

Existence, Uniqueness and Continuous Dependence of Solutions for a Noncontractive Caputo Fractional Delay Equation

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SUMMARY

This study establishes a complete qualitative analysis for a class of noncontractive Caputo fractional delay differential equations subject to integral boundary conditions. First, the existence of solutions is proved under mild assumptions using Sadovskii's fixed point theorem, with the noncontractive property explicitly characterized by the derived condition k_{op} . Uniqueness is then established via an innovative application of the fractional Grönwall inequality. Furthermore, a rigorous perturbation framework is developed to prove the continuous dependence of solutions on external parameters, ensuring structural robustness. The theoretical results are validated both analytically and through numerical simulations in MATLAB, which illustrate the system's distinct dynamical behavior across fractional orders $\alpha \in (0, 1)$, highlighting enhanced memory effects at lower orders and transitional dynamics at intermediate values. Our results improve, extend and provide a comprehensive analytical foundation for noncontractive fractional delay systems and contribute significantly to the qualitative theory of fractional differential equations with applications in hereditary and memory-dependent processes, as described in the literature.

Key words: Caputo fractional derivative, noncontractive systems, fractional Grönwall inequality, Sadovskii's fixed point theorem, perturbation analysis

1. Introduction

Studies and research in fractional differential equations are gaining significant attention because of their role as a superior modeling prototype over

their integer-order counterparts. This is due to their proficiency in capturing complex physical and biological phenomena endowed with memory and hereditary properties, as found in viscoelasticity, epidemiology, control theory, etc, Kilbas et al. (2006); Hilfer (2000); Kochubei and Luchko (2019); Podlubny (1999). The non-local nature of fractional operators provides a more accurate description of systems whose future state depends on their entire historical evolution Benchohra et al. (2023); Cui and Zhou (2023); Itatime et al. (2023). Within this rich landscape, fractional boundary value problems (FBVPs) incorporating delay terms and nonlocal conditions offer a particularly potent framework, as they can simultaneously model memory effects, time lags in the system's response, and integrated boundary data Katugampola (2011); Zhou (2024).

The foundational pillars of a well-posed dynamical system are the existence and uniqueness of solutions and their continuous dependence on initial and boundary data. While a substantial body of literature has been devoted to establishing existence results for various classes of FBVPs, often via fixed point theorems in a contractive setting Almezal et al. (2014); Georgiev and Zennir (2020); Yuri (2021), the investigation of *uniqueness* and *continuous dependence* remains comparatively less traversed, especially for systems that are inherently *noncontractive*. The standard Banach Contraction Principle, though powerful, imposes stringent Lipschitz conditions that are often too restrictive for many nonlinear and delay-type fractional dynamics. This limitation necessitates a shift to more sophisticated topological methods, such as Sadovskii's fixed point theorem for condensing maps, which can handle a broader class of problems where contractivity fails.

Yong-ki et al. (2022) considered existence and continuous dependence results by adding nonlocal conditions to a Caputo fractional evolution integer differential equation of order $r \in (1, 2)$ in a Hilbert space with delay. They applied Krasnoselskii's fixed point theorem and the Henry–Grönwall inequality with control conditions utilizing cosine and sine function operators.

Guerfi and Ardjouni (2022) investigated existence using the Schauder fixed point theorem and the uniqueness, continuous dependence and Ulam stability of mild solutions for the nonlinear iterative fractional differential equation:

$$\begin{cases} {}^c D_{0+}^\alpha x(t) = f(x^{[0]}, x^{[1]}(t), x^{[2]} \dots x^{[n]}(t)), & t \in J = [0, T], \\ x(0) = x'(0) = 0, \end{cases}$$

where ${}^c D_{0+}^\alpha$ is the Caputo fractional derivative of order $\alpha \in (1, 2)$ and f is a positive continuous function.

Li and Zhang (2020) discussed a class of semi-linear fractional evolution equations with infinite delay and an almost sectorial operator on an infinite interval in Banach space. They applied properties of analytic semigroups and Schauder's fixed point theorem. They also investigated the existence of a mild solution when the analytic semigroup lacks compactness by the Kuratowski measure of noncompactness and the Darbo–Sadovskii fixed point theorem. However, in their work they failed to explicitly investigate the noncontractive nature of their fractional derivative, as we undertake to do in this paper.

Hatime et al. (2023) studied existence, uniqueness and finite-time stability results for a fractional delayed Newton cooling law equation involving ψ -Caputo fractional derivatives of order $\alpha \in (0, 1)$:

$${}^c D_{0+}^\alpha u(s) = r(u(s) - u_a), \quad s > 0$$

subject to the initial condition $\varphi \in C^1([-\tau, 0], \mathbb{R})$, $u(s) = \varphi(s)$ for all $s \in [-\tau, 0]$, and their solution model was given as $u(s) = (u_0 - u_a)E_\alpha(r(\varphi(s) - \varphi(0))^\alpha) + u_a$, $s > 0$. They applied the Banach fixed point theorem and Henry–Grönwall-type retarded integral inequalities to establish the existence and uniqueness of solutions. Moreover, they used the heat transfer model to derive a new criterion for finite time stability with time delay, but did not investigate the perturbation sensitivity of their system.

Karthikeyan et al. (2023) demonstrated the existence, well-posedness and controllability of a second-order abstract differential system with state-delay using the combination of a measure of non-compactness and Sadovskii's fixed point theorem. However, they did not demonstrate the noncontractive nature of the solution operator utilized.

Karthikeyan et al. (2022a) investigated existence and uniqueness results for multi-point boundary value problems for sequential impulsive fractional differential equations involving the ψ -Hilfer fractional derivative, using the Banach fixed point theorem, Krasnoselskii's theorem and the nonlinear alternative of the Leray–Schauder fixed point theorem. Karthikeyan et al. (2022b) applied the contraction mapping concept and Schaefer's fixed point theorem to establish more generalized solutions for impulsive integro-differential equations involving Ulam–Hyers–Rassias stability results via a ψ -fractional derivative with boundary value conditions. Similarly, Buvaneshwari et al. (2024) applied Banach contraction and Schaefer's fixed point theorem to a coupled system of Caputo–Hadamard fractional differential equations, without investigating how noncontractivity and boundary conditions affected the system's local Lipschitz properties.

Sivaranjani et al. (2025) examined the controllability of an implicit fractional system including state and control delay in terms of $\mathbb{ABC}$, the non-local derivative of the form

$$\begin{cases} {}_0^{\mathbb{ABC}}D_t^\alpha[s(t) - g(t, s(t))] = As(t) + Bc(t - \tilde{t}) + f(t, s_{t,0} {}_0^{\mathbb{ABC}}D^\alpha s(t), c(t)), \\ \quad t \in [0, T] = \mathcal{J}, 0 < \alpha \leq 1, \\ s(t) = \varphi(t), \quad t \in [-\tilde{t}, 0], \\ c(t) = c_0(t), \quad t \in [-\tilde{t}, 0], \end{cases}$$

where ${}_0^{\mathbb{ABC}}D_t^\alpha$ represents the $\mathbb{ABC}$ fractional derivative with order α , $A : D(A) \subset \Omega \rightarrow \Omega$ is an infinitesimal generator of the α -resolvent family τ_α , and S_α for $t \geq 0$ is the solution operator on the Banach space $(\Omega, \|\cdot\|)$. k -set contraction mapping and the Darbo fixed point theorem were applied

to obtain the controllability results.

Inspired by the works above, we investigated a noncontractive Caputo fractional delay integral boundary value problem, a model which uniquely combines a Caputo derivative of order $\alpha \in (0, 1)$, with a persistent delay term $\tau > 0$, and a sophisticated nonlocal boundary condition defined by an integral over the solution's history. The distinction between our noncontractive framework and the conventional contractive one is fundamental: we worked under assumptions that do not guarantee that the mapping defined by our problem is a contraction, thereby venturing into a domain where classical tools like the Banach fixed point theorem are inapplicable. We developed the rigorous methodological framework, theorems, lemmas and propositions needed to establish the existence and uniqueness of a solution to a noncontractive Caputo fractional delay integral boundary value problem (CFDIBVP) in Banach space given by:

$$\begin{cases} {}^c D_{0+}^\alpha u(t) = f(t, u(t), u(t - \tau)) & t \in [0, 1], \quad \tau > 0 \\ u(0) = 0, u(1) = \int_0^1 g(s, u(s), u(s - \tau)) ds, \end{cases} \quad (1)$$

where ${}^c D_{0+}^\alpha$ is the Caputo derivative of order $\alpha \in (0, 1)$, $(t - \tau)$ is the delay term (with memory), and $f, g, u \in AC^m([0, 1], \mathbb{R}^n)$ are continuous functions with integral norm given by:

$$\|u\|_{AC^m} := \sum_{k=0}^{m-1} \sup_{t \in [0, 1]} \|u^{(k)}(t)\| + \int_0^1 \|u^{(m)}(t)\| dt, \text{ for } k = (0, \dots, m-1). \quad (2)$$

Thereby bridging existing gaps in the literature. Our contributions are four-fold. First, we establish the existence of solutions under the mild continuity and compactness conditions (H1)–(H3) by employing Sadovskii's fixed point theorem, effectively bypassing the need for contractivity. Second, we explicitly demonstrate that our system's operator is noncontractive, justifying the need for our advanced topological approach. Third, we prove the uniqueness

of solutions by developing a novel application of the fractional Grönwall inequality, a critical step that relies on careful norm estimations independent of contraction arguments. Finally, we perform a comprehensive perturbation analysis to establish the continuous dependence of solutions on the given functions f and g , ensuring that small changes in the model's parameters lead to only small changes in the solutions. This attribute is vital for the physical reliability and numerical stability of the model.

The paper is structured as follows. Section 2 recaps essential preliminaries. Section 3 is dedicated to the existence result. Section 4 details the noncontractive nature of our problem. Section 5 presents the uniqueness theorem via the fractional Grönwall inequality, and section 6 provides the framework for continuous dependence. We conclude with a summary of our findings.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used subsequently in this paper.

Definition 1. *Kilbas et al. (2006).* A function $f : [a, b] \rightarrow \mathbb{R} \in AC^m([a, b])$ is absolutely continuous if:

- i. $f^{(m-1)} \in AC([a, b])$,
- ii. $f^{(m)}$ exists almost everywhere and $f^{(m)} \in L^1([a, b])$. Hence,

$$f \in AC^m([a, b]) \iff f^{(m-1)} \text{ is absolutely continuous on } [a, b].$$

In the vector-valued case $AC^m([a, b], \mathbb{R}^n)$, each component satisfies the same conditions.

A standard norm on $AC^m([a, b], \mathbb{R}^n)$ is defined as

$$\|f\|_{AC^m} := \sum_{k=0}^{m-1} \sup_{t \in [a, b]} \|f^{(k)}(t)\| + \int_a^b \|f^{(m)}(t)\| dt, \quad (3)$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Equipped with this norm, $AC^m([a, b], \mathbb{R}^n)$ is a Banach space.

Definition 2. Podlubny (1999). The Riemann–Liouville fractional Integral of order $\alpha \in (0, 1)$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined as

$$(I_{0+}^\alpha f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad x \in [a, b]. \quad (4)$$

Definition 3. Caputo and Mainardi (1971). The Caputo fractional derivative of order $\alpha \in (0, 1)$ for a function $f \in AC^m([a, b], \mathbb{R}^n)$ is defined by

$$({}^c D_{a+}^\alpha f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad x \in [a, b],$$

where $m = [\alpha] + 1$ if $\alpha \notin \mathbb{N}_0$, otherwise $m = \alpha$ and $[\alpha]$ is the integer part of α that is not greater than α .

Definition 4. Zeidler (1986). Let $(X, \|\cdot\|)$ be a normed linear space, $x_0 \in X$ and $r > 0$; the closed ball of radius r centered at x_0 is defined by

$$\bar{B}_r(x_0) := \{y \in X : \|y - x_0\| \leq r\}. \quad (5)$$

Definition 5. Pata (2019)(Condensing Mapping) Let X be Banach space, a continuous map $T : D(T) \subset X \rightarrow X$ is condensing if, for every bounded subset $A \subset D(T)$ with $\alpha(A) > 0$, then:

$$\alpha(T(A)) < \alpha(A).$$

Definition 6 (Kuratowski Measure of non-compactness). Let $(X, \|\cdot\|)$ be a Banach space and let $A \subset X$ be bounded set. The Kuratowski measure of non-compactness of A , denoted by $\alpha(A)$ is defined by

$$\alpha(A) := \inf \{d > 0 : A \subset \cup_{k=1}^m A_k, \text{ diam}(A_k) \leq d, m \in \mathbb{N}\},$$

where the diameter of a set A_k is $\text{diam}(A_k) := \sup \{\|x - y\| : x, y \in A_k\}$.

Theorem 7. *Sadovskii (1967). (Sadovskii's Fixed Point Theorem) Let B_r be a convex, closed, and bounded subset of a Banach space X . If $T : B_r \rightarrow B_r$ is condensing, then T has at least one fixed point in B_r .*

Theorem 8. *Fractional Grönwall inequality*

Let $\alpha \in (0, 1)$, $t \in [0, T]$ and let $u(t)$, $a(t)$ and $b(t)$ be non-negative continuous functions on $[0, T]$, with $b(t)$ non-decreasing. Suppose:

$$u(t) \leq a(t) + b(t) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \quad \text{for all } t \in [0, T].$$

Then:

$$u(t) \leq a(t) + \int_0^t K(t-s) a(s) ds,$$

where,

$$K(t) = \frac{b(t)}{\Gamma(\alpha)} t^{\alpha-1} E_{\alpha, \alpha}(b(t)t^\alpha), \quad (6)$$

and $E_{\alpha, \beta}(\cdot)$ is the Mittag-Leffler function Agarwal et al. (2009); Pata (2019); Kilbas et al. (2006); McBride (1979).

3. Existence results and noncontractiveness

Theorem 9. *(Existence of a solution)*

Let $X = AC^1([0, 1], \mathbb{R}^n)$ with norm: $\|u\|_{AC^1} := \sup_{t \in [0, 1]} \|u(t)\| + \int_0^1 \|u'(t)\| dt$. For fixed $r, \tau > 0$, consider the closed, bounded and convex subset $B_r := \{u \in X : \|u\|_{AC^1} \leq r\}$. Assume $f, g : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous nonlinear functions, and let $\varphi : [-\tau, 0] \rightarrow B_r$ be a given delay function satisfying $u(0) = u_0 \in B_r$ and $\varphi(-\tau) = 0$ extend the delay by $u(t) = \varphi(t)$ for $t \in [-\tau, 0]$.

Define the operator $T : B_r \rightarrow X$:

$$(Tu)(t) := I^\alpha[f(s, u(s), u(s-\tau))](t) + h(t)(G(u) - A(u)),$$

$$\text{where } G(u) := \int_0^1 g(s, u(s), u(s-\tau)) ds,$$

$$A(u) := I^\alpha[F_u]$$

and h is the given boundary functional.

Assume:

(H1) Lipschitz continuity

There exist $L_f, L_g \geq 0$ such that for all $t \in [0, 1]$ and all u_i, v_i with $\|u_i\|, \|v_i\| \leq r$:

$$\begin{aligned} \|f(t, u_1, v_1) - f(t, u_2, v_2)\| &\leq L_f(\|u_1 - u_2\| + \|v_1 - v_2\|), \\ \|g(t, u_1, v_1) - g(t, u_2, v_2)\| &\leq L_g(\|u_1 - u_2\| + \|v_1 - v_2\|), \end{aligned}$$

with $h \in C([0, 1], \mathbb{R}^n)$ satisfying $h(0) = 0$, $h(1) = 1$, and $H = \sup_{t \in [0, 1]} \|h(t)\|$.

(H2) Self-map (invariance)

The operator $T : B_r \rightarrow B_r$ maps the ball into itself: such that $T(B_r) \subseteq B_r$.

(H3) Condensing decomposition

T is α -condensing with respect to the Kuratowski measure of non-compactness α :

$$\alpha(T(B)) < \alpha(B),$$

for every bounded $B \subseteq B_r$ with $\alpha(B) > 0$. Furthermore, T admits a decomposition $T = T_1 + T_2$, where T_1 is a k -contraction ($0 < k < 1$) and T_2 is compact (so $\alpha(T_2(B)) = 0$ for all bounded $B \subset B_r$). Therefore if conditions (H1) and (H3) hold, then the Caputo fractional delay integral boundary value problem.

$$\begin{cases} {}^c D_{0+}^\alpha f(s, u(s), u(s - \tau)) & u \in X \\ u(0) = 0, u(1) = \int_0^1 g(s, u(s), u(s - \tau)) ds \end{cases}$$

admits at least one solution $u \in AC^1([0, 1], \mathbb{R}^n)$ by Sadovskii's fixed point theorem. See Udo and Eze (2025) and the references therein.

3.1. Proof of Theorem (9)

Proof: [H1] – **Continuity of T:**

Let $\{u_n\} \subset B_r$ be a sequence such that $u_n \rightarrow u$ in the AC^1 norm. This implies uniform convergence of $u_n \rightarrow u$ and L^1 -convergence of the derivatives $u_n' \rightarrow u'$. Define the sequences:

$$\phi_n(s) := f(s, u_n(s), u_n(s - \tau)),$$

$$\phi(s) := f(s, u(s), u(s - \tau)).$$

By continuity of f and uniform convergence of u_n and the delay terms $u_n(s - \tau)$, it follows

that $\|\phi_n - \phi\|_{L^\infty} \rightarrow 0$.

- i. Convergence of the fractional integral term – we use the kernel estimate and linearity of I^α for each $t \in [0, 1]$:

$$\begin{aligned} \|I^\alpha[\phi_n](t) - I^\alpha[\phi]\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\phi_n(s) - \phi\| ds \\ &= \frac{\|\phi_n - \phi\|_{L^\infty}}{\Gamma(\alpha+1)} t^\alpha \rightarrow 0 \quad \text{uniformly in } t. \end{aligned}$$

- ii. Convergence of the boundary terms:

By the continuity of g and an analogous argument $g_n(s) = g(s, u_n(s), u_n(s - \tau))$ converges uniformly to $g(s, u(s), u(s - \tau))$. Consequently,

$$G(u_n) = \int_0^1 g_n(s) ds \rightarrow \int_0^1 g(s, u(s), u(s - \tau)) ds = G(u).$$

- iii. We consider the boundary residual function (h) , for $h \in AC^1([0, 1], \mathbb{R}^n)$ which is continuous and bounded. Set $H := \sup_{t \in [0, 1]} \|h(t)\| < \infty$, then for each t , we have:

$$\begin{aligned} &\|h(t)(G(u_n) - A(u_n)) - h(t)(G(u) - A(u))\| \\ &\leq H(|G(u_n) - G(u)| + |A(u_n) - A(u)|) \rightarrow 0 \end{aligned}$$

uniformly in t , since and $A(u_n) \rightarrow A(u)$.

iv. Convergence of the full operator:

$$\begin{aligned}
& \|Tu_n(t) - Tu(t)\| \\
& \leq \|I^\alpha[\phi_n](t) - I^\alpha[\phi](t)\| \\
& + \|h(t)\|(|G(u_n) - G(u)| + |A(u_n) - A(u)|) \\
& \leq \frac{\|\phi_n - \phi\|_{L^\infty}}{\Gamma(\alpha + 1)} + H(|G(u_n) - G(u)| + |A(u_n) - A(u)|) \rightarrow 0.
\end{aligned}$$

By uniform convergence of both terms

$$\sup_{t \in [0,1]} \|Tu_n(t) - Tu(t)\| \rightarrow 0 \quad \text{Thus } Tu_n \rightarrow Tu \text{ uniformly.}$$

For convergence of derivatives:

Differentiating the integral term almost everywhere.

For:

$$\int_0^1 \|(Tu_n)'(t) - (Tu)'(t)\| dt.$$

Then:

$$\frac{d}{dt} I^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} \phi'(s) ds - \frac{\phi(0)t^{\alpha-1}}{\Gamma(\alpha)}.$$

Therefore:

$$\begin{aligned}
(Tu_n)'(t) - (Tu)'(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)^{\alpha-1} (\phi_n'(s) - \phi'(s)) ds \\
&- \frac{\phi_n(0) - \phi(0)}{\Gamma(\alpha)} t^{\alpha-1} + h'(t)(G(u_n) - A(u_n)) \\
&- (G(u) - A(u)),
\end{aligned}$$

since $h(t)$ is independent of s and differentiates to $h'(t)$ while $G(u)$, $A(u)$ are constants with respect to t . We take the norm and integrate over

$t \in [0, 1]$ using the triangle inequality:

$$\begin{aligned}
& \int_0^1 \left\| (Tu_n)'(t) - (Tu)'(t) \right\| dt \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^1 \left\| \int_0^t (t-s)^{\alpha-1} (\phi_n'(s) - \phi'(s)) ds \right\| dt \\
& \quad + \frac{\|\phi_n(0) - \phi(0)\|}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} dt \\
& \quad + \int_0^1 \left\| h'(t) \right\| \left\| (G(u_n) - G(u)) - (A(u_n) - A(u)) \right\| dt.
\end{aligned}$$

We now estimate the three terms on the right-hand side separately:

Term 1 (double integral term):

Applying the Fubini theorem to exchange the order of integration.

$$\begin{aligned}
& \int_0^1 \left\| \int_0^t (t-s)^{\alpha-1} (\phi_n'(s) - \phi'(s)) ds \right\| dt \\
& \leq \int_0^1 \int_0^t (t-s)^{\alpha-1} \left\| \phi_n'(s) - \phi'(s) \right\| ds dt \\
& = \int_0^1 \left\| \phi_n'(s) - \phi'(s) \right\| \left(\int_0^t (t-s)^{\alpha-1} dt \right) ds \\
& = \int_0^1 \left\| \phi_n'(s) - \phi'(s) \right\| \cdot \frac{(1-s)^\alpha}{\alpha} ds.
\end{aligned} \tag{7}$$

Hence:

$$\begin{aligned}
& \frac{1}{\Gamma(\alpha)} \int_0^1 \left\| \int_0^t (t-s)^{\alpha-1} (\phi_n'(s) - \phi'(s)) ds \right\| dt \\
& \leq \frac{1}{\alpha \Gamma(\alpha)} \int_0^1 (1-s)^\alpha \left\| \phi_n'(s) - \phi'(s) \right\| ds,
\end{aligned} \tag{8}$$

since $0 \leq (1-s)^\alpha \leq 1$ on $[0, 1]$

$$\leq \frac{1}{\Gamma(\alpha+1)} \left\| \phi_n'(s) - \phi'(s) \right\|_{L^1(0,1)}. \tag{9}$$

Term 2 (delay function term):

We compute the t -integral

$$\int_0^1 t^{\alpha-1} dt = \frac{1}{\alpha}$$

Hence,
$$\frac{|\phi_n(0) - \phi(0)|}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} dt = \frac{|\phi_n(0) - \phi(0)|}{\Gamma(\alpha+1)}. \quad (10)$$

Term 3 (boundary residue term):

Factoring the t -independent difference:

$$\begin{aligned} & \int_0^1 \|h'(t)\| |(G(u_n) - G(u)) - (A(u_n) - A(u))| dt \\ & \|h'(t)\|_{L^1(0,1)} (|G(u_n) - G(u)| + |A(u_n) - A(u)|). \end{aligned}$$

We have already proved in continuity of T for the sup-norm that

$$G(u_n) \rightarrow G(u), \quad A(u_n) \rightarrow A(u) \quad \text{as } n \rightarrow \infty.$$

Since f and g are continuous and I^α is continuous from $L^1[0,1]$ into $C([0,1])$, then

$$\int_0^1 \|h'(t)\| |(G(u_n) - G(u)) - (A(u_n) - A(u))| dt \rightarrow 0.$$

Collecting all the three terms together, we obtain:

$$\begin{aligned} & \int_0^1 \|(Tu_n)'(t) - (Tu)'(t)\| dt \\ & \leq \frac{1}{\Gamma(\alpha+1)} \|\phi_n'(s) - \phi'(s)\|_{L^1(0,1)} + \frac{|\phi_n(0) - \phi(0)|}{\Gamma(\alpha+1)} \\ & + \|h'(t)\|_{L^1(0,1)} (|G(u_n) - G(u)| + |A(u_n) - A(u)|). \end{aligned}$$

We know that:

- i. $u_n \rightarrow u$ in AC^1 implies $u_n \rightarrow u$ uniformly and $u_n' \rightarrow u'$ in $L^1([0,1])$
- ii. For $f(s, u, v) \in C^1([0,1])$ on the relevant compact set, then by the chain rule:

$$\begin{aligned}
\phi'_n(s) - \phi'(s) &= f_s(s, \xi_n(s)) - f_s(s, \xi(s)) + f_u(s, \xi_n(s))u'_n(s) \\
&\quad - f_u(s, \xi(s))u'(s) + f_v(s, \xi_n(s))u'_n(s - \tau) \\
&\quad - f_v(s, \xi(s))u'(s - \tau),
\end{aligned} \tag{11}$$

where $\xi_n(s) = (u_n(s), u_n(s - \tau))$, and $\xi(s) = (u(s), u(s - \tau))$.

Therefore $u_n \rightarrow u$ uniformly and $u'_n \rightarrow u'$ in $L^1([0, 1])$ imply

$$\begin{aligned}
\|\phi'_n - \phi'\|_{L^1([0, 1])} &\rightarrow 0, \quad \phi_n(0) \rightarrow \phi(0). \\
\text{Therefore } \int_0^1 \|(Tu_n)'(t) - (Tu)'(t)\| dt &\rightarrow 0 \quad (\text{as } n \rightarrow \infty),
\end{aligned}$$

together with the previously established uniform convergence of $Tu_n \rightarrow Tu$ on $[0, 1]$.

We obtain:

$$\begin{aligned}
\|Tu_n - Tu\|_{AC^1} &= \sup_{t \in [0, 1]} \|Tu_n(t) - Tu(t)\|_{AC^1} \\
&\quad + \int_0^1 \|(Tu_n)'(t) - (Tu)'(t)\| dt \rightarrow 0.
\end{aligned} \tag{12}$$

Hence, T is continuous on B_r in the AC^1 -norm. $\square$

Lemma 3.1 (L^1 -estimate for the derivative of a fractional integral). *Let $\alpha \in (0, 1)$ and $\varphi \in L^\infty(0, 1)$, and set $(I^\alpha \varphi)(t) := \frac{1}{\Gamma(\alpha)} \int_0^1 (t - s)^{\alpha-1} \varphi(s) ds$, $t \in [0, 1]$, then $I^\alpha \varphi$ is absolutely continuous on $[0, 1]$, and its derivative exists almost everywhere and is given by:*

$$\frac{d}{dt}(I^\alpha \varphi)(t) = I^{\alpha-1} \varphi(t).$$

There is a finite constant $C(\alpha) > 0$, depending only on α , such that:

$$\int_0^1 \left| \frac{d}{dt}(I^\alpha \varphi)(t) \right| dt = \int_0^1 |I^{\alpha-1} \varphi|(t) dt \leq C(\alpha).$$

$$\text{Hence } \int_0^1 |(I^\alpha \varphi)'| dt \leq \frac{2}{\Gamma(\alpha + 1)} \|\varphi\|_\infty \quad (13)$$

Samko et al. (1993); Kilbas et al. (2006); Mardanov et al. (2022); Srivastava and Saxena (2001).

Proof [H2]: (Self-mapping)

Let $B_r \subset AC^1([0, 1], \mathbb{R}^n)$ be the closed ball $\|u\|_{AC^1} \leq r$, for every $u \in B_r$. By continuity of f and g from hypothesis [H1], the functions f and g attain maxima, hence there exist finite constants:

$$M_f = \max_{(t,u,v) \in K} \|f(t, u, v)\| > 0,$$

$$M_g = \max_{(t,u,v) \in K} \|g(t, u, v)\| > 0.$$

These constants are independent of the particular $u \in B_r$, hence we can now directly apply M_f and M_g in our proof of self-mapping.

i. Bound for $\|(Tu)(t)\|$ for each $t \in [0, 1]$ – the pointwise estimate is:

$$\begin{aligned} \|(Tu)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, u(s), u(s-\tau))\| ds \\ &\quad + \|h(t)\| \left(\int_0^1 (1-s)^{\alpha-1} \|f(s, u(s), u(s-\tau))\| ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^1 \|g(s, u(s), u(s-\tau))\| ds \right) \\ &\leq \frac{M_f t^\alpha}{\Gamma(\alpha+1)} + H \left(M_g + \frac{M_f}{\Gamma(\alpha+1)} \right) \quad \text{where } H = \sup_{t \in [0,1]} \|h(t)\|. \end{aligned}$$

ii. Evaluating the integral bound for $\|(Tu)(t)\|$:

$$\begin{aligned} \int_0^1 \|(Tu)(t)\| dt &\leq \frac{M_f}{\Gamma(\alpha+1)} \int_0^1 t^\alpha dt + H \left(M_g + \frac{M_f}{\Gamma(\alpha+1)} \right) \\ &= \frac{M_f}{\Gamma(\alpha+2)} + H \left(M_g + \frac{M_f}{\Gamma(\alpha+1)} \right). \end{aligned}$$

iii. The derivative of a fractional integral: $\|(Tu')'(t)\|$:

$$\begin{aligned}
\|(Tu')'(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \|f(s, u(s), u(s-\tau))\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \|f(s, u(s), u(s-\tau))\| ds \\
&\quad + \|h'(t)\| \left(\int_0^1 \|g(s, u(s), u(s-\tau))\| ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \|f(s, u(s), u(s-\tau))\| ds \right) \\
&\leq \frac{M_f t^{\alpha-1}}{\Gamma(\alpha)} + \frac{M_f}{\Gamma(\alpha+1)} + \|h'(t)\| \left(M_g + \frac{M_f}{\Gamma(\alpha+1)} \right). \quad (14)
\end{aligned}$$

iv. Integral bound for the derivative:

$$\begin{aligned}
\int_0^1 \|(Tu')'(t)\| ds &\leq \frac{M_f}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} dt + \left(\frac{M_f}{\Gamma(\alpha+1)} + M_g \right) \int_0^1 dt \\
&= \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \right) M_f + M_g \\
&= \frac{3M_f}{\Gamma(\alpha+1)} + M_g.
\end{aligned}$$

Step 3: (Combine for AC^1 - norm):

$$\begin{aligned}
\|u\|_{AC^1} &= \int_0^1 \|u(t)\| dt + \int_0^1 \|u'(t)\| dt \\
&\leq M_f \left(\frac{1}{\Gamma(\alpha+2)} + \frac{1}{2\Gamma(\alpha+1)} \right) + \frac{M_g}{2} + \frac{3M_f}{\Gamma(\alpha+1)} + M_g \\
&= M_f \left(\frac{1}{\Gamma(\alpha+2)} + \frac{7}{2\Gamma(\alpha+1)} \right) + \frac{3M_g}{2}.
\end{aligned}$$

Therefore, if:

$$M_f \left(\frac{1}{\Gamma(\alpha+2)} + \frac{7}{2\Gamma(\alpha+1)} \right) + \frac{3M_g}{2} \leq r, \quad (15)$$

then $\|Tu\|_{AC^1} \leq r$, proving $T(B_r) \subseteq B_r$. $\square$

Proof: ([H3] – condensing-decomposition $T = T_1 + T_2$)

We proved that the operator T is condensing with respect to the Kuratowski measure of non-compact α , for a bounded set $B_r \subset X := AC^1([0, 1], \mathbb{R}^n)$ in a Banach space such that we have $\alpha(B_r) > 0$ and $\alpha(T(B_r)) < \alpha(B_r)$.

Let $B_r = \{u \in X : \|u\|_{AC^1} \leq r\}$ be a closed, bounded and convex set. We define the decomposition as follows:

$$T = T_1 + T_2, \quad \text{where:}$$

$$T_1 u(t) = I^\alpha[f(s, u(s), u(s - \tau))](t),$$

$$T_2 u(t) = h(t)(G(u) - A(u)), \quad \text{where } G(u) = \int_0^1 g(s, u(s), u(s - \tau)) ds$$

$$A(u) = I^\alpha[f(s, u(s), u(s - \tau))](1).$$

where T_1 is Lipschitz and T_2 is compact, hence T is condensing with respect to the Kuratowski measure of non-compactness α , for $t \in [0, 1]$

Step 1: T_1 is Lipschitz on B_r in $\|u\|_{AC^1}$ -norm

Assume f and g satisfy the Lipschitz condition:

- i. Where f, g are continuous functions and satisfy Lipschitz conditions for all t . There exist nonnegative constants $M_{f_1}, M_{f_2}, M_{g_1}, M_{g_2}$ (depending on r) such for all $t \in [0, 1]$ and all $u_1, u_2, v_1, v_2 \in B_r$, we have the standard componentwise Lipschitz bounds

$$\|f(t, u_1, u_2) - f(t, v_1, v_2)\| \leq L_{f_1} (\|u_1 - v_1\| + L_{f_2} \|u_2 - v_2\|),$$

$$\|g(t, u_1, u_2) - g(t, v_1, v_2)\| \leq L_{g_1} (\|u_1 - v_1\| + L_{g_2} \|u_2 - v_2\|).$$

We set the combined constants:

$$L_f := L_{f_1} + L_{f_2},$$

$$L_g := L_{g_1} + L_{g_2}.$$

- ii. $u \in AC^1([0, T], \mathbb{R}^n)$ with $u(0) = 0$.

We evaluate:

$$\begin{aligned} \|T_1 u(t) - T_1 v(t)\| &\leq \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\|u(s) - v(s)\| + \|u(s-\tau) \\ &\quad - v(s-\tau)\|) ds. \end{aligned}$$

Split the integral at $s = \tau$:

- i. For $s \in [0, \min(t, \tau)]$, $\|u(s-\tau) - v(s-\tau)\| = 0$ (since $u = v = \phi$ on $[-\tau, 0]$).
- ii. For $s \in (0, \min(t, \tau)]$, $\|u(s-\tau) - v(s-\tau)\| \leq \|u - v\|_{AC^1}$.

Thus:

$$\begin{aligned} \|T_1 u(t) - T_1 v(t)\| &\leq \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u(s) - v(s)\| ds \\ &\quad + \int_{\min(t, \tau)}^t (t-s)^{\alpha-1} \|u(s-\tau) - v(s-\tau)\| ds \\ &\leq \frac{2L_f}{\Gamma(\alpha)} \|u - v\|_{AC^1} \int_0^t (t-s)^{\alpha-1} ds \\ &= \frac{2L_f}{\Gamma(\alpha)} \|u - v\|_{AC^1} \cdot \frac{t^\alpha}{\alpha} \\ &\leq \frac{2L_f}{\Gamma(\alpha+1)} \|u - v\|_{AC^1}. \end{aligned}$$

Thus integrating over $[0, 1]$:

$$\int_0^1 \|T_1 u - T_1 v\|_{AC^1} dt \leq \frac{2L_f}{\Gamma(\alpha+1)} \|u - v\|_{AC^1}. \quad (16)$$

Evaluating the derivative of $T_1 u$:

$$(T_1 u)'(t) = \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} f(s, s(u), u(s-\tau)) ds,$$

Applying the Lipschitz condition:

$$\begin{aligned} \|(T_1 u)'(t) - (T_1 v)'(t)\| &\leq \frac{(1-\alpha)L_f}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-2} \|u(s) - v(s)\| \right. \\ &\quad \left. + \|u(s-\tau) - v(s-\tau)\| \right) ds \\ &\leq \frac{2L_f}{\Gamma(\alpha-1)} \|u - v\|_{AC^1} \int_0^t (t-s)^{\alpha-2} ds. \end{aligned}$$

Evaluating the integrals:

$$\int_0^t (t-s)^{\alpha-2} ds = \frac{t^{\alpha-1}}{\alpha-1} \leq \frac{t^{\alpha-1}}{\alpha-1}, \quad t \in [0, 1],$$

$$\|(T_1 u)'(t) - (T_1 v)'(t)\|_{AC^1} \leq \left(\frac{2L_f t^{\alpha-1}}{\Gamma(\alpha)} \right) \|u - v\|_{AC^1} \frac{t^{\alpha-1}}{\alpha-1}.$$

Integrating over $[0, 1]$:

$$\begin{aligned} \int_0^1 \|(T_1 u)'(t) - (T_1 v)'(t)\| dt &\leq \frac{2L_f}{\Gamma(\alpha)} \|u - v\|_{AC^1} \int_0^1 \frac{t^{\alpha-1}}{\alpha-1} dt \\ &= \frac{2L_f}{\Gamma(\alpha)} \|u - v\|_{AC^1} \cdot \frac{1}{\alpha} \\ &\leq \frac{2L_f}{\Gamma(\alpha+1)} \|u - v\|_{AC^1} \end{aligned} \tag{17}$$

Combining equations (16) and (17) we obtain:

$$\begin{aligned} \|T_1 u - T_1 v\|_{AC^1} + \int_0^1 \|(Tu)' - (Tv)'\| dt \\ &\leq \left(\frac{2L_f}{\Gamma(\alpha+1)} + \frac{2L_f}{\Gamma(\alpha+1)} \right) \|u - v\|_{AC^1} \\ &\leq \frac{4L_f}{\Gamma(\alpha+1)} \|u - v\|_{AC^1}. \end{aligned}$$

Thus T_1 is a contraction if:

$$\frac{4L_f}{\Gamma(\alpha + 1)} < 1. \quad (18)$$

Step 2: T_2 (Compactness)

Let $B \subset B_r$ be bounded. We show that $T_2(B)$ is relatively compact in X . We have to prove the boundedness of $\varphi(B)$. With B being bounded in X and f, g continuous on B_r , then a set $\nu := \{\varphi(u) : u \in B\} \subset \mathbb{R}^n$ is bounded.

$$\text{Let } M_\varphi := \sup\{\|\varphi(u)\| : u \in B\} < \infty.$$

For $u \in B$, $(T_2u)(t) = h(t)\varphi(u)$, then

$$\frac{d}{dt}(T_2u)(t) = h'(t)\varphi(u).$$

Since h, h' are continuous on $[0, 1]$, we set:

$$H_0 := \sup_{t \in [0, 1]} \|h(t)\|,$$

$$H_1 := \sup_{t \in [0, 1]} \|h'(t)\|.$$

Then for every $u \in B$ and $t \in [0, 1]$,

$$\|T_2u(t)\| \leq H_0 M_\varphi,$$

$$\left\| \frac{d}{dt} T_2u(t) \right\| \leq H_1 M_\varphi.$$

Hence, $\{T_2u : u \in B\}$ is equibounded and has uniformly bounded derivatives (equi-Lipschitz). In particular, the family is equicontinuous and uniformly bounded in $C([0, 1], \mathbb{R}^n)$.

For finite-dimensional bound:

For each $u \in B$:

$$\begin{aligned} T_2 u(t) &= h(t)\varphi(u), \\ \text{then } \varphi(u) &= \sum_{j=1}^n C_j(u)\ell_j \quad (\text{in standard basis}), \\ \text{we have } T_2 u(t) &= \sum_{j=1}^n C_j(u)h(t)\ell_j, \end{aligned}$$

we have

$$T_2(B) \subset M := \{h(t)\ell_j, \dots, h(t)\ell_n\} \in C([0, 1], \mathbb{R}^n).$$

Consequently, bounded subsets of a finite-dimensional subspace are relatively compact in every Banach norm. In particular, they are relatively compact in $\|\cdot\|_{AC^1}$. Thus, $T_2(B)$ is relatively compact in X .

Therefore:

$$\alpha(T_2(B)) = 0. \tag{19}$$

Applying the Kuratowski inequality and condensing property (subadditivity and the two results above): For any bounded $B \subset B_r$, we have:

$$\begin{aligned} \alpha(T(B)) &= \alpha(T_1(B) + T_2(B)) \leq \alpha(T_1(B)) + \alpha(T_2(B)) \\ &\leq k\alpha(B) + 0 \\ &= K\alpha(B). \end{aligned} \tag{20}$$

Since T_1 is a contraction ($K < 1$) and T_2 is compact ($\alpha(T_2(B)) = 0$) thus T is condensing. Hence, under the given conditions and assuming f is Lipschitz with constant L_f satisfying

$$\frac{4L_f}{\Gamma(\alpha + 1)} < 1,$$

the operator T is condensing with decomposition $T = T_1 + T_2$

where T_1 is a contraction and T_2 is compact. □

Hence, by Sadovskii's fixed point theorem, there exists at least one fixed

point $u^* \in B_r$ such that:

$$T(u^*) = u^*.$$

This fixed point u^* satisfies the operator equation and is therefore a solution to the generalized Caputo fractional delay integral boundary value problem.

3.2. Proof of noncontractiveness

Theorem 10 (Noncontractive hypotheses). *Let $0 < \alpha < 1$ and let $B_r \subset AC^1([0, 1], \mathbb{R}^n)$ be the closed ball with the norm*

$$\|u\|_{AC^1} = \sup_{t \in [0, 1]} \|u(t)\| + \int_0^1 \|u'(t)\| dt.$$

Assume the nonlinearities $f, g : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy the point-wise Lipschitz bounds on B_r :

$$\begin{aligned} \|f(t, u_1, u_2) - f(t, v_1, v_2)\| &\leq L_{f_1} \|u_1 - v_1\| + L_{f_2} \|u_2 - v_2\|, \\ \|g(t, u_1, u_2) - g(t, v_1, v_2)\| &\leq L_{g_1} \|u_1 - v_1\| + L_{g_2} \|u_2 - v_2\|. \end{aligned} \quad (21)$$

Put

$$L := L_{f_1} + L_{f_2}, \quad G := L_{g_1} + L_{g_2}.$$

Define the operator $T : B_r \rightarrow AC^1([0, 1], \mathbb{R}^n)$ by the fractional integral representation used in the existence proof (the one for which a fixed point is a solution of the CFDIBVP). Then for all $u, v \in B_r$ we have

$$\|Tu - Tv\|_{AC^1} \leq k_{\text{op}} \|u - v\|_{AC^1},$$

where the computed operator constant is

$$k_{\text{op}} = \left(\frac{1}{\Gamma(\alpha + 2)} + \frac{7}{2\Gamma(\alpha + 1)} \right) L + \frac{3}{2} G. \quad (22)$$

Moreover: **(Why k_{op} is generally not a contraction.)** For $\alpha \in (0, 1)$ we have $0 < \Gamma(\alpha + 1) < 1$, hence $1/\Gamma(\alpha + 1) > 1$. Consequently

$$\frac{1}{\Gamma(\alpha + 2)} + \frac{7}{2\Gamma(\alpha + 1)} \geq \frac{7}{2\Gamma(\alpha + 1)} > \frac{7}{2} = 3.5 > 1,$$

and therefore the lower bound

$$k_{\text{op}} \geq \frac{7}{2}L + \frac{3}{2}G,$$

holds. In particular, unless the combined Lipschitz constants L and G are extremely small (so that $\frac{7}{2}L + \frac{3}{2}G < 1$), the value of k_{op} will not satisfy $k_{\text{op}} < 1$. For typical values of the local Lipschitz constants L, G the computed operator constant k_{op} will not satisfy $k_{\text{op}} < 1$. Thus Banach's contraction condition is usually too restrictive for this problem on $[0, 1]$. This motivates a Volterra or fractional Grönwall approach that exploits the convolutional kernel instead of a global operator norm estimate.

4. Uniqueness of solution

Uniqueness via fractional Grönwall inequality.

Let $u_1, u_2 \in B_r$ be two solutions that agree on the history $[-\tau, 0]$ and set

$$\psi(t) := u_1(t) - u_2(t), \quad t \in [-\tau, 1]. \quad (23)$$

Then $\psi(t) = 0$ for $t \in [-\tau, 0]$. Subtracting the two integral representations of the solutions yields, for $t \in [0, 1]$,

$$\begin{aligned} \psi(t) &= \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [f(s, u_1(s), u_1(s-\tau)) - f(s, u_2(s), u_2(s-\tau))] ds \\ &+ h(t) \int_0^1 [g(s, u_1(s), u_1(s-\tau)) - g(s, u_2(s), u_2(s-\tau))] ds - h(t)(A(u_1) - A(u_2)), \end{aligned} \quad (24)$$

where $|h(t)| \leq 1$ and $A(u) = I^\alpha[G_u]$ denotes the fractional integral correction appearing in the boundary term.

Take norms and apply (21). For each $t \in [0, 1]$,

$$\begin{aligned} \|\psi(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (L_{f_1} \|\psi(s)\| + L_{f_2} \|\psi(s-\tau)\|) ds \\ &\quad + \int_0^1 (L_{g_1} \|\psi(s)\| + L_{g_2} \|\psi(s-\tau)\|) ds + \|A(u_1) - A(u_2)\|. \end{aligned}$$

We estimate the A -term by the sup-norm bound for the fractional integral:

$$\begin{aligned} \|A(u_1) - A(u_2)\| &= \|I^\alpha[F_{u_1} - F_{u_2}]\|_\infty \leq \frac{1}{\Gamma(\alpha+1)} \sup_{s \in [0,1]} \|F_{u_1}(s) - F_{u_2}(s)\| \\ &\leq \frac{LW}{\Gamma(\alpha+1)}, \end{aligned}$$

where we have defined

$$W := \sup_{r \in [-\tau, 1]} \|\psi(r)\| \quad \text{and} \quad L = L_{f_1} + L_{f_2}.$$

Now we use the trivial bounds $\|\psi(s)\| \leq W$ and $\|\psi(s-\tau)\| \leq W$ to convert all purely delay terms to W . We also bound the integral-average term by GW with $G = L_{g_1} + L_{g_2}$. The pointwise estimate becomes (for every $t \in [0, 1]$)

$$\begin{aligned} \|\psi(t)\| &\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\psi(s)\| ds + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} W ds \\ &\quad + GW + \frac{LW}{\Gamma(\alpha+1)}. \end{aligned}$$

We combine the W -only terms and use $\int_0^t (t-s)^{\alpha-1} ds = t^\alpha/\alpha = t^\alpha/\Gamma(\alpha+1) \cdot \Gamma(\alpha)$ to obtain the simpler and conservative inequality

$$\|\psi(t)\| \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\psi(s)\| ds + \left(\frac{Lt^\alpha}{\Gamma(\alpha+1)} + G + \frac{L}{\Gamma(\alpha+1)} \right) W.$$

We now have:

$$u(t) \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + \left(\frac{Lt^\alpha}{\Gamma(\alpha+1)} + G + \frac{L}{\Gamma(\alpha+1)} \right) W. \quad (25)$$

Now we estimate the bracket by a convenient, simple grouping that yields a canonical Volterra inequality:

$$u(t) \leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + C_1(t)W,$$

where $C_1(t) = \frac{Lt^\alpha}{\Gamma(\alpha+1)} + G + \frac{L}{\Gamma(\alpha+1)}$

for all $t \in [0, 1]$; since $0 \leq t^\alpha \leq 1$ we have

$$C_1(1) = G + \frac{2L}{\Gamma(\alpha+1)}. \quad (26)$$

We take the uniform constant and define it as:

$$C_0 := \left(G + \frac{2L}{\Gamma(\alpha+1)} \right) W.$$

Thus equation (26) becomes the canonical form

$$u(t) := \|\psi(t)\| \leq C_0 + K \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad (27)$$

with the explicit choices

$$C_0 := \left(G + \frac{2L}{\Gamma(\alpha+1)} \right) W, \quad K := \frac{L}{\Gamma(\alpha)}.$$

By the fractional Grönwall lemma (Mittag-Leffler form) it follows that

$$u(t) \leq C_0 E_\alpha(Kt^\alpha) \leq C_0 E_\alpha(K) \quad (0 \leq t \leq 1).$$

Hence

$$\|\psi(t)\| \leq \left(G + \frac{2L}{\Gamma(\alpha + 1)}\right) W E_\alpha\left(\frac{L}{\Gamma(\alpha)}\right) \quad \text{for all } t \in [0, 1]. \quad (28)$$

By definition

$$W := \sup_{r \in [-\tau, 1]} \|\psi(r)\|$$

Taking the supremum of equation (28) over $t \in [0, 1]$ we have:

$$W = \sup_{t \in [0, 1]} \|\varphi(t)\| \leq \sup_{t \in [0, 1]} \left(G + \frac{2L}{\Gamma(\alpha + 1)}\right) W E_\alpha\left(\frac{L}{\Gamma(\alpha)}\right).$$

which gives the scalar inequality

$$W \leq \left(G + \frac{2L}{\Gamma(\alpha + 1)}\right) W E_\alpha\left(\frac{L}{\Gamma(\alpha)}\right).$$

If $W > 0$, we divide both sides by W to obtain

$$1 \leq \left(G + \frac{2L}{\Gamma(\alpha + 1)}\right) E_\alpha\left(\frac{L}{\Gamma(\alpha)}\right)$$

which contradicts the condition

$$\left(G + \frac{2L}{\Gamma(\alpha + 1)}\right) E_\alpha\left(\frac{L}{\Gamma(\alpha)}\right) < 1.$$

Showing that the restrictive condition holds, the bracketed factor is strictly less than 1 (< 1).

If $W = 0$, then $\sup_{r \in [-\tau, 1]} \|\psi(r)\| = 0$, hence $\psi(t) = 0$, for every $t \in [-\tau, 1]$.

Thus

$$u_1(t) - u_2(t) = 0, \quad \text{for all } t \in [0, 1], \quad \text{then}$$

$$u_1 = u_2 \text{ on } [0, 1]. \quad (29)$$

This proves uniqueness. $\square$

5. Continuous dependence of solutions

We will now establish the continuous dependence of solutions on the initial data, boundary data and nonlinear functions f and g through a perturbation analysis.

Let us consider two problems:

- i. Problem 1 (Original Problem) – equation (1) given by:

$$\begin{cases} {}^c D_{0+}^\alpha u(t) = f(t, u(t), u(t - \tau)) & \alpha \in (0, 1), \quad t \in [0, 1], \quad \tau > 0 \\ u(0) = 0, u(1) = \int_0^1 g(s, u(s), u(s - \tau)) ds, \end{cases}$$

- ii. Problem 2 (Perturbed Problem):

$$\begin{cases} {}^c D_{0+}^\alpha v(t) = f(t, v(t), v(t - \tau)) & \alpha \in (0, 1), \quad t \in [0, 1], \quad \tau > 0 \\ v(0) = \delta_0, v(1) = \int_0^1 g(s, v(s), v(s - \tau)) ds + \delta_1. \end{cases} \quad (30)$$

We want to show that if the perturbations are small, then the difference between the solutions u and v is small.

Assume the following:

- i. Lipschitz conditions:

$$\begin{aligned} \|f(t, u_1, u_2) - f(t, v_1, v_2)\| &\leq L_f (\|u_1 - v_1\| + \|u_2 - v_2\|), \\ \|g(t, u_1, u_2) - g(t, v_1, v_2)\| &\leq L_g (\|u_1 - v_1\| + \|u_2 - v_2\|), \end{aligned}$$

and similarly for $\tilde{f}$ and $\tilde{g}$ with the same constants L_f and L_g .

ii. Perturbation bounds:

$$\begin{aligned} \|f(t, u, v) - \tilde{f}(t, u, v)\| &\leq \epsilon_f, \\ \|g(t, u, v) - \tilde{g}(t, u, v)\| &\leq \epsilon_g, \quad \text{for all } t \in [0, 1] \text{ and } u, v \in B_r. \end{aligned}$$

iii. With small initial and boundary perturbations:

$$\|\delta_0\| \leq \epsilon_0, \quad \|\delta_1\| \leq \epsilon_1.$$

iv. Initial history perturbation: Let φ and ψ be the initial histories for u and v on $[-\tau, 0]$, respectively, and define:

$$\Delta = \sup_{t \in [-\tau, 0]} \|u(t) - v(t)\| = \sup_{t \in [-\tau, 0]} \|\varphi(t) - \psi(t)\|.$$

iv. Boundary function: The function $h(t)$ satisfies $h(0) = 0$, $h(1) = 1$ and $|h(t)| \leq H$, $|1 - h(t)| \leq H'$ for all $t \in [0, 1]$. Basically, $h(t) = t$, so $H = 1$ and $H' = 1$.

5.1. Integral formulation and difference estimate

The solutions u and v satisfy the integral equations:

i. For u :

$$\begin{aligned} u(t) = I^\alpha [f(s, u(s), u(s - \tau))](t) + h(t) &\left(\int_0^1 g(s, u(s), u(s - \tau)) ds \right. \\ &\left. - I^\alpha [f(s, u(s), u(s - \tau))](1) \right). \end{aligned}$$

ii. For v :

$$\begin{aligned} v(t) = I^\alpha [\tilde{f}(s, v(s), v(s - \tau))](t) + h(t) &\left(\int_0^1 \tilde{g}(s, v(s), v(s - \tau)) ds \right. \\ &\left. - I^\alpha [\tilde{f}(s, v(s), v(s - \tau))](1) \right) + \delta_0(1 - h(t)) + h(t)\delta_1. \end{aligned}$$

Define the difference $w(t) = u(t) - v(t)$. Then:

$$\begin{aligned} w(t) &= I^\alpha[f(s, u(s), u(s - \tau)) - \tilde{f}(s, v(s), v(s - \tau))](t) \\ &\quad + h(t) \int_0^1 [g(s, u(s), u(s - \tau)) - \tilde{g}(s, v(s), v(s - \tau))] ds \\ &\quad - h(t) I^\alpha[f(s, u(s), u(s - \tau)) - \tilde{f}(s, v(s), v(s - \tau))](1) \\ &\quad - \delta_0(1 - h(t)) - h(t)\delta_1. \end{aligned}$$

Splitting the differences:

$$\begin{aligned} f(s, u(s), u(s - \tau)) - \tilde{f}(s, v(s), v(s - \tau)) &= \\ [f(s, u(s), u(s - \tau)) - f(s, v(s), v(s - \tau))] &+ \\ + [f(s, u(s), u(s - \tau)) - \tilde{f}(s, v(s), v(s - \tau))], \end{aligned}$$

$$\begin{aligned} g(s, u(s), u(s - \tau)) - \tilde{g}(s, v(s), v(s - \tau)) &= \\ [g(s, u(s), u(s - \tau)) - g(s, v(s), v(s - \tau))] &+ \\ + [g(s, u(s), u(s - \tau)) - \tilde{g}(s, v(s), v(s - \tau))]. \end{aligned}$$

Let:

$$\begin{aligned} \Delta f(s) &= f(s, u(s), u(s - \tau)) - \tilde{f}(s, v(s), v(s - \tau)), \quad \|\Delta f(s)\| \leq \epsilon_f, \\ \Delta g(s) &= g(s, u(s), u(s - \tau)) - \tilde{g}(s, v(s), v(s - \tau)), \quad \|\Delta g(s)\| \leq \epsilon_g, \end{aligned}$$

Then:

$$\begin{aligned} w(t) &= I^\alpha[f(s, u(s), u(s - \tau)) - f(s, v(s), v(s - \tau)) + \Delta f(s)](t) \\ &\quad + h(t) \int_0^1 [g(s, u(s), u(s - \tau)) - g(s, v(s), v(s - \tau)) + \Delta g(s)] ds \\ &\quad - h(t) I^\alpha[f(s, u(s), u(s - \tau)) - f(s, v(s), v(s - \tau)) + \Delta f(s)](1) \\ &\quad - \delta_0(1 - h(t)) - h(t)\delta_1. \end{aligned}$$

Norm estimate and Grönwall inequality

Let $\phi(t) = \|w(t)\|$ and $W = \sup_{t \in [-\tau, 1]} \phi(t)$. Then:

$$\begin{aligned} \phi(t) &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [L_f(\phi(s) + \phi(s-\tau)) + \epsilon_f] ds \\ &\quad + H \int_0^1 [L_g(\phi(s) + \phi(s-\tau)) + \epsilon_g] ds \\ &\quad + H \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} [L_f(\phi(s) + \phi(s-\tau)) + \epsilon_f] ds + H' \epsilon_0 + H \epsilon_1. \end{aligned}$$

Using $\phi(s) \leq W$ and $\phi(s-\tau) \leq W$, and evaluating the integrals:

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha} \leq \frac{1}{\alpha} = \frac{1}{\Gamma(\alpha+1)},$$

$$\int_0^t (1-s)^{\alpha-1} ds = \frac{1}{\alpha} = \frac{1}{\Gamma(\alpha+1)},$$

we obtain:

$$\begin{aligned} \phi(t) &\leq \frac{L_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) ds + W \left[\frac{L_f}{\Gamma(\alpha+1)} + 2HL_g + \frac{2HL_f}{\Gamma(\alpha+1)} \right] \\ &\quad + \epsilon_f \left[\frac{1}{\Gamma(\alpha+1)} + \frac{H}{\Gamma(\alpha+1)} \right] + H\epsilon_g + H' \epsilon_0 + H\epsilon_1. \end{aligned}$$

Define constants:

$$\begin{aligned} C_1 &= \frac{L_f}{\Gamma(\alpha)}, \\ C_2 &= L_f \left(\frac{1+2H}{\Gamma(\alpha+1)} \right) + 2HL_g, \\ C_3 &= \epsilon_f \frac{1+H}{\Gamma(\alpha+1)} + H\epsilon_g + H' \epsilon_0 + H\epsilon_1. \end{aligned}$$

Then:

$$\phi(t) \leq C_3 + C_2 W + C_1 \int_0^t (t-s)^{\alpha-1} \phi(s) ds.$$

By the fractional Grönwall inequality:

$$\phi(t) \leq (C_3 + C_2 W) E_\alpha(C_1 t^\alpha) \leq (C_3 + C_2 W) E_\alpha C_1.$$

$$\text{Let } M = \sup_{t \in [0,1]} \phi(t). \text{ Then } M \leq (C_3 + C_2 W) E_\alpha C_1.$$

Since $W = \max(\Delta, M)$, we consider two cases:

i. If $W = \Delta$, then:

$$M \leq (C_3 + C_2 W) E_\alpha C_1.$$

ii. If $W = M$, then

$$M \leq \frac{C_3 E_\alpha(C_1)}{1 - C_2 E_\alpha(C_1)} \quad \text{provided } 1 - C_2 E_\alpha(C_1) > 0.$$

Thus:

$$M \leq \max \left\{ \Delta, \frac{C_3 E_\alpha(C_1)}{1 - C_2 E_\alpha(C_1)} \right\}.$$

We now state the Continuous Dependence Theorem:

Theorem 11 (Continuous Dependence). *From the above derivations and assumptions, if:*

$$1 - \left(L_f \frac{1+2H}{\Gamma(\alpha+1)} + 2HL_g \right) E_\alpha \left(\frac{L_f}{\Gamma(\alpha)} \right) > 0,$$

then the difference between the solutions u and v satisfies:

$$\sup_{t \in [0,1]} \|u(t) - v(t)\| \leq \max \left\{ \Delta, \frac{\left(\epsilon_f \frac{1+H}{\Gamma(\alpha+1)} + H\epsilon_g + H'\epsilon_0 + H\epsilon_1 \right) E_\alpha \left(\frac{L_f}{\Gamma(\alpha)} \right)}{1 - \left(L_f \frac{1+2H}{\Gamma(\alpha+1)} + 2HL_g \right) E_\alpha \left(\frac{L_f}{\Gamma(\alpha)} \right)} \right\}$$

For $H = 1$ and $H' = 1$ respectively

$$\leq \max \left\{ \Delta, \frac{\left(\epsilon_f \frac{2}{\Gamma(\alpha+1)} + \epsilon_g + \epsilon_0 + \epsilon_1 \right) E_\alpha \left(\frac{L_f}{\Gamma(\alpha)} \right)}{1 - \left(L_f \frac{3}{\Gamma(\alpha+1)} + 2L_g \right) E_\alpha \left(\frac{L_f}{\Gamma(\alpha)} \right)} \right\}. \quad (31)$$

In particular, if $\epsilon_f, \epsilon_g, \epsilon_0, \epsilon_1, \Delta \rightarrow 0$, then

$$\sup_{t \in [0,1]} \|u(t) - v(t)\| \rightarrow 0. \quad \square$$

Example:

We now discuss the following problem for analytical verifications. Given the Caputo fractional delay integral value problem on a bounded, close and convex domain $B_r := \{u \in X : \|u\|_{AC^1} \leq r\}$.

Assume $f, g : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous nonlinear functions, and let $\varphi : [-\tau, 0] \rightarrow B_r$ be a given delay function satisfying $u(0) = u_0 \in B_r$ and $\varphi(-\tau) = 0$ extend the delay by $u(t) = \varphi(t)$ for $t \in [-\tau, 0]$, and let a continuous perturbation function $\Psi(t)$ on the interval $[0, 1]$ be defined by a family of perturbations $\Psi_\epsilon(t) = \epsilon \cdot \Psi(t)$, where $\Psi \in C([0, 1])$ is a fixed function and $\epsilon > 0$.

$$\begin{cases} {}^c D_{0+}^\alpha u(t) = -\lambda u(t) + \mu \sin(u(t - \tau)) + \frac{1}{10} \left(\frac{\sin(u(t))}{1+|u(t)|} \right) + \Psi(t) \\ u(0) = 0, u(1) = \frac{1}{4} \left(\int_0^1 \sin(u(s)) ds \right) \end{cases} \quad (32)$$

for $t \in [0, 1]$, $\tau > 0$ where ${}^c D_{0+}^\alpha$ is a Caputo derivative with order $\alpha \in (0, 1)$, $\lambda = 2.0$, $\mu = 1.2$, and $\Psi(t) = \epsilon \cdot \sin(2\pi t)$.

We show that:

1. The solution operator T is noncontractive.

2. Using Lipschitz properties and the fractional Grönwall inequality we show that the solution depends continuously on the data. We show that small changes in the perturbation function $\Psi(t)$ lead to small changes in the solution.
3. We then develop suitable MATLAB code to plot the graphs of our solutions' phase portraits.

Solutions:

1. **The solution operator T is defined through the equivalent integral equation:**

$$(Tu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u(s-\tau)) ds \\ + t \left[\frac{1}{4} \int_0^1 \sin(u(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s), u(s-\tau)) ds \right]$$

where:

$$f(t, u(t), u(t-\tau)) = -2.0u(t) + 1.2 \sin(u(t-\tau)) + 0.1 \left(\frac{\sin(u(t))}{1+|u(t)|} \right) + \Psi(t).$$

Lipschitz constant analysis

- i. For $-2.0u(t)$: Lipschitz Constant = 2.0.
- ii. For $1.2 \sin(u(t-\tau))$: Lipschitz Constant = 1.2
- iii. For $0.1 \left(\frac{\sin(u(t))}{1+|u(t)|} \right)$: Lipschitz Constant = 0.2

Total Lipschitz Constant $L_f = 2.0 + 1.2 + 0.2 = 3.4$. The Lipschitz constant for T is bounded by:

$$L_T \leq \frac{2L_f}{\alpha\Gamma(\alpha)} + \frac{1}{4}, \text{ for } \alpha = 0.5, \Gamma(0.5) = \sqrt{\pi} \equiv 1.772 \text{ such that } L_T \leq \frac{2 \times 3.4}{0.5 \times 1.772} + 0.25 = 7.92 > 1.$$

Since $L_T > 1$ for all $\alpha \in (0, 1)$, the operator is noncontractive.

- iv. $\Psi(t) = \epsilon \sin(2\pi t) \implies \Psi(0.5) = 0.055$, where $t = 0.5$, $\epsilon = 1$.

2. **Continuous dependence on perturbation via fractional Grönwall inequality.**

Let u_1 and u_2 be solutions corresponding to the perturbations Ψ_1 and

Ψ_2 with $\|\Psi_1 - \Psi_2\|_\infty \leq \epsilon$.

The difference $\delta(t) = u_1(t) - u_2(t)$ satisfies ${}^c D_{0+}^\alpha \delta(t) = [f(t, u_1(t), u_1(t - \tau)) - f(t, u_2(t), u_2(t - \tau))] + [\Psi_1 - \Psi_2]$; using the Lipschitz property and fractional Grönwall inequality we obtain:

$\|u_1 - u_2\|_\infty \leq C_\epsilon$, where $C > 0$ is a constant independent of ϵ , proving the continuous dependence of the solutions.

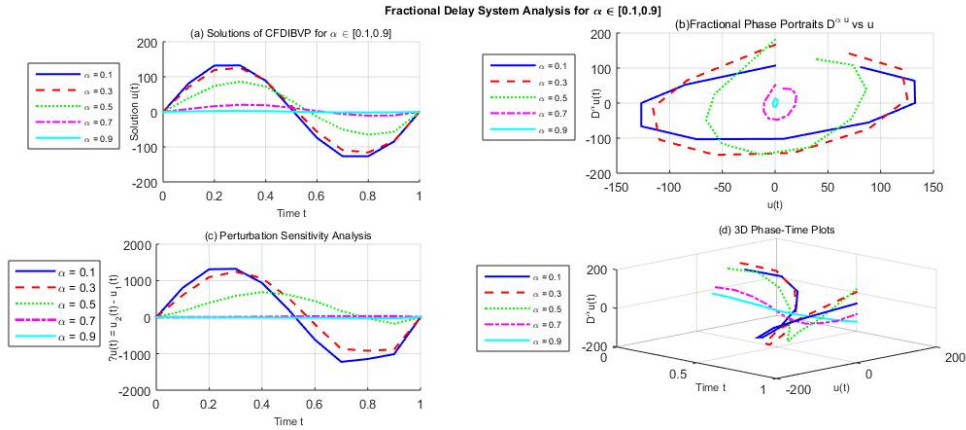


Figure 1. Caputo fractional delay system analysis

6. Discussion and conclusion

Figure 1(a) is a graphical display of solution trajectories of CFDBVP for $\alpha \in [0.1, 0.9]$, indicating the existence of solutions and well-posedness. Fig. 1(b) shows fractional phase portraits of the fractional derivatives of solutions ${}^c D^\alpha u$ plotted against the solutions u for $\alpha \in [0.1, 0.9]$. The phase portraits reveal evolving attractor structures, confirming the system's noncontractive nature through complex trajectory patterns. Fig 1(c) is the graphical display of the system Continuous Dependence Perturbation Sensitivity Analysis confirming that perturbations in the data lead to bounded changes in the solutions, indicating stability. Fig 1(d) is a 3D display of a phase-time plot for $\alpha \in [0.1, 0.9]$. This shows that lower α values (0.1 – 0.3) produce

more damped responses, characteristic of strong hereditary properties in a fractional system. However, intermediate and higher values ($0.5 - 0.9$) produce classical oscillatory patterns with faster response time.

In conclusion, this paper establishes a comprehensive and rigorous framework for analyzing a noncontractive Caputo fractional delay system with integral boundary conditions. The key contributions of this work include: successful proof of the existence of solutions for a noncontractive fractional delay equation via Sadovskii's fixed point theorem, uniqueness and continuous dependence of solutions through a fractional Grönwall inequality, and perturbation sensitivity analysis that ensures physical reliability and analytic stability while the noncontractive paradigm extends applicability beyond conventional Lipschitz-restrictiveness. Finally, this work provides a robust theoretical foundation for modeling complex processes with memory effects and delay interactions in a Banach space of the absolutely continuous space $AC^1([0, 1], \mathbb{R}^n)$.

Recommendations for future work

1. **Theoretical extensions:** The existence and uniqueness framework can be extended to a broader class of nonlinearities or delay structures such as the Hölder continuity, weak Lipschitz or one-sided Lipschitz conditions.
2. **Stability analysis:** Researchers can advance this field of research by exploring and developing Lyapunov-like functionals or a frequency-domain method for stability analysis of other fractional derivatives.
3. **Transition from finite delay to infinite delay systems:** Researchers may extend the scope of our results into phase space with unbounded or infinite delay to investigate results on the existence of solutions, uniqueness, and continuous dependence.

Acknowledgments

We sincerely thank the editor and reviewers for the meticulous review of our manuscript and the well-articulated feedback that improves our results.

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