3922,0.3414, 0.3711 ,0.4288, 0.4077 ,0.3939 ,0.4076 ,0.4079, 0.4408, 0.4046, 0.3836, 0.3545, 0.3275 ,0.3162 ,0.3150, 0.3053 ,0.2930, 0.2790 ,0.2685 ,0.2588, 0.2492, 0.2481 ,0.2453, 0.2355, 0.2285, 0.2241, 0.2193, 0.2162, 0.2153, 0.2129, 0.2098, 0.2037, 0.2066, 0.2087, 0.2038, 0.2029, 0.2023, 0.1993, 0.1962, 0.1711, 0.1678, 0.1646, 0.1629, 0.1613, 0.1544, 0.1510, 0.1484, 0.1465, 0.1453, 0.1436, 0.1420, 0.1396, 0.1372

The data in the third set (Data III) refer to maximum flood levels reported in Dumonceaux and Antle (1973). They are:

0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.3235, 0.269, 0.740, 0.418, 0.412, 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.

The parameters of the PUG, UGD, UPL, Kum and beta distributions are estimated via the maximum likelihood method. In order to compare the fits of the four models to each of the three data sets, we make reference to the minus estimated value of the log-likelihood function (L), Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov-Smirnov Statistic (KS), Cramer-von Mises statistic (W) and Anderson-Darling statistic (A). The model that gives the minimum values of the statistics is taken to be the most suitable distribution for the data among the considered models. Table 5.1 contains the maximum likelihood estimates of the parameters of the five fitted distributions alongside the estimated values of the log-likelihood functions, AIC,

BIC, KS, W and A. The numerical results in Table 5.1 reveal that among the five models, PUGD is associated with smallest AIC, BIC, KS, W and A values for each data. As a consequence, it is the best among the five models fitted to the three data sets.

Table 5.1. Maximum likelihood estimates of the parameters of the models fitted to the three data sets and the associated numerical results.

Da -ta	Distri bution	Estimate	SE	L	AIC	BIC	KS	CVM	AD	P-value
I	PUGD	$\hat{\beta} = 0.52$	0.14	22.433	-40.8660	-38.5950	0.0737	0.0159	0.1529	0.9987
		$\hat{\lambda} = 0.36$	0.23							
	UGD	$\hat{\lambda} = 0.0493$	0.01	17.1423	-32.2846	-31.1491	0.2394	0.4890	3.3855	0.1207
	UPLD	$\hat{\beta} = 0.62$	0.10	20.2149	-36.4298	-34.1588	0.1273	0.0664	0.6257	0.8053
		$\hat{\lambda} = 3.41$	0.66							
	Kum	$\hat{\beta} = 0.68$	0.14	20.3296	-36.6592	-34.3883	0.1393	0.0989	0.5755	0.7122
		$\hat{\lambda} = 2.94$	0.96							
	Beta	$\hat{\beta} = 0.63$	0.16	20.0285	-36.0571	-33.7861	0.1542	0.1265	0.6886	0.5918
		$\hat{\lambda} = 3.23$	1.06							
II	PUG	$\hat{\beta} = 2.49$	0.27	58.9822	-113.9644	-109.5851	0.0814	0.0952	0.7633	0.7731
		$\hat{\lambda} = 0.03$	0.02							
	UGD	$\hat{\lambda} = 0.45$	0.05	41.1341	-80.2682	-78.0786	0.2381	1.1225	6.3334	0.0011
	UPL	$\hat{\beta} = 1.79$	0.16	51.8590	-99.7181	-95.3388	0.1493	0.1965	1.6517	0.1055
		$\hat{\lambda} = 4.55$	0.58							
	Kum	$\hat{\beta} = 2.68$	0.27	55.2113	-106.4226	-102.0433	0.1271	0.2054	1.2432	0.2371
		$\hat{\lambda} = 22.13$	6.69							
	Beta	$\hat{\beta} = 4.90$	0.83	57.5743	-111.1486	-106.7692	0.1148	0.1783	1.0518	0.3494
		$\hat{\lambda} = 12.79$	2.23							
III	PUG	$\hat{\beta} = 3.62$	0.72	16.3068	-28.6136	-26.6221	0.1498	0.0523	0.2952	0.7604
		$\hat{\lambda} = 0.04$	0.03							
	UGD	$\hat{\lambda} = 0.89$	0.17	8.2504	-14.5008	-13.5051	0.3225	0.5372	2.7364	0.0312
	UPL	$\hat{\beta} = 1.49$	0.23	9.7671	-15.5342	-13.5428	0.2246	0.2325	1.4461	0.2654
		$\hat{\lambda} = 1.54$	0.28							
	Kum	$\hat{\beta} = 3.36$	0.60	12.8662	-21.7324	-19.7409	0.2109	0.1635	0.9321	0.336
		$\hat{\lambda} = 11.79$	5.36							
	Beta	$\hat{\beta} = 6.76$	2.09	14.0622	-24.1245	-22.1330	0.1988	0.1236	0.7327	0.4081
		$\hat{\lambda} = 9.11$	2.85							

Estimated pdfs and cdfs of the fitted distributions are plotted in Figure 5.1 for each of the data sets. In each case, it can be seen that PUGD outperforms the other models.

6. Conclusion

This article deals with a new useful generalization of the unit Garima distribution, called the power unit Garima distribution (PUGD). Visual inspection of the plots of the pdf of PUGD reveals that it can be negatively or positively skewed and

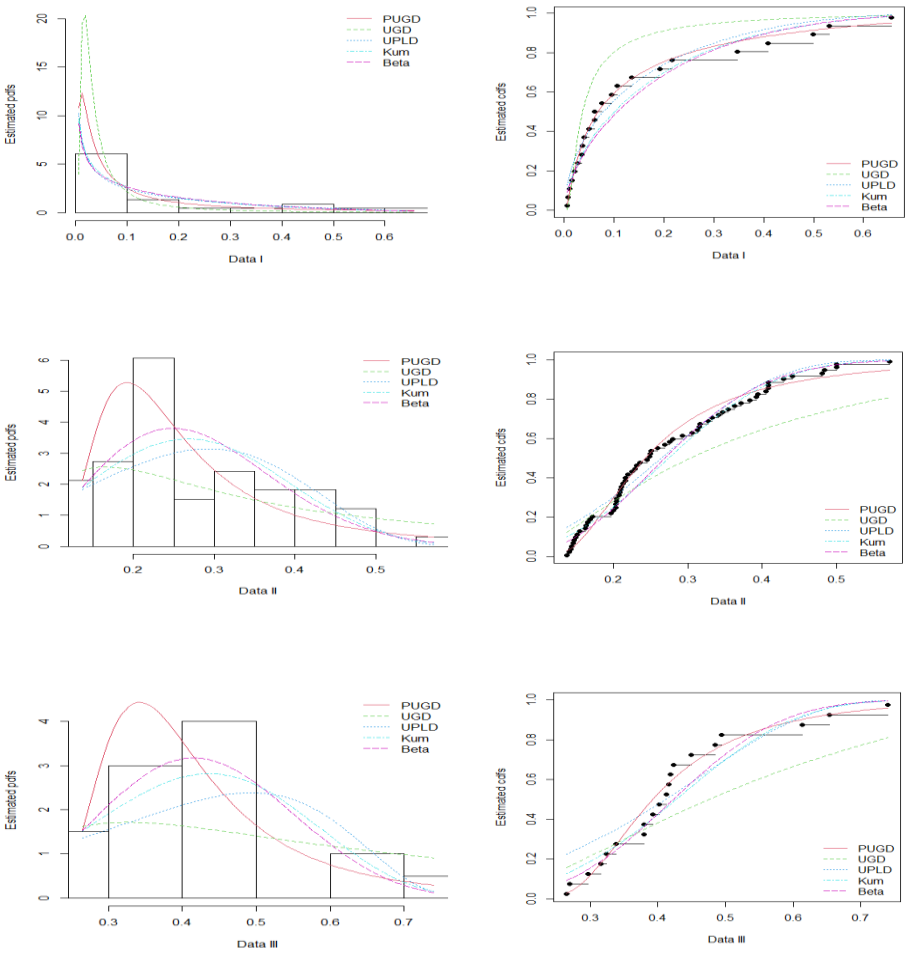


Figure 5.1. Plots of the pdfs and cfds of the distributions fitted to the three data sets

unimodal. On the other hand, the hazard rate function can be nondecreasing. The properties of PUGD expressed in closed form include the mode, quantiles, raw moments, variance, incomplete moments, conditional moments, inverse moments, and Lorenz and Bonferroni curves, among others. Point estimation approaches such as the maximum likelihood, least squares, weighted least squares, Anderson–Darling and Cramer–von Mises methods were discussed and

their performance was compared via a simulation procedure. The obtained simulation results favor the maximum likelihood technique. By way of numerical illustrations, the flexibility as well as the applicability of the new model was illustrated by comparing its fits to three real-world data sets with those of the unit power Lindley, Kumaraswamy and beta distributions. On the basis of the goodness of fit statistics, PUGD was established to be the best for these data compared with the other competing distributions.

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