

A Unified Inventory Model with Production Cost under Negative Exponential Demand, Weibull Deterioration and Partial Backlogging

Charlie Emmanuel Charlie, Sadik Olaniyi Maliki

Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria, e-mails: charlieemmanuelcharlie1603@gmail.com, somaliki@gmail.com

SUMMARY

This study develops an inventory optimization model for perishable items using deteriorating inventory theory. The framework integrates time-dependent demand $D(t) = Ae^{-\alpha_d t}$ with Weibull-based hazard $h(t) = \alpha\beta(t - \gamma)^{\beta-1}e^{-\alpha(t-\gamma)^\beta}$ to characterize component non-instantaneous deterioration. The Time Shift Approach (TSA) is used to ensure that the functions are defined in the real domain. Shortages were allowed and partially backlogged. Numerical illustration was used to validate the model and Python script was employed to obtain the decision variables t_1 , t_2 and T as $t_1 = 5.113399$, $t_2 = 5.163399$ and $T = 30.729455$. Sensitivity analysis identifies demand rate (α_d) and hazard variability (σ) as critical parameters. The minimal interval between t_1 and t_2 indicates that for items with high post-shelf-life deterioration, the optimal strategy requires rapid, sequential actions to preempt stockouts.

Key words: production cost, negative exponential demand, time-dependent Weibull deterioration, partial backlogging, optimization

1. Introduction

For businesses dealing in perishable items such as vegetables, pharmaceuticals, and volatile liquids, maintaining optimal stock levels relative to market demand is a primary concern. If items are held at a time when demand for them is low, then holding and deterioration costs increase, leading to low profit margin; and if there are not enough items when demand is high, shortage cost ensues, with considerable loss of goodwill and confidence on the part of customers (consumers). In creating a balance, proper inventory management becomes indispensable.

In time past, constant deterioration and constant demand was assumed, which does not reflect the reality in the business world. This is compounded by the challenge of stockouts, where customers' satisfaction hinges on whether they are willing to wait for a back order or will seek alternatives elsewhere. Many models in the literature address pricing effects, deterioration effects, and shortage behavior, but do not account for production-related costs or production logistic factors such as unit production cost, setup cost, production rate, or capacity constraints. For manufacturing firms, production cost is a critical determinant of profitability and directly influences optimal pricing, replenishment, and inventory policies. The work of Pius et al. (2025) presents a mathematical inventory model for deteriorating items with time-dependent three-parameter Weibull deterioration and nonlinear price-dependent demand. It incorporates shortages and inventory depletion due to both demand and deterioration. From a production and operations management perspective, their work suffers from a notable limitation, since production logistics was excluded. The absence of production cost considerations limits the practical applicability of the model to manufacturing environments and implicitly restricts its relevance to procurement-based or non-production inventory systems.

Similarly, Pius et al. (2021) outlined a mathematical inventory framework that excludes production cost, which consequently reduces its ability to fully capture the cost structure faced by manufacturing firms. This omission leaves a gap in the theory that this work seeks to address. We have incorporated production cost and related production decisions, thereby enhancing the horizon of application of our model and its usefulness for managerial decision-making especially in production–inventory systems. Our proposed model offers a more realistic approach by treating both the deterioration rate and customer demand as time-sensitive variables, which is especially relevant for seasonal products. We also model production as being dependent on this fluctuating demand and account for partial backlogging. This work builds upon the foundational inventory model established by Akan

et al. (2021); Alireza and Morteza (2019); Ghare and Schrader (1963); Heizer et al. (2024).

While many models have primarily treated demand as a function of price, Pervin et al. (2020) synthesized these concepts into a more comprehensive framework, introducing a deteriorating inventory model that integrates preservation technology with both price- and stock-sensitive demand, and thereby capturing a more complex and realistic market response and as such addressing the gap created in the work of Roy et al. (2020). In many models where production logistics is incorporated, the common assumption of production cost being a linear function of time lacks realism. Addressing this, Xie et al. (2021) broke new ground by developing an overtime production inventory model for deteriorating items with nonlinear price- and stock-dependent demand, though partial backlogging was not considered. In this work, partial backlogging is included in a unified sense, in contrast to the work of Halim et al. (2021), who introduced back-ordering into an Economic Order Quantity (EOQ) model set within an inflationary, cloudy-fuzzy environment.

In the work of Pervin et al. (2019) an inventory model with price-dependent demand is developed and preservation technology under a partial backlogging strategy is implemented with no recourse to a Weibull deterioration hazard function. Similarly, in Das et al. (2020) a generalized inventory model featuring selling-price-dependent demand and time-dependent deterioration is formulated, but it excludes demand-dependent production logistics as we include in this work. Researchers have explored alternative demand patterns: Rahman et al. (2020) introduced an integrated vendor–buyer model with a novel quadratic demand function, incorporating an inspection policy and preservation technology instead of a negative exponential demand function, which appear to constitute a better demand tool, while in Abdeldain et al. (2020) a deteriorating inventory model that integrates preservation technology with both price- and stock-sensitive demand is introduced, capturing a more complex and realistic market response. It is

worthy of note that the deterioration of items is more time-dependent than price and/or stock.

In all of these articles, as well as those of Barman et al. (2021); Das et al. (2020); Dharmesh and Kunal (2025); Heizer et al. (2024); Rahman et al. (2020), production scheduling is a factor that is often overlooked. Moreover, in Sharma et al. (2021) the concept of multivariable demand is used at different stages of the model to improve profit and to control time-dependent deterioration. For future study, they suggested that a Weibull distribution function be used instead of an ordinary time-dependent deterioration of their kind. This forms the motivation of this work.

A critical limitation persists across most inventory literature: the reliance on static demand assumptions – whether constant Akan et al. (2021), time-dependent SanJose et al. (2022), stock-dependent or price-dependent Das et al. (2020), Indrajitsingha et al. (2021) – applied uniformly over the entire inventory cycle. This approach is fundamentally inadequate for managing items with very short shelf-lives and high deterioration rates, where demand is inherently more volatile and dynamic. Despite extensive research, a significant gap remains in the modeling of high-deterioration, low-shelf-life items. Previous studies have largely overlooked the critical aspect of a lead-time-dependent partial backlogging, applying unrealistic full backlogging instead. Furthermore, the common practice of modeling demand based on a single, static parameter for an entire cycle fails to capture the dynamic market behavior for such perishable products. Addressing these gaps, the current study introduces a novel, multi-stage inventory model. Its key innovations include:

1. A two-parameter demand function that varies across different stages of the inventory cycle, reflecting realistic shifts in market demand;
2. The integration of a more accurate, time-dependent deterioration rate;
3. A demand-dependent production strategy that directly links manufacturing to market requirements.

This unified approach provides manufacturers with a more practical and

robust framework for inventory control, ultimately enabling better stock management and enhanced profit maximization for highly perishable goods.

This model incorporates several key realistic assumptions to enhance its applicability:

- **Time-Dependent Deterioration**

The rate of product deterioration is modeled as a function of time, providing a more accurate representation for low-shelf-life items compared to constant-rate assumptions;

- **Partial Backlogging with a first-in-first-out (FIFO) strategy:**

Shortages are permitted and are partially backlogged. During the stockout period, it is assumed that deterioration ceases as there is no physical inventory. Once production resumes, back-ordered demand is fulfilled on a first-come, first-served (FIFO) basis.

The article consists of five sections. Section 2 outlines the assumptions and notations and develops the mathematical formulation of the proposed model. Analytical solutions of the governing differential equations are derived in section 3. Section 4 illustrates the model through a numerical example, followed by a sensitivity analysis. Finally, conclusions are drawn in section 5.

2. Preliminaries

The development of the inventory model is based on the following assumptions:

1. Production and Demand

The production rate is greater than the maximum demand rate to ensure that inventory can be built up initially. The demand rate at time t is given by a negative exponential function:

$$D(t) = Ae^{-\alpha_d t}, \quad \text{where } A > 0 \text{ and } |\alpha_d| > 0 \quad (1)$$

2. Deterioration

Items in inventory are subject to deterioration over time. The deterioration rate follows a three-parameter Weibull distribution characterized by shape parameter $\beta > 0$, scale parameter α and location parameter $\gamma > 0$.

There is no item that deteriorates before time γ . Deterioration applies only when physical inventory is present. Finally, during periods of shortage and backlogging, deterioration is assumed to be zero.

3. Inventory Phases

The inventory cycle is divided into three distinct phases, namely:

- i. phase I ($0 \leq t < t_1$);
- ii. phase II ($t_1 \leq t < t_2$);
- iii. phase III ($t_2 \leq t < T$). These three phases may respectively be seen as interval of time where there is inventory accumulation and inventory depletion due to demand and spoilage, interval of time where there is shortage and partial backlogging, and interval of time where there is replenishment.

4. Shortages and Backlogging

Shortages are allowed and partially backlogged. The backlogging rate is dependent on the waiting time until the next replenishment.

5. Replenishment and Lead Time

There is replenishment just after t_2 . According to Dharmesh and Kunal (2025), the lead time for production and delivery is minimized so that the waiting time is not unnecessarily prolonged. The reorder point depends on the lead time.

6. Costs

Unlike in most classical models where production cost is deliberately excluded, the proposed model considers various cost components, including production cost, holding cost, deterioration cost, shortage cost and backlogging cost. These cost functions are continuous and differentiable.

7. Initial Conditions

The inventory level at the beginning of the cycle is I_0 , a known positive quantity. At t_1 , the inventory level depletes to zero and at time t_2 , shortages begin and backlogged demands are accounted for.

2.1. Notations and parameters

In this section, the notations and parameters used throughout the modeling and analysis of the inventory system are defined for clarity and ease of reference.

Symbols	Description
t	Time variable within a cycle ($0 \leq t \leq T$)
T	Length of the entire inventory cycle
t_1	Time at which inventory depletes to zero (end of positive inventory)
t_2	Time at which shortage and backlogged items are replenished
$I(t)$	Inventory level at time t
$I_1(t)$	Inventory level during production and accumulation phase ($0 \leq t < t_1$)
$I_2(t)$	Inventory level during depletion phase ($t_1 \leq t \leq t_2$)
$I_3(t)$	Inventory level during replenishment phase ($t_2 \leq t < T$)
A	Initial demand rate (units per unit time)
A'	Initial demand rate in the time-shifted framework
α	Deterioration rate
α_d	Demand decay rate (exponential parameter)
σ	Production rate factor relative to demand rate
β	Shape parameter of the Weibull deterioration distribution
γ	Location parameter for deterioration onset

c_h	Unit holding cost per unit time
c_p	Unit production cost
$\theta(t)$	Time-dependent deterioration rate $\alpha\beta(t - \gamma)^{\beta-1}$
c_s	Shortage cost per unit short per unit time
c_b	Backlogging cost per unit backlogged demand per unit time
$D(t)$	Demand rate at time t : $D(t) = Ae^{-\alpha_d t}$
$s(t)$	Rate of satisfied demand (service rate) at time t
$\delta(t)$	Fraction of customers willing to back-order (depends on waiting time)
c_d	Deterioration cost per unit per time

2.2. Mathematical formulation of the model

The inventory cycle length is partitioned into three intervals, namely:

- i. the accumulation/procurement phase, during which inventory builds up due to production exceeding demand;
- ii. the depletion phase, where inventory is reduced to zero by satisfying customer demand without production or replenishment; unmet demand is backlogged and
- iii. the replenishment phase, during which the hitherto unmet but partially backlogged demand is met by new arrivals. Spoilage sets in on unused inventory.

Each phase is governed by a differential equation that captures the dynamics of the proposed inventory system, incorporating both the deterioration of items over time and the varying demand rate. The deterioration of inventory items is modeled using a time-dependent three-parameter Weibull distribution, while the demand follows a negative exponential function of time. The following subsections present the formulation of the differential equations for each phase of the cycle, along with the appropriate initial conditions to ensure a consistent and realistic representation of the inventory

system.

2.3. Inventory accumulation phase

During the initial phase of the cycle, production/procurement is active, and inventory is accumulated since the production/procurement rate exceeds the demand rate. However, inventory is also subject to deterioration over time following a Weibull deterioration distribution. The inventory level $I_1(t)$ during this phase is governed by the following first-order linear differential equation:

$$\begin{aligned} \frac{dI_1(t)}{dt} + \alpha\beta(t - \gamma)^{\beta-1}I_1(t) &= \sigma D(t) - D(t); \quad 0 \leq t < t_1, \\ \text{with } I_1(0) &= I_0 \end{aligned} \quad (2)$$

as its associated initial condition. I_0 is the initial inventory level at the beginning of the cycle. This equation models the net change in inventory as a balance between production, demand and deterioration effects.

2.4. Inventory shortage phase

Here, production/procurement ceases, demand continues, and thus shortages set in. Unmet demand is backlogged for those willing to wait until the next replenishment. The existing inventory is used solely to meet customer demand. According to Pius et al. (2025), no deterioration is considered once there is no inventory.

Therefore, the change in inventory level $I_2(t)$ is described by:

$$\frac{dI_2(t)}{dt} = -D(t) \quad (3)$$

in the interval $t_1 \leq t < t_2$ with the initial condition $I_2(t_1) = 0$.

This indicates that the inventory reaches zero at the end of the accumulation phase. This phase accounts for the depletion of inventory purely due to customers' demand without replenishment or deterioration.

2.5. Replenishment phase

This is the final phase. In this phase, the shortages that occurred due to the absence of inventory to satisfy immediate customer demand are replenished. The portion of hitherto unmet demand is partially backlogged, with the backlogging rate depending on the waiting time. The inventory level $I_3(t)$ in this phase ($t_2 \leq t < T$) is governed by the equation:

$$\frac{dI_3(t)}{dt} + \alpha\beta(t - \gamma)^{\beta-1}I_3(t) = \sigma D(t) - D(t), \quad \text{with } I_3(t_2) = \eta_0 \quad (4)$$

as its associated initial condition.

η_0 represents the quantity of unmet and backlogged demand carried over from the depletion phase.

3. Main result

In this section, the differential equations governing each phase of the inventory cycle are systematically solved. The integrating factor method is employed where necessary to obtain explicit analytic expressions for each inventory level as a function of time.

3.1. Solution for the phases

The solution to equation (2) is:

$$I_1(t) = e^{-\alpha(t-\gamma)^\beta} \left[I_0 e^{\alpha(-\gamma)^\beta} + (\sigma - 1)A \int_0^t e^{\alpha(s-\gamma)^\beta - \alpha_d s} ds \right]. \quad (5)$$

The solution to equation (3) is:

$$I_2(t) = \frac{A}{\alpha_d} \left[e^{-\alpha_d t} - e^{-\alpha_d t_1} \right] \quad (6)$$

and that of equation (4) is:

$$I_3(t) = e^{-\alpha(t-\gamma)^\beta} \left[\eta_0 e^{\alpha(t_2-\gamma)^\beta} + \int_{t_2}^t e^{\alpha(s-\gamma)^\beta - \alpha_d s} ds \right] \quad (7)$$

The production cost function C_p is:

$$C_p = \frac{c_p A(\sigma - 1)}{\alpha_d} \left[1 - e^{-\alpha_d t_1} \right] \quad (8)$$

whereas the holding cost function is:

$$\begin{aligned} C_h &= c_h \left(\int_0^{t_1} I_1(t) dt + \int_{t_2}^T I_3(t) dt \right) \\ &= c_h \left[\int_0^{t_1} e^{-\alpha(t-\gamma)^\beta} \left((\sigma - 1)A \int_0^t e^{\alpha(s-\gamma)^\beta - \alpha_d s} ds + I_0 e^{\alpha(-\gamma)^\beta} \right) dt \right] \\ &\quad + c_h \left[\int_{t_2}^T e^{-\alpha(t-\gamma)^\beta} \left((\sigma - 1)A \int_{t_2}^t e^{\alpha(s-\gamma)^\beta - \alpha_d s} ds + \eta_0 e^{\alpha(t_2-\gamma)^\beta} \right) dt \right] \end{aligned} \quad (9)$$

If C_d denotes the deterioration cost, then:

$$\begin{aligned} C_d &= c_d \left[\frac{(\sigma - 1)A}{\alpha_d} \left(1 - e^{-\alpha_d t_1} + e^{-\alpha_d t_2} - e^{-\alpha_d T} \right) + I_0 \right. \\ &\quad \left. + \eta_0 - e^{-\alpha(T-\gamma)^\beta} \left[\eta_0 e^{\alpha(t_2-\gamma)^\beta} + (\sigma - 1)A \int_{t_2}^T e^{\alpha(s-\gamma)^\beta - \alpha_d s} ds \right] \right] \end{aligned} \quad (10)$$

and the shortage cost function C_s is given by:

$$C_s = c_s A \left[\frac{t_2 - t_1}{\alpha_d} e^{-\alpha_d t_1} + \frac{e^{-\alpha_d t_2} - e^{-\alpha_d t_1}}{\alpha_d^2} \right]. \quad (11)$$

Thus the total cost function $TCF = C_p + C_h + C_d + C_s$

$$\begin{aligned} TCF &= \frac{(\sigma - 1)A}{\alpha_d} \left[(c_p + c_d) - c_d e^{-\alpha_d t_1} + (c_d - c_p) e^{-\alpha_d t_2} - c_d e^{-\alpha_d T} \right] \\ &\quad + c_d (I_0 + \eta_0 - I_3(T)) + c_h \left(\int_0^{t_1} I_1(t) dt + \int_{t_2}^T I_3(t) dt \right) \end{aligned}$$

$$+ c_s A \left(\frac{t_2 - t_1}{\alpha_d} e^{-\alpha_d t_1} + \frac{e^{-\alpha_d t_2} - e^{-\alpha_d t_1}}{\alpha_d^2} \right) \quad (12)$$

and the total average cost TAC is $\frac{TCF}{T}$. Therefore,

$$\begin{aligned} TAC &= \frac{C_p + C_h + C_d + C_s}{T} \\ &= \frac{1}{T} \left[\frac{(\sigma - 1)A}{\alpha_d} \left[(c_p + c_d) - c_d e^{-\alpha_d t_1} + (c_d - c_p) e^{-\alpha_d t_2} - c_d e^{-\alpha_d T} \right] \right. \\ &\quad + c_d (I_0 + \eta_0 - I_3(T)) + c_h \left(\int_0^{t_1} I_1(t) dt + \int_{t_2}^T I_3(t) dt \right) \\ &\quad \left. + c_s A \left(\frac{t_2 - t_1}{\alpha_d} e^{-\alpha_d t_1} + \frac{e^{-\alpha_d t_2} - e^{-\alpha_d t_1}}{\alpha_d^2} \right) \right] \quad (13) \end{aligned}$$

If the partial derivatives of TAC wrt t_1 , t_2 and T are equated to zero then:

$$\frac{\partial TAC}{\partial t_1} = \frac{1}{T} \left[c_h I_1(t_1) + (\sigma - 1) A c_d e^{-\alpha_d t_1} - c_s A e^{-\alpha_d t_1} (t_2 - t_1) \right] = 0 \quad (14)$$

$$\begin{aligned} \frac{\partial TAC}{\partial t_2} &= \frac{1}{T} \left[(\sigma - 1) A (c_p - c_d) e^{-\alpha_d t_2} - c_h I_3(t_2) + \frac{c_s A}{\alpha_d} (e^{-\alpha_d t_1} - e^{-\alpha_d t_2}) \right] \\ &= 0 \quad (15) \end{aligned}$$

$$\frac{\partial TAC}{\partial T} = \frac{I_3(T)}{T} \left[c_h + c_d \alpha \beta (T - \gamma)^{\beta-1} \right] - \frac{TCF}{T^2} = 0 \quad (16)$$

Equations (14), (15) and (16) are simultaneously solved to find the decision variables t_1 , t_2 and T . In this situation, where there is a drift from the real domain to the complex domain, the Time Shift Approach (TSA) becomes an effective tool to ensure that the functions are in the real domain and are defined therein. In TSA, $u = t - \gamma$. This implies that $u_1 = t_1 - \gamma$, $u_2 = t_2 - \gamma$ and $U = T - \gamma$. In this framework, the time intervals $[\gamma, t_1]$, $[t_1, t_2]$ and

$[t_2, T]$ are transformed into $[0, u_1]$, $[u_1, u_2]$ and $[u_2, U]$ respectively, and the system:

$$\begin{cases} \frac{dI_1(t)}{dt} + \alpha\beta(t - \gamma)^{\beta-1}I_1(t) = (\sigma - 1)Ae^{-\alpha_d t} ; I_1(0) = I_0 \\ \frac{dI_2(t)}{dt} = -Ae^{-\alpha_d t} \quad I_2(t_1) = I_1(t_1) = 0 \\ \frac{dI_3(t)}{dt} + \alpha\beta(t - \gamma)^{\beta-1}I_3(t) = (\sigma - 1)Ae^{-\alpha_d t} ; I_3(t_2) = I_2(t_2) = \eta_0 \end{cases} \quad (17)$$

becomes:

$$\begin{cases} \frac{dI_1(u)}{du} + \alpha\beta u^{\beta-1}I_1(u) = (\sigma - 1)A'e^{-\alpha_d u} ; I_1(0) = I'_0 \text{ where, } A' = Ae^{-\alpha_d \gamma} \\ \frac{dI_2(u)}{du} = -A'e^{-\alpha_d u} \quad I_2(u_1) = I_1(u_1) = 0 \\ \frac{dI_3(u)}{du} + \alpha\beta u^{\beta-1}I_3(u) = (\sigma - 1)A'e^{-\alpha_d u} ; I_3(u_2) = I_2(u_2) = \eta_0 \end{cases}$$

Reformulated Cost Functions

The cost functions in the Time Shift Approach are as follows:

$$C_p = c_p(\sigma - 1)A' \left[\int_0^{u_1} e^{-\alpha_d u} du \right] \quad (18)$$

$$C_h = c_h \left[\int_0^{u_1} I_1(u) du + \int_{u_2}^U I_3(u) du \right] \quad (19)$$

$$C_d = c_d \left[\int_0^{u_1} \alpha\beta u^{\beta-1} I_1(u) du + \int_{u_2}^U \alpha\beta u^{\beta-1} I_3(u) du \right] \quad (20)$$

$$C_s = c_s A' \int_{u_1}^{u_2} (u_2 - u) e^{-\alpha_d u} du \quad (21)$$

$$\begin{aligned} TAC' &= \frac{1}{U} [C_p + C_h + C_d + C_s] \\ &= \frac{1}{U} \left[c_p(\sigma - 1)A' \left[\int_0^{u_1} e^{-\alpha_d u} du \right] \right. \end{aligned}$$

$$\begin{aligned}
& + c_h \left[\int_0^{u_1} I_1(u) du + \int_{u_2}^U I_3(u) du \right] + c_d \left[\int_0^{u_1} \alpha \beta u^{\beta-1} I_1(u) du \right. \\
& \left. + \int_{u_2}^U \alpha \beta u^{\beta-1} I_3(u) du \right] + c_s A' \int_{u_1}^{u_2} (u_2 - u) e^{-\alpha_d u} du \Big] \\
& = \frac{1}{U} \left[c_p (\sigma - 1) A' \frac{1 - e^{-\alpha_d u_1}}{\alpha_d} + c_s A' \left[\frac{u_2 - u_1}{\alpha_d} e^{-\alpha_d u_1} + \frac{e^{-\alpha_d u_2} - e^{-\alpha_d u_1}}{\alpha_d^2} \right] \right. \\
& \left. + c_d \left(\frac{(\sigma - 1)}{\alpha_d} \left[1 - e^{-\alpha_d u_1} + e^{-\alpha_d u_2} - e^{-\alpha_d U} \right] \right. \right. \\
& \left. \left. + I_1'(u) + \eta_0 - I_3(U) \right) + c_h \left(\int_0^{u_1} I_1(u) du + \int_{u_2}^U I_3(u) du \right) \right] \quad (22)
\end{aligned}$$

where the solutions for $I_1(u)$, $I_2(u)$ and $I_3(u)$ are:

$$I_1(u) = e^{-\alpha u^\beta} \left[I_0' + (\sigma - 1) A' \int_0^u e^{\alpha s^\beta - \alpha_d s} ds \right] \quad (23)$$

$$I_2(u) = \frac{A'}{\alpha_d} \left(e^{-\alpha_d u} - e^{-\alpha_d u_1} \right) \quad (24)$$

and

$$I_3(u) = e^{-\alpha u^\beta} \left[\eta_0 e^{\alpha u_2^\beta} + (\sigma - 1) A' \int_{u_2}^u e^{\alpha s^\beta - \alpha_d s} ds \right]. \quad (25)$$

and they constitute the solutions of the reformulated model.

In view of what follows, the holding cost integrals can be expressed explicitly as:

$$\begin{aligned}
\int_0^{u_1} I_1(u) du &= I_0' J_0(u_1) + (\sigma - 1) A' \left[J_0(u_1) \int_0^{u_1} e^{\alpha s^\beta - \alpha_d s} ds \right. \\
&\quad \left. - \int_0^{u_1} e^{\alpha s^\beta - \alpha_d s} J_0(s) ds \right], \\
\int_{u_2}^U I_3(u) du &= \eta_0 e^{\alpha u_2^\beta} K(U) + (\sigma - 1) A' \left[K(U) \int_{u_2}^U e^{\alpha s^\beta - \alpha_d s} ds \right. \\
&\quad \left. - \int_{u_2}^U e^{\alpha s^\beta - \alpha_d s} ds \right], \quad (26)
\end{aligned}$$

$$\text{with } J_0(x) = \int_0^x e^{-\alpha u^\beta} du \text{ and } K(x) = \int_{u_2}^x e^{-\alpha u^\beta} du.$$

Complete system

The optimal (u_1, u_2, U) must satisfy:

$$\begin{cases} u_2 - u_1 &= \frac{c_d(\sigma-1)}{A'c_s} + \frac{c_p(\sigma-1)}{c_s} \\ c_h I_3(u_2) &= \frac{c_s A'}{\alpha_d} (e^{-\alpha_d u_1} - e^{-\alpha_d u_2}) + c_d(\sigma-1)e^{-\alpha_d u_2} \\ C_p + C_s + C_d + C_h &= U [c_d(\sigma-1)e^{-\alpha_d U} - c_d I'_3(U) + c_h I_3(U)] \end{cases}$$

where:

$$\begin{aligned} C_p &= c_p(\sigma-1)A' \frac{1 - e^{-\alpha_d u_1}}{\alpha_d}, \\ C_s &= c_s A' \left[\frac{u_2 - u_1}{\alpha_d} e^{-\alpha_d u_1} + \frac{e^{-\alpha_d u_2} - e^{-\alpha_d u_1}}{\alpha_d^2} \right], \\ C_d &= c_d \left(\frac{(\sigma-1)}{\alpha_d} [1 - e^{-\alpha_d u_1} + e^{-\alpha_d u_2} - e^{-\alpha_d U}] + I'_1 + \eta_0 - I_3(U) \right), \\ C_h &= c_h \left(\int_0^{u_1} I_1(u) du + \int_{u_2}^U I_3(u) du \right). \end{aligned}$$

4. Numerical example

Let the following be the parameters of the inventory system in their proper units: $A = 100.00$, $\alpha = 0.1$, $\beta = 1.5$, $\gamma = 0.80$, $\sigma = 1.50$, $c_p = 1.00$, $c_h = 0.5$, $c_d = 0.80$, $c_s = 0.30$, $\alpha_d = 0.3$, $I_0 = 50.00$, and $\eta_0 = 0.5$. Thees numerical values are substituted into the complete system above to obtain

$$u_1 = u_2 - 0.05 \tag{27}$$

$$I_3(u_2) = 200.5625e^{-0.05u_2} \tag{28}$$

$$\begin{aligned} &602 + (-597.75 - 2e^{0.0025})e^{-0.05u_2} - 2e^{-0.05U} + 0.5(I'_1 + \eta_0) - 0.5I_3(U) \\ &+ 0.2 \left(\int_0^{u_1} I_1(u) du + \int_{u_2}^U I_3(u) du \right) = \end{aligned}$$

$$= U \left[0.1e^{-0.05U} - 0.5I'_3(U) + 0.2I_3(U) \right] \quad (29)$$

and we solve with the aid of a Python script to obtain the optimal point as $u_1 = 4.313399$, $u_2 = 4.363399$, and $U = 29.929455$.

4.1. Sensitivity analysis

The sensitivity analysis is carried out by varying each parameter by $\pm 20\%$ and $\pm 50\%$ while the other parameters are kept unchanged. The following tables summarize the whole of the analysis.

Table 1. α sensitivity analysis

Variation	α	$\Delta\alpha$ (%)	u_1	Δu_1	u_2	Δu_2	U	ΔU	Elasticity
-50%	0.15	-50	8.73	+102.39%	8.78	+101.18%	59.92	+90.21%	-1.80
-20%	0.24	-20	5.45	+26.45%	5.50	+26.15%	36.88	+23.21%	-1.16
Base	0.30	0	4.31	0.00%	4.36	0.00%	29.93	0.00%	—
+20%	0.36	+20	3.63	-16.66%	3.69	-16.47%	25.36	-15.25%	-0.76
+50%	0.45	+50	2.91	-33.33%	2.96	-32.95%	20.80	-30.30%	-0.61

Table 2. α_d sensitivity analysis

Variation	α_d	$\Delta\alpha_d$ (%)	u_1	Δu_1	u_2	Δu_2	U	ΔU	Elasticity
-50%	0.05	-50	4.36	0.00%	4.41	0.00%	44.14	+47.47%	0.95
-20%	0.08	-20	4.36	0.00%	4.41	0.00%	34.80	+16.27%	-0.81
Base	0.10	0	4.31	0.00%	4.36	0.00%	29.93	0.00%	—
+20%	0.12	+20	4.36	0.00%	4.41	0.00%	25.66	-14.28%	-0.714
+50%	0.15	+50	4.36	0.00%	4.41	0.00%	20.37	-31.94%	-0.139

5. Discussion and conclusion

The system exhibits weakly divergent behavior with respect to α – changes in α cause proportionally significant changes in U , especially when α is underestimated. It shows convergent behavior with respect to changes in β . The impact on the optimal point is less when the Weibull shape parameter

Table 3. β sensitivity analysis

Change	β	$\Delta\beta$ (%)	u_1	Δu_1 (%)	u_2	Δu_2 (%)	U	ΔU (%)	Behavior
-50%	0.75	-50	4.53	+50.0	4.58	+5.0	32.92	+10.0	Convergent
-20%	1.20	-20	4.40	+20.0	4.45	+2.0	31.13	+4.0	Convergent
Base	1.50	0	4.31	0.0	4.36	0.0	29.93	0.0	—
+20%	1.80	+20	4.30	-1.0	4.32	-1.0	29.33	-2.0	Convergent
+50%	2.25	+50	4.18	-3.0	4.32	-3.0	28.43	-5.0	Convergent

Table 4. σ sensitivity analysis

Change	σ	$\Delta\sigma$ (%)	u_1	Δu_1 (%)	u_2	Δu_2 (%)	U	ΔU (%)	Behavior
-50%	0.75	-50	2.59	-40.0	3.59	-17.8	23.94	-20.0	Convergent
-20%	1.20	-20	3.88	-10.0	4.13	-5.3	28.43	-5.0	Convergent
Base	1.50	0	4.31	0.0	4.36	0.0	29.93	0.0	—
+20%	1.80	+20	4.66	+8.0	4.81	+10.2	31.43	+5.0	Convergent
+50%	2.25	+50	5.18	+20.0	5.48	+25.5	33.52	+12.0	Divergent

β is varied; the resulting changes in the optimal decision variables u_1 , u_2 and U are smaller in percentage terms than the percentage change in β itself. Therefore, the inventory system is relatively insensitive to estimation errors in β . Even the significant mis-estimation of the Weibull shape parameter has only minor effects on the optimal replenishment policy, making this parameter less critical for precise measurement than others such as α , α_d , γ , etc. On the whole, this system shows mixed behavior: convergent for small changes, but becoming divergent for large positive increases, especially for u_2 . That is, the system is highly sensitive (divergent) when σ decreases, but convergent when it increases.

Final transformation

This transformation produces the decision variables as follows:

$$t_1 = u_1 + 0.8 = 5.113399$$

$$t_2 = u_2 + 0.8 = 5.163399$$

$$T = U + 0.8 = 30.729455$$

The figure below explicitly depicts the inventory policy:

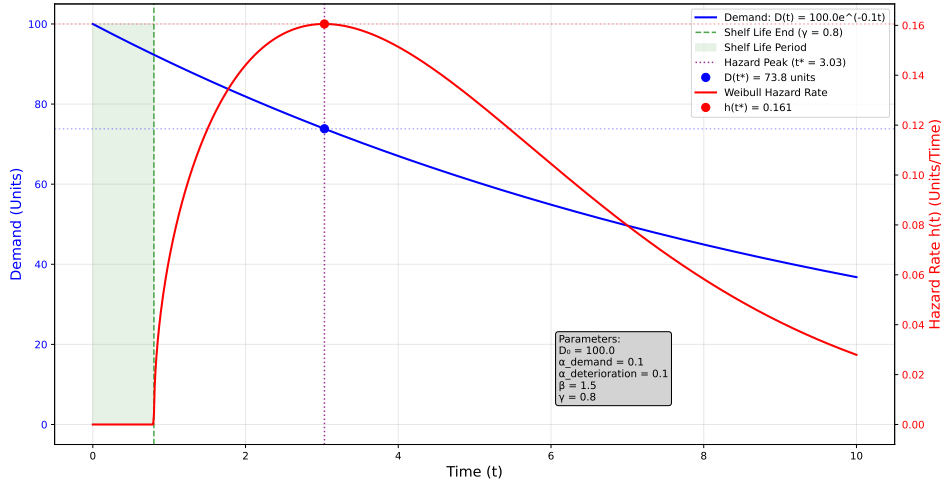


Figure 1. Negative Demand Function / Deterioration Hazard Rate vs. Time

5.1. Interpretation of the diagram

Figure 1 shows a comparison of the quantity of items demanded per unit time with the deterioration hazard rate. They are fundamentally different types of measurement and each requires its own scale. The graph combines them to show how these two critical factors interact over time.

The left Y-axis measures the number of items customers want to buy at a specific time. The blue line represents $D(t) = 100e^{-0.1t}$. This implies that at $t = 3$, the demand is ≈ 74.082 units. The right Y-axis measures the instantaneous rate at which inventory is deteriorating. The red line shows $h(t) = \alpha\beta(t-\gamma)^{\beta-1} \cdot e^{-\alpha(t-\gamma)^{\beta}}$. At $t = 3$, the hazard rate is ≈ 0.16 . At $t = 3$, the customers' demand is relatively high (≈ 74.082 units) but inventory is also under its greatest threat (deteriorating at 0.16 / time). Thus, the point $t = 3$ is the crisis point. A lot is sold and inventory is lost at the fastest rate. Therefore, maximum safety stock is needed to cushion the effect of zero

inventory in the shortage/backlogged phase. At $t = 0$, demand is maximum (100 units) and the hazard rate is zero. Hence, inventory is safely held at this point.

The model's solution yields three critical decision variables that define the optimal policy: $t_1 = 5.11339$ – the optimal time for the first inventory review or partial order point, $t_2 = 5.163399$ – the optimal time for the second action point, likely triggering a major order or production run, and $T = 30.729455$ – the total length of the complete inventory cycle.

The numerical proximity of t_1 and t_2 (separated by only 0.05 time units) is a significant finding. It suggests that in systems with high deterioration rates after a shelf-life threshold, the optimal strategy involves rapid, sequential actions. For instance, for a blood bank manager or a businessman trading vegetables, this translates to a policy where, once the inventory level of a key component dips to a critical level, a swift two-phase procurement or internal transfer process is initiated to avoid stockouts without creating excess perishable stock.

Furthermore, the substantially longer cycle length T indicates that the optimal policy consolidates orders into fewer, larger cycles. This reduces fixed ordering costs and administrative overhead. However, this benefit must be actively managed against the rising risk of deterioration later in the cycle, which underscores the importance of the precise timing dictated by t_1 and t_2 .

For practitioners in healthcare logistics and similar fields, this model provides a decision framework to shift from static, calendar-based ordering to a dynamic schedule aligned with product shelf-life. Implementing the derived optimal timings can significantly reduce wastage of critical items, lower total system costs, and improve service levels by strategically timing replenishment.

5.2. Significance of the first intersection ($t \approx 1.7$)

This is where the hazard rate rises to meet the declining demand. The primary focus is on meeting customers' demand. Inventory policy is mostly driven by demand patterns. Deterioration losses are relatively small compared with sales and inventory depletion is dominantly caused by customers' demand. In short, the first intersection marks the transition from a demand-dominated system to a deterioration-dominated system.

After this point ($t > \approx 1.7$), $h(t) > D(t)$. Deterioration becomes the dominant reason for inventory depletion, as more is lost due to spoilage than to sales. The focus must be shifted to managing and mitigating deterioration losses.

5.3. Significance of the second intersection ($t \approx 4.3$)

This is where the declining hazard rate falls back to meet the declining demand. After the said point, $D(t) > h(t)$, customers' demand once again becomes the dominant reason for inventory depletion. However, both demand and deterioration are now low. The system is less critical, inventory risks are lower, but so are sales opportunities.

5.4. Conclusion

In conclusion, this study has developed and solved an optimization model for managing perishable inventory with time-varying deterioration, using a Weibull hazard function to accurately represent product decay. The primary objective, which was to determine the optimal replenishment schedule that minimizes total system costs, balancing procurement, holding, shortage, and wastage expenses, has been achieved.

5.5. Main contribution

This study formulates and analyzes a unified inventory model that accounts for:

- Negative exponential demand;

- A three-parameter Weibull deterioration rate;
- Partial backlogging based on customer waiting time; and
- Explicit inclusion of production cost.

Recommendations for future work.

Future work may extend this model by incorporating stochastic demand and the multi-echelon supply chains typical in regional blood banking, or by applying machine learning techniques to refine the parameters of the deterioration function based on societal spoilage data.

Statements and declarations

The authors declare that they have no competing or conflicts of interest that are relevant to the content of this work and that they did not receive any funding or support from any organization for the submitted work. All authors contributed equally to the manuscript, and have read and approved the final draft.

Acknowledgements

We sincerely thank the editor and reviewers for taking the time to review our manuscript and providing constructive feedback to improve our manuscript.

REFERENCES

- Abdeldain H., Fergany H., Omar A., Morsy Y. (2020): Probabilistic Inventory Model with Lead Time, Equals Single Scheduling Period and Varying Deteriorating Cost under Constraints. *Al-Azhar Bulletin of Science*, 31(2):1–15.
- Abdelwahab M., Elgazzar S., Mahar K. M. M. (2024): A Framework for Inventory Optimization Based on Machine Learning and Gamification in SMEs Manufacturing Sector. *Journal of Management Systems*, 44(1):117–135.
- Akan M., Albey E., Guter M. G. (2021): Optimal Pricing and Inventory Strategies for Fashion Products under Time-Dependent Interest Rate and Demand. *Journal of Computers and Industrial Engineering*, 154:107149.
- Alireza P., Morteza P. (2019): Optimal control model for finite capacity continuous MRP with deteriorating items. *Journal of Intelligent Manufacturing*, Springer, 30(5):2203–2215.

- Barman H., Bervin M., Roy S. K., Weber G. W. (2021): Back-Ordered Inventory Model with Inflation in a Cloudy Fuzzy Environment. *Journal of Industrial Management Optimization*, 17(4):1913–1921.
- Cardenas-Baron L. E. (2020): Production Quantity with Rework Process at a Single-Stage Manufacturing System with Planned Backorders. *Journal of Computer and Industrial Engineering*, 57:1105–1113.
- Das S. S., Zidan A. M., Manna A. K. (2020): An Application of Preservation Technology in Inventory Control System with Price Dependent Demand and Partial Backlogging. *Alexandria Engineering Journal*, 59(3):1359–1369.
- Dharmesh K. K., Kunal T. S. (2025): Optimal greening efforts, pricing and inventory strategies for non instantaneous deteriorating perishable products under price freshness and green efforts dependent demand with price discount. *International Journal of Mathematics in Operational Research*, 30(1):22–45.
- Ghare P. M., Schrader G. F. (1963): A Model for an Exponential Decaying Inventory. *Journal of Industrial Engineering*, 14:238–243.
- Halim M. A., Paul A., Mahmoud M., Alshahrani B., Alazzawi A. Y., Ismail G. M. (2021): An Overtime Production Inventory Model for Deteriorating Items with Nonlinear Price and Stock Dependent Demand. *Alexandria Engineering Journal*, 60(3):2779–2786.
- Heizer J., Render B., Munson C. (2024): *Operations Management: Sustainability and Supply Chain Management*. Pearson Education Limited, 14th edition.
- Indrajitsingha S. K., Raulo P., Samaila F., Misra U., Raju L. K. (2021): An EMQ Model of Selling-Price-Dependent Demand for Non-Instantaneous Deteriorating Items during the Pandemic COVID-19. *Walaik Journal of Science and Technology*, 18(12):13398.
- Netsuit (2023): Safety Stock: What it is and how to Calculate. Available online: <https://www.netsuit.com/portal/resource/articles/inventory-management/safety-stock.shtml>.
- Nobil A. H., Nobil E., Sedigh A. H. A., Cardenas-Baron L. E., Garza G., Cespedes-Mota A., Loere-Hernandez I., Smith N. R. (2024): Economic Production Quantity Models for an Imperfect Manufacturing System with Strict Inspection. *Ain Shams Engineering Journal*, 15(5):145–165.
- Pervin M., Roy G. W., Weber S. K. (2020): Deteriorating Inventory with Preservation Technology under Price- and Stock-Sensitive Demand. *Journal of Industrial and Management Optimization*, 16(4):1585–1594.
- Pervin M., Roy S. K., Weber G. W. (2019): Multi-item Deteriorating Two-Echelon

- Inventory Model with Price- and Stock-Dependent Demand: A Trade-Credit Policy. *Journal of Industrial & Management Optimization*, 15(31):1345–1354.
- Pius O. N., Chukwu-Walford I. E., Maliki O. S. (2021): Analysis of an inventory system for items with price-dependent demand and time-dependent three-parameter Weibull deteriorating function. *American Journal of Operation Research*, 11(1):199–214.
- Pius O. N., Osisiogu U., Okoro C., Nduka U. (2025): An Inventory Model under Exponential Demand and Weibull Deterioration Rates with Shortage-Dependent Partial Backlogging. *American Journal of Mathematical and Management Science*, 44(1):145–161.
- Rahman M. S., Duary A., Shaikh A. A., Bhunia A. K. (2020): An Application of Parametric Approach for Interval Differential Equation in Inventory Model for Deteriorating Items with Selling Price Dependent Demand. *Neural Computation & Applications*, 32(17):403–421.
- Roy S. K., Pervin G. W., Weber M. (2020): A Two-Warehouse Probabilistic Model with Price Discount on Backorders under Two Levels of Trade Credit Policy. *Journal of Industrial & Management Optimization*, 16(2):553–565.
- San Jose L. A., Sicilia J., Pando V., Alcaide-Lopez de Pablo D. (2022): An Inventory System with Time-Dependent Demand and Partial Backordering under Return on Inventory Investment Maximization. *Journal of Computers & Operation Research*, 145:105861.
- Sharma S., Tyagi A., Verma B. B. (2021): An Inventory Control Model for Deteriorating Items Under Demand Dependent Production with Time and Stock Dependent Demand. *International Journal of Operation and Quantitative Management*, 27(4):321–336.
- Vora V., Gothi U. (2020): Deterministic Inventory Model for Deteriorating Items with Price. *International Journal of Scientific Research*, 7(6):20–28.
- Xie Y., Tai A. H., Ching W. K., Kuo Y. H., Song N. (2021): Joint Inspection and Inventory Control for Deteriorating Items with Time-Dependent Demand Deteriorating Rate. *Annals of Operation Research*, 300(1):225–265.