

Notes on parametric functions in the linear model

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SUMMARY

Starting from a simple Gauss–Markov model, this paper presents notes about criteria for the estimability of parametric functions of the vector of interest in linear models. The results obtained for the transformed models are compared with known results from the literature.

Key words: estimable parametric functions, linear univariate model, optimal design

1. Introduction

The theory of the linear univariate model is well known in the literature and has been developed over the years. One of its uses is to determine the optimal plan of an experiment, based on the information matrix for the estimation of the parameters of interest (Iriante et al., 2020; Graczyk and Ceranka, 2023). Moreover, the experiments can be described by linear univariate models, and the unknown parameters are estimated and verified statistically. When the experiments become more complex, and more characteristics are observed, the univariate models are replaced by multivariate models. Nevertheless, the estimation and statistical inference can still be based on the theory of univariate models (Walkowiak, 1987; Markiewicz and Szczepańska, 2007).

Estimability criteria in linear models have been considered by several authors. For example, Baksalary and Kala (1978) presented necessary and sufficient conditions for estimability in a general linear model, Alalouf and Styan (1979) introduced several characterizations of estimability, while further research covered more specific aspects of estimability on linear models, such as partitioned linear models (Seely and Birkes, 1980), testing of general linear hypotheses (Hu and Shi, 2008), or models with restrictions (Wawrzosek and Wesołowska-Janczarek, 2009).

In the simple linear model $\mathbf{y} = \mathbf{A}\boldsymbol{\beta}_1 + \mathbf{e}$, where the matrix $\mathbf{A}$ is not of full column rank, the solution of the normal equation is not unique. In that case it is worthwhile to consider criteria of estimability for the parametric vector $\mathbf{K}'\boldsymbol{\beta}_1$ such that $\mathbf{K}'\hat{\boldsymbol{\beta}}_1$ is unique. In the next section, some notes are presented on the criteria for estimability in linear models and transformed linear models. Throughout this work, bold lower case denotes vectors and bold upper case denotes matrices. Greek letters will be used to denote parameters. The symbol $\mathbf{A}^-$ will denote a general inverse of a matrix. $\mathcal{R}(\mathbf{A})$ will stand for the range of $\mathbf{A}$ and $\mathcal{R}^\perp(\mathbf{A})$ for the orthogonal complement of $\mathcal{R}(\mathbf{A})$. $\mathcal{N}(\mathbf{A})$ denotes the null space of $\mathbf{A}$. The matrix $\mathbf{Q}_{A_2} = \mathbf{I} - \mathbf{A}(\mathbf{A}'\mathbf{A})^- \mathbf{A}'$ will be the orthogonal projector onto the subspace $\mathcal{R}^\perp(\mathbf{A})$.

2. Estimable parametric function

Let us consider a simple Gauss–Markov model

$$\mathbf{y} = \mathbf{A}_1\boldsymbol{\beta}_1 + \mathbf{e}, \quad \mathbf{e} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n) \quad (1)$$

where $\mathbf{A}_1 \in \mathbb{R}_{n \times p}$ is a known nonzero matrix, while $\mathbf{e} \in \mathbb{R}^n$ is an n -dimensional vector of random errors with $E(\mathbf{e}) = \mathbf{0}$ and the known variance-covariance matrix $\sigma^2\mathbf{I}_n$. The vector $\mathbf{y} \in \mathbb{R}^n$ is an observable random vector, which depends linearly on the vector of parameters of interest, $\boldsymbol{\beta}_1 \in \mathbb{R}^p$, and represents measurements on a single response variable.

Many applications of the model (1) are based on the estimation of the vector $\boldsymbol{\beta}_1$. It is well known that the least squares estimator or maximum likelihood estimator of $\boldsymbol{\beta}_1$ in model (1) is the arbitrary solution of a normal equation of the form

$$\mathbf{A}_1'\mathbf{A}_1\boldsymbol{\beta}_1 = \mathbf{A}_1'\mathbf{y}.$$

If $\text{rank}(\mathbf{A}_1) = p$ then the estimator of $\boldsymbol{\beta}_1$ has the form

$$\hat{\boldsymbol{\beta}}_1 = (\mathbf{A}_1'\mathbf{A}_1)^{-1}\mathbf{A}_1'\mathbf{y}$$

In the case $\text{rank}(\mathbf{A}_1) < p$ the general solution can be written as

$$\hat{\boldsymbol{\beta}}_1 = (\mathbf{A}_1'\mathbf{A}_1)^-\mathbf{A}_1'\mathbf{y}$$

for some choice of $(\mathbf{A}_1'\mathbf{A}_1)^-$. Then the estimator $\hat{\boldsymbol{\beta}}_1$ is not unique and the estimation of parametric functions $\mathbf{k}_i'\boldsymbol{\beta}_1, i = \{1, 2, \dots, s\}$, $\mathbf{k}_i \in \mathbb{R}^p$ may be of interest.

Let $\mathbf{K}'\beta_1$, $\mathbf{K} = [\mathbf{k}_1 : \mathbf{k}_2 : \dots : \mathbf{k}_s]$, $\mathbf{K} \in \mathbb{R}_{p \times s}$ be the vector of parametric functions and $\{\mathbf{L}\mathbf{y} : \mathbf{L} \in \mathbb{R}_{s \times p}\}$ be the set of all linear statistics of $\mathbf{K}'\beta_1$. Then the estimability of $\mathbf{K}'\beta_1$ is specified by the following definition.

Definition 2.1 (Baksary, 1984)

The vector of parametric functions $\mathbf{K}'\beta_1$ is said to be estimable if it has a linear unbiased estimator, i.e. there exists $\mathbf{L}$ such that

$$E(\mathbf{L}\mathbf{y}) = \mathbf{L}\mathbf{A}_1\beta_1 = \mathbf{K}'\beta_1 \text{ for all } \beta_1 \in \mathbb{R}^p.$$

Moreover, the subspace of the vector of estimable parametric functions is given below.

Theorem 2.1 (Baksalary, 1984)

The vector of parametric functions $\mathbf{K}'\beta_1$ is estimable if and only if there exists $\mathbf{L} \in \mathbb{R}^{n \times p}$ such that $\mathbf{K} = \mathbf{A}_1'\mathbf{L}'$, meaning $\mathcal{R}(\mathbf{K}) \subset \mathcal{R}(\mathbf{A}_1')$.

From the above theorem it follows that $\mathbf{K}'\hat{\beta}_1 = \mathbf{K}'(\mathbf{A}_1'\mathbf{A}_1)^{-}\mathbf{A}_1'\mathbf{y}$ in model (1) is unique and invariant for all $(\mathbf{A}_1'\mathbf{A}_1)^{-}$ if and only if $\mathcal{R}(\mathbf{K}) \subset \mathcal{R}(\mathbf{A}_1')$.

If the effect of the nuisance parameters on the observable random vector, $\mathbf{y}$, is considered, then the model (1) can be extended to a Gauss–Markov model with nuisance parameters (see Baksalary, 1984):

$$\mathbf{y} = \mathbf{A}_1\beta_1 + \mathbf{A}_2\beta_2 + \mathbf{e}, \quad \mathbf{e} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n), \quad (2)$$

where $\mathbf{A}_2 \in \mathbb{R}_{n \times q}$ is the known nonzero matrix and $\beta_2 \in \mathbb{R}^q$ is an unknown vector of nuisance parameters.

In the model (2) the vector of parametric functions $\mathbf{K}'\beta_1$ is estimable if and only if $\mathcal{R}(\mathbf{K}) \subset \mathcal{R}(\mathbf{A}_1'\mathbf{Q}_{A_2})$ (Baksalary, 1984). Moreover, the classes of parametric functions are identical in model (1) and (2) if and only if the subspaces $\mathcal{R}(\mathbf{A}_1)$, $\mathcal{R}(\mathbf{A}_2)$ are disjoint,

$$\mathcal{R}(\mathbf{A}_1) \cap \mathcal{R}(\mathbf{A}_2) = \{\mathbf{0}\} \quad (3)$$

(Baksalary, 1984). The condition $\mathcal{R}(\mathbf{A}_1) \cap \mathcal{R}(\mathbf{A}_2) = \{\mathbf{0}\}$ is equivalent to $\mathcal{R}(\mathbf{A}_1'\mathbf{Q}_{A_2}) = \mathcal{R}(\mathbf{A}_1')$ (Puntanen et al., 2011, chapter 16). The dimension of the space $\mathcal{R}(\mathbf{A}_1) \cap \mathcal{R}(\mathbf{A}_2)$ shows how the nuisance parameters diminish the subspace.

If $\text{rank}(\mathbf{A}_2) < n$ then the model (2) can be transformed to the model given below:

$$\mathbf{Q}_{A_2}\mathbf{y} = \mathbf{Q}_{A_2}\mathbf{A}_1\beta_1 + \mathbf{Q}_{A_2}\mathbf{e}, \quad \mathbf{e} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n), \quad (4)$$

where $\mathbf{Q}_{A_2} = \mathbf{I}_q - \mathbf{A}_2(\mathbf{A}_2'\mathbf{A}_2)^{-}\mathbf{A}_2'$.

From Theorem 2.1, the vector of linear parametric functions $\mathbf{K}'\boldsymbol{\beta}_1$ is estimable in model (4) if and only if $\mathcal{R}(\mathbf{K}) \subset \mathcal{R}(\mathbf{A}_1'\mathbf{Q}_{A_2})$. This means that under condition (3) the same parametric functions $\mathbf{K}'\boldsymbol{\beta}_1$ are estimable in models given by (1), (2) and (4).

Let us consider a more general transformation of model (2),

$$\mathbf{H}\mathbf{y} = \mathbf{H}\mathbf{A}_1\boldsymbol{\beta}_1 + \mathbf{H}\mathbf{e}, \quad \mathbf{e} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n), \quad (5)$$

where $\mathbf{H} \in \mathcal{R}_{s \times n}$, $s \geq n - r(\mathbf{A}_2)$ is any matrix of maximum rank such that $\mathbf{H}\mathbf{A}_2 = \mathbf{0}$ (Markiewicz, 2001). The condition for the estimability of $\mathbf{K}'\boldsymbol{\beta}_1$ is formulated in the following theorem.

Theorem 2.2

The vector of parametric functions $\mathbf{K}'\boldsymbol{\beta}_1$ is estimable in model (5) if and only if $\mathcal{R}(\mathbf{K}) \subset \mathcal{R}(\mathbf{A}_1'\mathbf{Q}_{A_2})$.

Proof:

From Theorem 2.1, the vector of parametric functions $\mathbf{K}'\boldsymbol{\beta}_1$ is estimable in model (5) if and only if $\mathcal{R}(\mathbf{K}) \subset \mathcal{R}(\mathbf{A}_1'\mathbf{H}')$. Observe that if $\mathbf{H}\mathbf{A}_2 = \mathbf{0}$ then $\mathcal{R}(\mathbf{A}_2) \subset \mathcal{N}(\mathbf{H})$. This condition is equivalent to $\mathcal{R}(\mathbf{A}_2) \subset \mathcal{R}^\perp(\mathbf{H}')$, which implies $\mathcal{R}(\mathbf{H}') \subset \mathcal{R}(\mathbf{Q}_{A_2})$. From the assumption that $\mathbf{H}$ has maximum rank, we obtain $\mathcal{R}(\mathbf{H}') = \mathcal{R}(\mathbf{Q}_{A_2})$, which implies that $\mathcal{R}(\mathbf{A}_1'\mathbf{H}') = \mathcal{R}(\mathbf{A}_1'\mathbf{Q}_{A_2})$.

Corollary 2.1

The subspace of the vector of estimable parametric functions $\mathbf{K}'\boldsymbol{\beta}_1$ in model (5) is the same as in model (2).

In the case $\text{rank}(\mathbf{A}_2) = n$ none of $\mathbf{K}'\boldsymbol{\beta}_1$ is estimable in model (5). Some approaches to nonestimable functions have been given by Majumdar and Mitra (1980) and Baksalary and Markiewicz (1990).

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