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An index for discriminating the direction of anti-sum-asymmetry for square contingency tables

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Summary

A square contingency table is a special case of a two-way contingency table wherein the row and column variables consist of the same classification. In square contingency tables, observations tend to be concentrated in the main diagonal cells of the table or nearby, and independence between row and column variables often does not hold. The anti-sum-symmetry model has a symmetrical structure with respect to the anti-diagonal cells of the table. The anti-sum-symmetry model should be applied to square contingency tables with ordered categories. The existing index, which can measure the degree of departure from the anti-sum-symmetry model, cannot discriminate between perfect upper and lower anti-sum-asymmetries. To address this issue, we propose here a directional index which can discriminate between the two types of anti-sum-asymmetries. The proposed index can be interpreted as an index for measuring the degree of perfect anti-sum-asymmetry. We derive an approximately unbiased estimator and an approximate confidence interval for the proposed index. The utility of the proposed index is demonstrated through a real data analysis.

Key words: anti-diagonal cell, approximately unbiased estimator, asymmetry, discrimination, ordered category

1. Introduction

A square contingency table is a special case of a two-way contingency table wherein the row and column variables consist of the same classification. In square contingency tables, observations tend to be concentrated in the main diagonal cells of the table or nearby, and independence between row and column variables often does not hold. The symmetry (S) model (Bowker, 1948), the marginal homogeneity (MH) model (Stuart, 1955), and the sum-symmetry model (Yamamoto, Tanaka and Tomizawa, 2013; Ando, 2022) have been proposed as models with a symmetrical structure between row and column variables specific to square contingency tables. The conditional symmetry model (McCullagh, 1978), the extended marginal homogeneity model (Tomizawa, 1984; Tomizawa, 1993), and the conditional sum-symmetry model (Yamamoto et al., 2013; Ando, 2021a; Ando, 2021b) have been proposed as models with an asymmetrical structure between row and column variables specific to square contingency tables.

When a model with a symmetrical structure does not fit the presented data well, we are interested in (i) measuring the degree of departure from the model, and (ii) applying a model with an asymmetrical structure. This paper focuses on an index for measuring the degree of departure from the model.

Henceforth, we will consider an $R \times R$ square contingency table with ordered categories. Let π_{ij} denote the probability that an observed frequency will fall in the (i, j)th cell of the table $(i = 1, 2, \ldots, R; j = 1, 2, \ldots, R)$. As asymmetrical structures between row and column variables, we may consider a perfect upper asymmetry (i.e., $\pi_{ij} > 0$ and $\pi_{ji} = 0$ for $i < j$) and a perfect lower asymmetry (i.e., $\pi_{ij} = 0$ and $\pi_{ji} > 0$ for $i < j$).

The S model is defined by $\pi_{ij} = \pi_{ji}$ for $i \neq j$. Tomizawa, Miyamoto and Hatanaka (2001) proposed the index Φ_{S} for measuring the degree of departure from the S model. Note that $\Phi_{\mathrm{S}} = 0$ if and only if the S model holds. However, this index cannot discriminate between perfect upper and lower asymmetries. This is because the index Φ_{S} takes a value of one under perfect upper and lower asymmetries. To discriminate between the two types of asymmetries, Tahata, Miyazawa and Tomizawa (2010) proposed the index Ψ_{S}. This index takes a value of $+1$ under perfect lower asymmetry, and a value of -1 under perfect upper asymmetry. Note that if the S model holds then $\Psi_{\mathrm{S}} = 0$, but the converse is not necessarily true. Therefore, Ψ_{S} can be interpreted as an index for measuring the degree of perfect asymmetry. For details of Φ_{S} and Ψ_{S}, see the Appendix.

The MH model is defined by $\pi_{i+} = \pi_{+i}$ for $i = 1, 2, \ldots, R$, where $\pi_{i+} = \sum_{j=1}^{R} \pi_{ij}$ and $\pi_{+i} = \sum_{j=1}^{R} \pi_{ji}$. Tomizawa, Miyamoto and Ashihara (2003) proposed the index Φ_{MH} for measuring the degree of departure from the MH model. Note that $\Phi_{\mathrm{MH}} = 0$ if and only if the MH model holds. However, this index cannot discriminate between perfect upper and lower marginal-asymmetries. This is because the index Φ_{MH} takes a value of one under perfect upper and lower marginal-asymmetries. To discriminate between the two types of marginal-asymmetries, Yamamoto, Ando and Tomizawa (2011) proposed the index Ψ_{MH}. This index takes a value of $+1$ under perfect lower marginal-asymmetry, and a value of -1 under perfect upper marginal-asymmetry. Note that if the MH model holds then $\Psi_{\mathrm{MH}} = 0$, but the converse is not necessarily true. Therefore, Ψ_{MH} can be interpreted as an index for measuring the degree of perfect marginal-asymmetry. For details of Φ_{MH} and Ψ_{MH}, see the Appendix.

Models in which symmetrical or asymmetrical structures exist with respect to the anti-diagonal cells of the table rather than the main-diagonal cells include the anti-sum-symmetry (ASS) and anti-conditional sum-symmetry (ACSS) models (Ando, 2021c; Ando, 2023). As anti-asymmetrical structures between row and column variables, we may consider a perfect upper anti-sum-asymmetry (i.e., $\pi_{ij} > 0$ and $\pi_{j^*i^*} = 0$ for $i + j < R + 1$) and a perfect lower anti-sum-asymmetry (i.e., $\pi_{ij} = 0$ and $\pi_{j^*i^*} > 0$ for $i + j < R + 1$), where $i^* = R + 1 - i$ and $j^* = R + 1 - j$. Note that the anti-diagonal cells are the cells (i, j) satisfying $i + j = R + 1$, while the main-diagonal cells are the cells (i, j) satisfying $i = j$.

Ando (2024) proposed the index Φ_{ASS} for measuring the degree of departure from the ASS model. Note that $\Phi_{\mathrm{ASS}} = 0$ if and only if the ASS model holds. However, this index cannot discriminate between perfect upper and lower anti-sum-asymmetries. This is because the index Φ_{ASS} takes a value of one under perfect upper and lower anti-sum-asymmetries. When we want to distinguish between the two types of anti-sum-asymmetries and measure the degree of departure from the ASS model, it is not appropriate to use the index Φ_{ASS}. To address this issue, we propose in this paper a directional index which can discriminate between the two types of anti-sum-asymmetries. The proposed index takes a value of $+1$ under perfect lower anti-sum-asymmetry, and a value of -1 under perfect upper anti-sum-asymmetry. Note that if the ASS model holds then the proposed index is equal to zero, but the converse is not necessarily true. When we want to measure the degree of departure from the ASS model, it is not appropriate to use the proposed index. Note that the proposed index can be interpreted as an index for measuring the degree of perfect anti-sum-asymmetry.

The remainder of the paper is organized as follows. In section 2, we introduce the ASS model, the ACSS model and the index Φ_{ASS}, and propose a directional index which can discriminate between two types of anti-sum-asymmetries. In section 3, we derive an approximately unbiased estimator and an approximate confidence interval for the proposed index. In section

4, the utility of the proposed index is demonstrated through a real data analysis. Section 5 contains concluding remarks.

2. Methods

2.1. Model

Let X and Y be row and column variables, respectively, and $\Pr(X = i, Y = j) = \pi_{ij}$ $(i = 1, 2, \ldots, R; j = 1, 2, \ldots, R)$. For $t = 2, 3, \ldots, R$, let

$$\begin{aligned} U_t &= \Pr(X + Y = t) = \textstyle\sum_{i+j=t} \pi_{ij} \\ L_t &= \Pr(X + Y = 2(R + 1) - t) = \textstyle\sum_{i+j=2(R+1)-t} \pi_{ij}. \end{aligned}$$

Note that the anti-diagonal cells are the cells (i, j) satisfying $i + j = R + 1$.

The ASS model (Ando, 2021c) is defined by

$$U_t = L_t \quad \text{for } t = 2, 3, \ldots, R.$$

The ACSS model (Ando, 2021c) is defined by

$$U_t = \tau L_t \quad \text{for } t = 2, 3, \ldots, R.$$

Note that the ASS model is a special case of the ACSS model obtained by putting $\tau = 1$.

2.2. Existing index

Assume that $U_t + L_t > 0$ for $t = 2, 3, \ldots, R$, Ando (2024) proposed the index Φ_{ASS}, which can measure the degree of departure from the ASS model, as follows:

$$\Phi_{\text{ASS}} = \tfrac{1}{\log 2} \textstyle\sum_{t=2}^{R} \left[U_t^* \log \left(\tfrac{U_t^*}{2^{-1}(U_t^* + L_t^*)} \right) + L_t^* \log \left(\tfrac{L_t^*}{2^{-1}(U_t^* + L_t^*)} \right) \right],$$

where

$$0 \log 0 = 0, \quad U_t^* = \frac{U_t}{\delta}, \quad L_t^* = \frac{L_t}{\delta} \quad \text{and} \quad \delta = \sum_{s=2}^{R} (U_s + L_s).$$

The index Φ_{ASS} has the following properties: (i) $0 \leqslant \Phi_{\text{ASS}} \leqslant 1$, (ii) $\Phi_{\text{ASS}} = 0$ if and only if the ASS model holds (i.e., $U_t = L_t$ for $t = 2, 3, \ldots, R$), and (iii) $\Phi_{\text{ASS}} = 1$ if and only if $U_t = 0$ and $L_t > 0$ or $U_t > 0$ and $L_t = 0$ for $t = 2, 3, \ldots, R$. Therefore, the index Φ_{ASS} cannot discriminate between perfect upper and lower anti-sum-asymmetries (i.e., perfect upper anti-sum-asymmetry: $U_t > 0$ and $L_t = 0$ for $t = 2, 3, \ldots, R$, and perfect lower anti-sum-asymmetry: $U_t = 0$ and $L_t > 0$ for $t = 2, 3, \ldots, R$). Note that (i) $U_t > 0$ and $L_t = 0$ for $t = 2, 3, \ldots, R$ if and only if $\pi_{ij} > 0$ and $\pi_{j^*i^*} = 0$ for $i + j < R + 1$, (ii) $U_t = 0$ and $L_t > 0$ for $t = 2, 3, \ldots, R$ if and only if $\pi_{ij} = 0$ and $\pi_{j^*i^*} > 0$ for $i + j < R + 1$.

Under the ACSS model, the index Φ_{ASS} can be expressed as

$$\Phi_{\mathrm{ASS}} = \frac{1}{(\tau + 1)\log 2}\left[\tau \log\left(\frac{2\tau}{\tau + 1}\right) + \log\left(\frac{2}{\tau + 1}\right)\right].$$

When we take the limit as $\tau \to 0$, we have for the index Φ_{ASS}:

$$\lim_{\tau \to 0} \Phi_{\mathrm{ASS}} = 1.$$

When we take the limit as $\tau \to \infty$, we have for the index Φ_{ASS}:

$$\lim_{\tau \to \infty} \Phi_{\mathrm{ASS}} = 1.$$

2.3. Proposed index

To discriminate between perfect upper and lower anti-sum-asymmetries, we assume that $U_t + L_t > 0$ for $t = 2, 3, \ldots, R$, and propose a directional index Ψ_{ASS} as follows:

$$\Psi_{\mathrm{ASS}} = \frac{4}{\pi}\sum_{t=2}^{R}(U_t^* + L_t^*)\left(\theta_t - \frac{\pi}{4}\right),$$

where

$$\theta_t = \arccos\left(\frac{U_t}{\sqrt{(U_t)^2 + (L_t)^2}}\right).$$

Note that the range of θ_t, for $t = 2, 3, \ldots, R$, is $0 \leqslant \theta_t \leqslant \pi/2$. The index Ψ_{ASS} has the following properties: (i) $-1 \leqslant \Psi_{\mathrm{ASS}} \leqslant 1$, (ii) if the ASS model holds then $\Psi_{\mathrm{ASS}} = 0$, but the converse is not necessarily true, (iii) $\Psi_{\mathrm{ASS}} = -1$ if and only if we have perfect upper anti-sum-asymmetry (i.e., $\pi_{ij} > 0$ and $\pi_{j^*i^*} = 0$ for $i + j < R + 1$), and (iv) $\Psi_{\mathrm{ASS}} = 1$ if and only if we have perfect lower anti-sum-asymmetry (i.e., $\pi_{ij} = 0$ and $\pi_{j^*i^*} > 0$ for $i + j < R + 1$). Therefore, the index Ψ_{ASS} can discriminate between perfect upper and lower anti-sum-asymmetries. If the value of Ψ_{ASS} is positive, this would indicate that $X+Y$ is expected to be greater rather than smaller than the midpoint $R+1$.

Under the ACSS model, the index Ψ_{ASS} can be expressed as

$$\Psi_{\mathrm{ASS}} = \frac{4}{\pi}\arccos\left(\frac{\tau}{\sqrt{\tau^2+1}}\right) - 1.$$

Therefore, under the ACSS model, Ψ_{ASS} is equal to zero if and only if the ASS model holds. When we take the limit as $\tau \to 0$, we have for the index Ψ_{ASS}:

$$\lim_{\tau\to 0}\Psi_{\mathrm{ASS}} = 1.$$

When we take the limit as $\tau \to \infty$, we have for the index Ψ_{ASS}:

$$\lim_{\tau\to\infty}\Psi_{\mathrm{ASS}} = -1.$$

Figure 1 sketches the value of Ψ_{ASS} in $0.1 \leqslant \tau \leqslant 10$ under the ACSS model.

3. Estimation of the proposed index

Let x_{ij} be the observed frequency in the (i,j)th cell of the table, where N $(=\sum\sum x_{ij})$ is sample size. Let $\boldsymbol{\pi}, \boldsymbol{x}$ and $\boldsymbol{p}$ be $R^2\times 1$ vectors as follows:

$$\begin{aligned}
\boldsymbol{\pi} &= (\pi_{11},\ldots,\pi_{1R},\pi_{21},\ldots,\pi_{2R},\ldots,\pi_{R1},\ldots,\pi_{RR})^\top,\\
\boldsymbol{x} &= (x_{11},\ldots,x_{1R},x_{21},\ldots,x_{2R},\ldots,x_{R1},\ldots,x_{RR})^\top \quad \text{and}\\
\boldsymbol{p} &= N^{-1}\boldsymbol{x} = (p_{11},\ldots,p_{1R},p_{21},\ldots,p_{2R},\ldots,p_{R1},\ldots,p_{RR})^\top,
\end{aligned}$$

where "$\top$" is a symbol indicating the transpose of a vector or matrix.

Let a random vector $\boldsymbol{x}$ be distributed in a multinomial distribution with parameters $(N,\boldsymbol{\pi})$. The value $\sqrt{N}(\boldsymbol{p}-\boldsymbol{\pi})$ asymptotically (as $N\to\infty$) approaches a multivariate normal distribution $\mathcal{N}(\mathbf{0},\mathbf{Diag}(\boldsymbol{\pi})-\boldsymbol{\pi}\boldsymbol{\pi}^\top)$, where $\mathbf{0}$ is the $R^2\times 1$ zero vector and $\mathbf{Diag}(\boldsymbol{\pi})$ is the diagonal matrix with $\boldsymbol{\pi}$ as diagonal elements; see for example, Bishop, Fienberg and Holland (2007, Section 14.6.3).

Using the Taylor expansion, the plug-in estimator $\hat{\Psi}_{\mathrm{ASS}}$, which is given by Ψ_{ASS} with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$, is expressed as follows:

$$\hat{\Psi}_{\mathrm{ASS}} = \Psi_{\mathrm{ASS}} + \left(\frac{\partial\Psi_{\mathrm{ASS}}}{\partial\boldsymbol{\pi}^\top}\right)(\boldsymbol{p}-\boldsymbol{\pi}) + o(\|\boldsymbol{p}-\boldsymbol{\pi}\|),$$

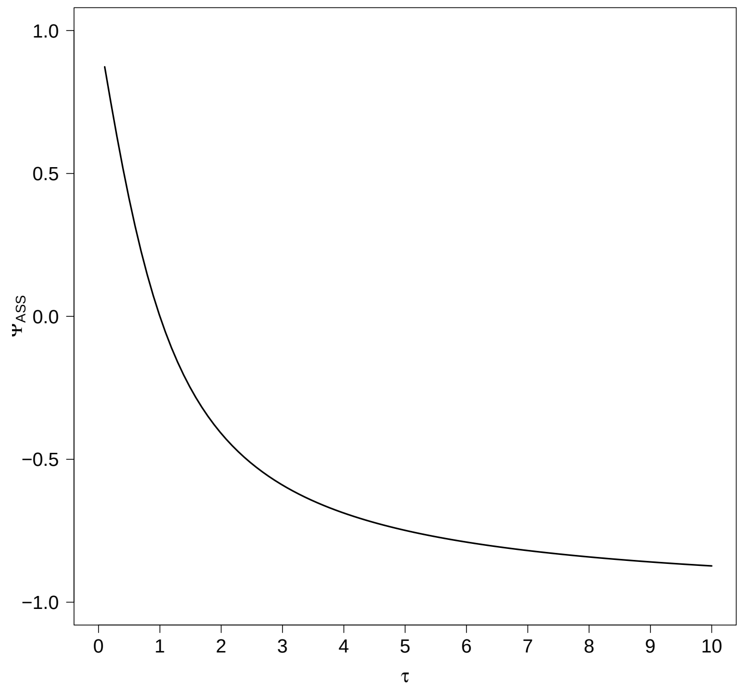

Figure 1. Values of the index Ψ_{ASS} in $0.1 \leqslant \tau \leqslant 10$ under the ACSS model

where

$$\frac{\partial \Psi_{\text{ASS}}}{\partial \boldsymbol{\pi}^{\top}} = \left.\frac{\partial \hat{\Psi}_{\text{ASS}}}{\partial \boldsymbol{p}^{\top}}\right|_{\boldsymbol{p}=\boldsymbol{\pi}}.$$

Therefore, the plug-in estimator $\hat{\Psi}_{\text{ASS}}$ is a consistent estimator and an asymptotically unbiased estimator.

By the delta method (Bishop et al., 2007, Section 14.6.3), the value $\sqrt{N}(\hat{\Psi}_{\text{ASS}} - \Psi_{\text{ASS}})$ asymptotically (as $N \to \infty$) approaches a normal distribution $\mathcal{N}(0, \sigma^2[\hat{\Psi}_{\text{ASS}}])$, where

$$\begin{aligned} \sigma^2[\hat{\Psi}_{\text{ASS}}] &= \left(\frac{\partial \Psi_{\text{ASS}}}{\partial \boldsymbol{\pi}^{\top}}\right)(\mathbf{Diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^{\top})\left(\frac{\partial \Psi_{\text{ASS}}}{\partial \boldsymbol{\pi}^{\top}}\right)^{\top} \\ &= \sum_{k=1}^{R}\sum_{l=1}^{R}\pi_{kl}\left(\frac{\partial \Psi_{\text{ASS}}}{\partial \pi_{kl}}\right)^2 - \left\{\sum_{k=1}^{R}\sum_{l=1}^{R}\pi_{kl}\left(\frac{\partial \Psi_{\text{ASS}}}{\partial \pi_{kl}}\right)\right\}^2, \end{aligned} \tag{1}$$

with

$$\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}} = \begin{cases} \frac{1}{\delta}\left(\theta_{k+l} - \frac{L_{k+l}(U_{k+l}+L_{k+l})}{U_{k+l}^2+L_{k+l}^2}\right) - \frac{\Psi_{\mathrm{ASS}}+1}{\delta} & \text{for } k+l<R+1, \\ 0 & \text{for } k+l=R+1, \\ \frac{1}{\delta}\left(\theta_{k^*+l^*} + \frac{U_{k^*+l^*}(U_{k^*+l^*}+L_{k^*+l^*})}{U_{k^*+l^*}^2+L_{k^*+l^*}^2}\right) - \frac{\Psi_{\mathrm{ASS}}+1}{\delta} & \text{for } k+l>R+1. \end{cases}$$

Note that $k^* = R+1-k$ and $l^* = R+1-l$, that is $k^*+l^* = 2(R+1)-(k+l)$.

Since

$$\begin{aligned}\sum_{k=1}^{R}\sum_{l=1}^{R}\pi_{kl}\left(\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}}\right) &= \sum_{k+l<R+1}\pi_{kl}\left(\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}}\right) + \sum_{k+l>R+1}\pi_{kl}\left(\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}}\right) \\ &= \sum_{t=2}^{R}\left[\sum_{k+l=t}\pi_{kl}\left(\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}}\right) + \sum_{k^*+l^*=t}\pi_{kl}\left(\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}}\right)\right] \\ &= \Psi_{\mathrm{ASS}} + 1 - (\Psi_{\mathrm{ASS}}+1) \\ &= 0,\end{aligned}$$

the equation (1) can be expressed as follows:

$$\begin{aligned}\sigma^2[\hat{\Psi}_{\mathrm{ASS}}] &= \sum_{k=1}^{R}\sum_{l=1}^{R}\pi_{kl}\left(\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}}\right)^2 \\ &= \sum_{k+l<R+1}\pi_{kl}\left(\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}}\right)^2 + \sum_{k+l>R+1}\pi_{kl}\left(\frac{\partial \Psi_{\mathrm{ASS}}}{\partial \pi_{kl}}\right)^2.\end{aligned}$$

From properties (iii) and (iv) of the index Ψ_{ASS}, since $\sigma^2[\hat{\Psi}_{\mathrm{ASS}}]$ is equal to zero when $\Psi_{\mathrm{ASS}} = -1$ and $\Psi_{\mathrm{ASS}} = 1$, the asymptotic normality of $\sqrt{N}(\hat{\Psi}_{\mathrm{ASS}} - \Psi_{\mathrm{ASS}})$ cannot be applicable. Therefore, when Ψ_{ASS} takes a value close to -1 or 1, the accuracy of the approximation may be poor.

The plug-in estimator $\widehat{\sigma^2[\hat{\Psi}_{\mathrm{ASS}}]}$ is given by $\sigma^2[\hat{\Psi}_{\mathrm{ASS}}]$ with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$. As $\widehat{\sigma^2[\hat{\Psi}_{\mathrm{ASS}}]}$ converges in probability to $\sigma^2[\hat{\Psi}_{\mathrm{ASS}}]$, $\sqrt{N}(\hat{\Psi}_{\mathrm{ASS}} - \Psi_{\mathrm{ASS}})$ asymptotically (as $N \to \infty$) approaches a normal distribution $\mathcal{N}(0, \widehat{\sigma^2[\hat{\Psi}_{\mathrm{ASS}}]})$. Therefore, an approximate $100(1-\alpha)\%$ confidence interval for Ψ_{ASS} is given by

$$\hat{\Psi}_{\mathrm{ASS}} \pm z_{\alpha/2}\frac{\widehat{\sigma[\hat{\Psi}_{\mathrm{ASS}}]}}{\sqrt{N}},$$

where $z_{\alpha/2}$ is the percentage point from the standard normal distribution that corresponds to a two-tail probability equal to α.

4. Real data analysis

The real datasets in Table 1 were analyzed to illustrate the utility of the proposed index Ψ_{ASS}. These real datasets containing distance vision records were taken from Tan (2017, p. 78, p. 139). The dataset in Table 1(a) records the right and left eye distance vision of 635 patients from an eye clinic. The vision of an eye is classified into four grades: the higher the grade, the better the vision. The dataset in Table 1(b) records the right and left eye vision grades of 239 students undergoing eyesight tests.

Table 1. Datasets of distance vision of patients and students; source: Tan (2017, p. 78, p. 139)

		Left eye grades				
Candidate	Right eye grades	(1)	(2)	(3)	(4)	Total
(a) Patients	Lowest grade (1)	50	21	24	35	130
	(2)	42	22	12	42	118
	(3)	56	32	52	16	156
	Highest grade (4)	67	82	20	62	231
	Total	215	157	108	155	635
(b) Students	Lowest grade (1)	10	16	9	17	52
	(2)	23	11	7	22	63
	(3)	21	8	12	25	66
	Highest grade (4)	18	10	17	13	58
	Total	72	45	45	77	239

Such distance vision data have been analyzed with respect to the S model or its related models; see for example Bowker (1948), Stuart (1955), Bishop et al. (2007) and Tan (2017). However, for these data, it may be appropriate to evaluate whether symmetrical and asymmetrical structures exist with respect to the anti-diagonal cells of the table rather than the main-diagonal cells. This is because, for distance vision data as given in Table 1, there is a natural tendency to evaluate an individual's visual acuity as the sum of the two eyes.

First, we evaluate the goodness-of-fit of the ASS model. The values of the likelihood ratio statistic G^2 for the datasets in Tables 1(a) and (b) are 28.68 ($> \chi^2_{(0.05;3)} = 7.81$) and 0.61 ($< \chi^2_{(0.05;3)} = 7.81$), respectively. Therefore, the ASS model is a poor fit for the dataset in Table 1(a), while it is a good fit for the dataset in Table 1(b). Since the test statistics G^2 cannot capture the direction of deviance, we are interested in discriminating between upper and lower anti-sum-asymmetries for the dataset in Table 1(a) using the proposed index Ψ_{ASS}.

From Table 2, for patients, the confidence interval for Ψ_{ASS} is positive. Therefore, for patients, the structure of lower anti-sum-asymmetry exists, and an individual's visual acuity as the sum of the two eyes tends to be below the reference value (i.e., $R+1=5$). We see from the positive values of the confidence interval that for an arbitrary patient, total vision – being the sum of the right eye and left eye grades – tends to be better instead of worse than the average total vision. On the other hand, for students, the confidence interval for Ψ_{ASS} includes zero. Therefore, from the results on the goodness-of-fit of the ASS model and the confidence interval for Ψ_{ASS}, the structure of anti-sum-asymmetry does not exist for students.

Table 2. Values of $\hat{\Psi}_{\text{ASS}}$, estimated approximate standard error of $\hat{\Psi}_{\text{ASS}}$, and approximate 95% confidence interval of Ψ_{ASS}, applied to Table 1

Estimate $\hat{\Psi}_{\text{ASS}}$	Standard error $\widehat{\sigma[\hat{\Psi}_{\text{ASS}}]}/\sqrt{N}$	Confidence interval $\hat{\Psi}_{\text{ASS}} \pm z_{\alpha/2}\widehat{\sigma[\hat{\Psi}_{\text{ASS}}]}/\sqrt{N}$ (Lower limit, Upper limit)
(a) Patients		
0.1508	0.0440	(0.0646, 0.2370)
(b) Students		
0.0605	0.0743	(−0.0851, 0.2061)

5. Concluding remarks

Ando (2024) proposed the index Φ_{ASS}, which can measure the degree of departure from the ASS model. Note that $\Phi_{\text{ASS}}=0$ if and only if the ASS model holds. However, Φ_{ASS} cannot discriminate between perfect upper and lower anti-sum-asymmetries. Perfect upper anti-sum-asymmetry implies $\pi_{ij}>0$ and $\pi_{j^*i^*}=0$ for $i+j<R+1$, and perfect lower anti-sum-asymmetry implies $\pi_{ij}=0$ and $\pi_{j^*i^*}>0$ for $i+j<R+1$, where $i^*=R+1-i$ and $j^*=R+1-j$. To address this issue, in this paper we have proposed the index Ψ_{ASS}, which can discriminate between the two kinds of perfect anti-sum-asymmetries.

We gave a plug-in estimator of Ψ_{ASS}, which is approximately the unbiased estimator of Ψ_{ASS}. Moreover, we derived the approximate confidence interval for Ψ_{ASS}. The utility of the proposed index was demonstrated through a real data analysis.

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Appendix

Let, for $i < j$,

$$G_{ij} = \Pr(X \leqslant i, Y \geqslant j) = \sum_{s=1}^{i} \sum_{t=j}^{R} \pi_{st} \quad \text{and}$$

$$G_{ji} = \Pr(X \geqslant j, Y \leqslant i) = \sum_{s=j}^{R} \sum_{t=1}^{i} \pi_{st}.$$

The S model can be expressed as follows:

$$G_{ij} = G_{ji} \quad \text{for } i < j;$$

see Tomizawa et al. (2001) and Ando et al. (2017).

Assume that $G_{ij} + G_{ji} > 0$ for $i < j$. Tomizawa et al. (2001) proposed the index Φ_{S}, which can measure the degree of departure from the S model, as follows:

$$\Phi_{\mathrm{S}} = \frac{1}{\log 2} \sum_{i<j} \left[G_{ij}^{*} \log \left(\frac{G_{ij}^{*}}{2^{-1}(G_{ij}^{*} + G_{ji}^{*})} \right) + G_{ji}^{*} \log \left(\frac{G_{ji}^{*}}{2^{-1}(G_{ij}^{*} + G_{ji}^{*})} \right) \right].$$

where

$$0 \log 0 = 0, \quad G^*_{ij} = \frac{G_{ij}}{\delta_1}, \quad G^*_{ji} = \frac{G_{ji}}{\delta_1} \quad \text{and} \quad \delta_1 = \sum_{s<t} (G_{st} + G_{ts}) .$$

The index Φ_{S} has the following properties: (i) $0 \leqslant \Phi_{\mathrm{S}} \leqslant 1$, (ii) $\Phi_{\mathrm{S}} = 0$ if and only if the S model holds (i.e., $G_{ij} = G_{ji}$ for $i < j$), and (iii) $\Phi_{\mathrm{S}} = 1$ if and only if $G_{ij} = 0$ and $G_{ji} > 0$ for $i < j$ or $G_{ij} > 0$ and $G_{ji} = 0$ for $i < j$. Therefore, the index Φ_{S} cannot discriminate between perfect upper and lower asymmetries (i.e., perfect upper asymmetry: $G_{ij} > 0$ and $G_{ji} = 0$ for $i < j$, and perfect lower asymmetry: $G_{ij} = 0$ and $G_{ji} > 0$ for $i < j$). Note that (i) $G_{ij} > 0$ and $G_{ji} = 0$ for $i < j$ if and only if $\pi_{ij} > 0$ and $\pi_{ji} = 0$ for $i < j$, (ii) $G_{ij} = 0$ and $G_{ji} > 0$ for $i < j$ if and only if $\pi_{ij} = 0$ and $\pi_{ji} > 0$ for $i < j$.

To discriminate between perfect upper and lower asymmetries, assume that $G_{ij} + G_{ji} > 0$ for $i < j$. Tahata et al. (2010) proposed the index Ψ_{S} as follows:

$$\Psi_{\mathrm{S}} = \frac{4}{\pi} \sum_{i<j} \left(G^*_{ij} + G^*_{ji} \right) \left(\theta^{\mathrm{S}}_{ij} - \frac{\pi}{4} \right),$$

where

$$\theta^{\mathrm{S}}_{ij} = \arccos \left(\frac{G_{ij}}{\sqrt{(G_{ij})^2 + (G_{ji})^2}} \right).$$

The index Ψ_{S} has the following properties: (i) $-1 \leqslant \Psi_{\mathrm{S}} \leqslant 1$, (ii) if the S model holds then $\Psi_{\mathrm{S}} = 0$, but the converse is not necessarily true, (iii) $\Psi_{\mathrm{S}} = -1$ if and only if we have perfect upper asymmetry (i.e., $\pi_{ij} > 0$ and $\pi_{ji} = 0$ for $i < j$), and (iv) $\Psi_{\mathrm{S}} = 1$ if and only if we have perfect lower asymmetry (i.e., $\pi_{ij} = 0$ and $\pi_{ji} > 0$ for $i < j$). Therefore, the index Ψ_{S} can discriminate between perfect upper and lower asymmetries.

The MH model can be expressed as follows:

$$G_{i,i+1} = G_{i+1,i} \quad \text{for } i = 1, 2, \ldots, R-1;$$

see Tomizawa et al. (2003) and Ando et al. (2021).

Assume that $G_{i,i+1} + G_{i+1,i} > 0$ for $i = 1, 2, \ldots, R-1$. Tomizawa et al. (2003) proposed the index Φ_{MH}, which can measure the degree of departure from the MH model, as follows:

$$\Phi_{\mathrm{MH}} = \frac{1}{\log 2} \sum_{i=1}^{R-1} \left[G^*_{i,i+1} \log \left(\frac{G^*_{i,i+1}}{2^{-1}(G^*_{i,i+1} + G^*_{i+1,i})} \right) \right. \\ \left. + G^*_{i+1,i} \log \left(\frac{G^*_{i+1,i}}{2^{-1}(G^*_{i,i+1} + G^*_{i+1,i})} \right) \right].$$

where

$$0 \log 0 = 0, \quad G^*_{i,i+1} = \frac{G_{i,i+1}}{\delta_2}, \quad G^*_{i+1,i} = \frac{G_{i+1,i}}{\delta_2} \quad \text{and}$$
$$\delta_2 = \sum_{s=1}^{R-1} \left(G_{s,s+1} + G_{s+1,s} \right).$$

The index Φ_{MH} has the following properties: (i) $0 \leqslant \Phi_{\mathrm{MH}} \leqslant 1$, (ii) $\Phi_{\mathrm{MH}} = 0$ if and only if the MH model holds (i.e., $G_{i,i+1} = G_{i+1,i}$ for $i = 1, 2, \dots, R-1$), and (iii) $\Phi_{\mathrm{MH}} = 1$ if and only if $G_{i,i+1} = 0$ and $G_{i+1,i} > 0$ or $G_{i,i+1} > 0$ and $G_{i+1,i} = 0$ for $i = 1, 2, \dots, R-1$. Therefore, the index Φ_{MH} cannot discriminate between perfect upper and lower marginal-asymmetries (i.e., perfect upper marginal-asymmetry: $G_{i,i+1} > 0$ and $G_{i+1,i} = 0$ for $i = 1, 2, \dots, R-1$, and perfect lower marginal-asymmetry: $G_{i,i+1} = 0$ and $G_{i+1,i} > 0$ for $i = 1, 2, \dots, R-1$). Note that (i) $G_{i,i+1} > 0$ and $G_{i+1,i} = 0$ for $i = 1, 2, \dots, R-1$ if and only if $\pi_{ij} > 0$ and $\pi_{ji} = 0$ for $i < j$, (ii) $G_{i,i+1} = 0$ and $G_{i+1,i} > 0$ for $i = 1, 2, \dots, R-1$ if and only if $\pi_{ij} = 0$ and $\pi_{ji} > 0$ for $i < j$.

To discriminate between perfect upper and lower marginal-asymmetries, assume that $G_{i,i+1} + G_{i+1,i} > 0$ for $i = 1, 2, \dots, R-1$. Yamamoto et al. (2011) proposed the index Ψ_{MH} as follows:

$$\Psi_{\mathrm{MH}} = \frac{4}{\pi} \sum_{i=1}^{R-1} \left(G^*_{i,i+1} + G^*_{i+1,i} \right) \left(\theta_i^{\mathrm{MH}} - \frac{\pi}{4} \right),$$

where

$$\theta_i^{\mathrm{MH}} = \arccos \left(\frac{G_{i,i+1}}{\sqrt{(G_{i,i+1})^2 + (G_{i+1,i})^2}} \right).$$

The index Ψ_{MH} has the following properties: (i) $-1 \leqslant \Psi_{\mathrm{MH}} \leqslant 1$, (ii) if the MH model holds then $\Psi_{\mathrm{MH}} = 0$, but the converse is not necessarily true,

(iii) $\Psi_{\mathrm{MH}} = -1$ if and only if we have perfect upper marginal-asymmetry (i.e., $\pi_{ij} > 0$ and $\pi_{ji} = 0$ for $i < j$), and (iv) $\Psi_{\mathrm{MH}} = 1$ if and only if we have perfect lower marginal-asymmetry (i.e., $\pi_{ij} = 0$ and $\pi_{ji} > 0$ for $i < j$). Therefore, the index Ψ_{MH} can discriminate between perfect upper and lower marginal-asymmetries.