

# Exact ranges of certain generalized polynomials with rational parameters

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**Abstract** For

$$f_{\alpha}(n) = \lfloor \alpha^2 n \rfloor - \lfloor \alpha \lfloor \alpha n \rfloor \rfloor, \quad n \in \mathbb{N},$$

we give an exact finite description of  $\text{Range}(f_{\alpha})$  for every positive rational parameter  $\alpha$ . In particular, this settles all supra-unitary rational cases and proves Conjecture 3.17 from the authors' previous work [1].

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## 1 Introduction

This work represents a companion to and natural continuation of the authors' paper [1] which considered the precise description of the ranges of certain generalized polynomials depending on a positive parameter  $\alpha$ , given by

$$f_{\alpha}(n) = \lfloor \alpha^2 n \rfloor - \lfloor \alpha \lfloor \alpha n \rfloor \rfloor, \quad n \in \mathbb{N}.$$

Generalized polynomials of this type arise in a natural way when considering dilated floor functions and their commutators, which in turn are motivated by investigations of digital straight lines; see, for example, [5] and the discussion in [1, Introduction]. While several answers regarding the range of  $f_{\alpha}$  for rational and irrational parameters  $\alpha$  were provided in [1], a few questions remained open. The recent work of Dubickas [3] settled the conjectures related to the irrational case, thus completing the description of  $\text{Range}(f_{\alpha})$  for every positive irrational parameter  $\alpha$ .

The goal of this paper is to complete the description of  $\text{Range}(f_{\alpha})$  for every positive rational parameter  $\alpha$ . With the exception of the sub-unitary rational parameter case

and several particular cases of supra-unitary parameters  $\alpha$  that appeared in [1], a result unifying all possible rational cases proved elusive. A nice insight into settling the rational parameter case came from ChatGPT, but the formulations and arguments presented below are due to the authors. In fact, the essential tools required for the exact characterization were already developed in [1].

Throughout this paper, we will follow the notation used in [1]. By  $\mathbb{Z}$  we denote the set of integers, by  $\mathbb{N}$  the set of positive integers, and by  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  the set of nonnegative integers. Given a real number  $x$ ,  $\lfloor x \rfloor$  and  $\lceil x \rceil$  denote the floor and ceiling of  $x$ , respectively; see [4, Chapter 3] for the history and properties of these functions. The fractional part of  $x \in \mathbb{R}$  is defined by

$$\langle x \rangle := x - \lfloor x \rfloor,$$

so that  $0 \leq \langle x \rangle < 1$ . Here we opt, as in [2], for the notation  $\langle x \rangle$  for the fractional part of a real number, rather than the more common  $\{x\}$ , since braces are consistently used for set notation; in particular,  $\{x\}$  denotes a singleton set.

For integers  $p \leq q$ , we write

$$\llbracket p, q \rrbracket := \{m \in \mathbb{Z} : p \leq m \leq q\}.$$

We will also use the following dilation and shift operations on a set: given a set  $A \subset \mathbb{R}$ , and given two scalars  $x, s \in \mathbb{R}$ , we let

$$x + sA := \{x + sa : a \in A\}.$$

## 2 Preliminaries

In this section we prove three results and recall one lemma from [1] that are needed in the proof of Theorem 3.1 below.

**Lemma 2.1.** *Let  $b \in \mathbb{N}$ . The function  $\psi_b : \llbracket 0, b-1 \rrbracket^2 \rightarrow \llbracket 0, b^2-1 \rrbracket$  defined by*

$$\psi_b(x, y) = bx + y, \quad (x, y) \in \llbracket 0, b-1 \rrbracket^2,$$

*is a bijection.*

*Proof.* Since both the domain and codomain have cardinality  $b^2$ , it suffices to prove that  $\psi_b$  is an injection. Let  $x_1, x_2, y_1, y_2 \in \llbracket 0, b-1 \rrbracket$  be such that  $\psi_b(x_1, y_1) = \psi_b(x_2, y_2)$ . Then

$$b(x_1 - x_2) = y_1 - y_2.$$

Since  $b$  divides  $y_1 - y_2$  and  $|y_1 - y_2| \leq b-1$ , it follows that  $y_1 = y_2$  and hence  $x_1 = x_2$ .  $\square$

Given a positive integer  $c \in \mathbb{N}$  we define the remainder function  $\rho_c : \mathbb{Z} \rightarrow \llbracket 0, c-1 \rrbracket$  by

$$\rho_c(x) = x - c \left\lfloor \frac{x}{c} \right\rfloor, \quad x \in \mathbb{Z}.$$

**Lemma 2.2.** *Let  $a, c \in \mathbb{N}$  be such that  $\gcd(a, c) = 1$ . The function  $\phi_{a,c} : \llbracket 0, c-1 \rrbracket \rightarrow \llbracket 0, c-1 \rrbracket$  defined by*

$$\phi_{a,c}(x) = \rho_c(ax) = ax - c \left\lfloor \frac{ax}{c} \right\rfloor, \quad x \in \llbracket 0, c-1 \rrbracket,$$

*is a bijection.*

*Proof.* By the definition of the floor function, we have

$$ax - c < \left\lfloor \frac{ax}{c} \right\rfloor c \leq ax,$$

proving that  $\phi_{a,c}(x) \in \llbracket 0, c-1 \rrbracket$  for all  $x \in \llbracket 0, c-1 \rrbracket$ .

To prove that  $\phi_{a,c}$  is an injection, let  $x_1, x_2 \in \llbracket 0, c-1 \rrbracket$  be such that  $x_1 \leq x_2$  and  $\phi_{a,c}(x_1) = \phi_{a,c}(x_2)$ . Then

$$c \left( \left\lfloor \frac{ax_2}{c} \right\rfloor - \left\lfloor \frac{ax_1}{c} \right\rfloor \right) = a(x_2 - x_1).$$

Since  $\gcd(a, c) = 1$ , by Euclid's lemma, the last displayed equality yields that  $c$  divides  $x_2 - x_1$ , and since  $x_2 - x_1 \in \llbracket 0, c-1 \rrbracket$  we have  $x_1 = x_2$ , proving that  $\phi_{a,c} : \llbracket 0, c-1 \rrbracket \rightarrow \llbracket 0, c-1 \rrbracket$  is an injection. As  $\llbracket 0, c-1 \rrbracket$  is finite, it is a bijection.  $\square$

**Remark 2.1.** If  $a, c, r \in \mathbb{N}$  are such that  $\gcd(a, c) = 1$  and  $c$  divides  $a-r$ , then  $\phi_{a,c} = \phi_{r,c}$ .

**Remark 2.2.** Let  $a, c \in \mathbb{N}$  satisfy  $\gcd(a, c) = 1$ , and let  $x, y \in \llbracket 0, c-1 \rrbracket$ . Then  $y = \phi_{a,c}(x)$  if and only if  $ax = c \left\lfloor \frac{ax}{c} \right\rfloor + y$ . Equivalently, using  $x = \phi_{a,c}^{-1}(y)$  we have

$$a\phi_{a,c}^{-1}(y) = c \left\lfloor \frac{a\phi_{a,c}^{-1}(y)}{c} \right\rfloor + y$$

Combining the previous two lemmas yields the following result.

**Corollary 2.3.** *Let  $a, b \in \mathbb{N}$  be such that  $\gcd(a, b) = 1$ . The function  $\Xi_{a,b} : \llbracket 0, b-1 \rrbracket^2 \rightarrow \llbracket 0, b^2-1 \rrbracket$  defined by*

$$\Xi_{a,b}(x, y) = \phi_{a,b^2}^{-1}(\psi_b(\phi_{a,b}^{-1}(x), y)), \quad x, y \in \llbracket 0, b-1 \rrbracket,$$

*is a bijection.*

The next lemma appears as Lemma 3.1 in [1]. Here, we find it convenient to extend the domain of  $f_{a/b}$  to include 0 with  $f_{a/b}(0) = 0$ ; this does not change the range of  $f_{a/b}$  since  $f_{a/b}(b^2) = 0$ .

**Lemma 2.4.** *Let  $a \in \mathbb{N}$  and  $b \in \mathbb{N}$  be relatively prime. Then*

$$\text{Range}(f_{a/b}) = f_{a/b}(\llbracket 0, b^2-1 \rrbracket).$$

### 3 Main result

**Theorem 3.1.** *Let  $\alpha = a/b$ , where  $a, b \in \mathbb{N}$ ,  $\gcd(a, b) = 1$ . We have  $\text{Range}(f_{a/b}) = \{0\}$  if and only if  $b = 1$  or  $a = 1$ . If  $b \in \mathbb{N} \setminus \{1\}$ , then*

$$\text{Range}(f_{a/b}) = \bigcup_{j=0}^{\mu_{a,b}} \left( j + \left\lfloor \frac{a}{b^2} \right\rfloor \mathbb{Y}_j \right), \quad (3.1)$$

where

$$\mu_{a,b} = \left\lfloor \frac{b^2 \left\langle \frac{a}{b^2} \right\rangle - 1}{b + 1} \right\rfloor \in \llbracket 0, b - 1 \rrbracket \quad (3.2)$$

and, for  $j \in \llbracket 0, \mu_{a,b} \rrbracket$ ,

$$\mathbb{Y}_j = \left\{ k \in \llbracket 0, b - 1 \rrbracket : (j - 1) + \frac{1}{b} \leq \left\langle \frac{a}{b^2} \right\rangle k < j + 1 \right\}$$

are nonempty sets.

*Proof.* If  $b = 1$ , then trivially we have  $\text{Range}(f_{a/b}) = \{0\}$ . For the remainder of the proof, we assume that  $b \in \mathbb{N} \setminus \{1\}$ .

**Step 1.** We use the bijection  $\Xi_{a,b} : \llbracket 0, b - 1 \rrbracket^2 \rightarrow \llbracket 0, b^2 - 1 \rrbracket$  defined in Corollary 2.3 and prove that for all  $n \in \llbracket 0, b^2 - 1 \rrbracket$  we have

$$f_{a/b}(n) = \left\lfloor \frac{ay}{b^2} + \frac{x}{b} \right\rfloor \quad \text{where} \quad (x, y) = \Xi_{a,b}^{-1}(n) \in \llbracket 0, b - 1 \rrbracket^2. \quad (3.3)$$

Assume  $\Xi_{a,b}(x, y) = n$ . Then, applying  $\phi_{a,b^2}$  to both sides of  $\Xi_{a,b}(x, y) = n$  we obtain  $\phi_{a,b^2}(n) = \psi_b(\phi_{a,b}^{-1}(x), y)$ . The definition of  $\psi_b$  yields  $\phi_{a,b^2}(n) = b\phi_{a,b}^{-1}(x) + y$ , and by the definition of  $\phi_{a,b^2}$  we have

$$an - b^2 \left\lfloor \frac{an}{b^2} \right\rfloor = b\phi_{a,b}^{-1}(x) + y.$$

Setting  $m = \left\lfloor \frac{an}{b^2} \right\rfloor$  we have

$$an = b^2 m + b\phi_{a,b}^{-1}(x) + y.$$

Consequently,

$$\left\lfloor \frac{a^2}{b^2} n \right\rfloor = \left\lfloor \frac{a}{b^2} (b^2 m + b\phi_{a,b}^{-1}(x) + y) \right\rfloor = am + \left\lfloor \frac{a\phi_{a,b}^{-1}(x)}{b} + \frac{ay}{b^2} \right\rfloor, \quad (3.4)$$

and

$$\left\lfloor \frac{a}{b} n \right\rfloor = \left\lfloor bm + \phi_{a,b}^{-1}(x) + \frac{y}{b} \right\rfloor = bm + \phi_{a,b}^{-1}(x).$$

Hence

$$\left\lfloor \frac{a}{b} \left\lfloor \frac{a}{b} n \right\rfloor \right\rfloor = \left\lfloor \frac{a}{b} (bm + \phi_{a,b}^{-1}(x)) \right\rfloor = am + \left\lfloor \frac{a\phi_{a,b}^{-1}(x)}{b} \right\rfloor. \quad (3.5)$$

Subtracting (3.5) from (3.4) we get

$$f_{a/b}(n) = \left\lfloor \frac{a\phi_{a,b}^{-1}(x)}{b} + \frac{ay}{b^2} \right\rfloor - \left\lfloor \frac{a\phi_{a,b}^{-1}(x)}{b} \right\rfloor. \quad (3.6)$$

Since, by Remark 2.2, we have

$$a\phi_{a,b}^{-1}(x) = b \left\lfloor \frac{a\phi_{a,b}^{-1}(x)}{b} \right\rfloor + x,$$

and thus

$$\frac{a\phi_{a,b}^{-1}(x)}{b} = \left\lfloor \frac{a\phi_{a,b}^{-1}(x)}{b} \right\rfloor + \frac{x}{b},$$

the equality in (3.6) becomes

$$f_{a/b}(n) = \left\lfloor \frac{x}{b} + \frac{ay}{b^2} \right\rfloor.$$

For arbitrary  $x, y \in \llbracket 0, b-1 \rrbracket$ , setting  $n = \Xi_{a,b}(x, y)$  we have that (3.3) holds. Therefore,

$$\text{Range}(f_{a/b}) = \left\{ \left\lfloor \frac{ak}{b^2} + \frac{l}{b} \right\rfloor : k, l \in \llbracket 0, b-1 \rrbracket \right\}. \quad (3.7)$$

**Step 2.** By the definition of the floor function we have

$$\left\lfloor \frac{ak}{b^2} \right\rfloor \leq \frac{ak}{b^2} < \left\lfloor \frac{ak}{b^2} \right\rfloor + 1 \quad \text{and} \quad \frac{ak}{b^2} + \frac{b-1}{b} < \frac{ak}{b^2} + 1 < \left\lfloor \frac{ak}{b^2} \right\rfloor + 2,$$

thus

$$\left\lfloor \frac{ak}{b^2} \right\rfloor \leq \frac{ak}{b^2} + \frac{b-1}{b} < \left\lfloor \frac{ak}{b^2} \right\rfloor + 2.$$

Consequently, for all  $l \in \llbracket 0, b-1 \rrbracket$  we have

$$\left\lfloor \frac{ak}{b^2} \right\rfloor \leq \frac{ak}{b^2} + \frac{l}{b} < \left\lfloor \frac{ak}{b^2} \right\rfloor + 2.$$

Therefore, by the definition of the floor function, for all  $l \in \llbracket 0, b-1 \rrbracket$  we have

$$\left\lfloor \frac{ak}{b^2} + \frac{l}{b} \right\rfloor = \left\lfloor \frac{ak}{b^2} \right\rfloor \quad \text{or} \quad \left\lfloor \frac{ak}{b^2} + \frac{l}{b} \right\rfloor = \left\lfloor \frac{ak}{b^2} \right\rfloor + 1,$$

or, equivalently, for all  $l \in \llbracket 0, b-1 \rrbracket$  we have

$$\left\lfloor \frac{ak}{b^2} + \frac{l}{b} \right\rfloor = \left\lfloor \frac{ak}{b^2} \right\rfloor \quad \text{or} \quad \left\lfloor \frac{ak}{b^2} + \frac{l}{b} \right\rfloor = \left\lfloor \frac{ak}{b^2} + \frac{b-1}{b} \right\rfloor.$$

Now

$$\text{Range}(f_{a/b}) = \left\{ \left\lfloor \frac{ak}{b^2} \right\rfloor, \left\lfloor \frac{ak}{b^2} + \frac{b-1}{b} \right\rfloor : k \in \llbracket 0, b-1 \rrbracket \right\} \quad (3.8)$$

follows from (3.7). By the definition of the fractional part, we have

$$\frac{a}{b^2} = \left\lfloor \frac{a}{b^2} \right\rfloor + \left\langle \frac{a}{b^2} \right\rangle,$$

thus, for all  $k \in \llbracket 0, b-1 \rrbracket$ , we get

$$\frac{ak}{b^2} = \left\lfloor \frac{a}{b^2} \right\rfloor k + \left\langle \frac{a}{b^2} \right\rangle k.$$

Therefore,

$$\left\lfloor \frac{ak}{b^2} \right\rfloor = \left\lfloor \frac{a}{b^2} \right\rfloor k + \left\lfloor \left\langle \frac{a}{b^2} \right\rangle k \right\rfloor$$

and

$$\left\lfloor \frac{ak}{b^2} + \frac{b-1}{b} \right\rfloor = \left\lfloor \frac{a}{b^2} \right\rfloor k + \left\lfloor \left\langle \frac{a}{b^2} \right\rangle k + \frac{b-1}{b} \right\rfloor.$$

Thus, setting

$$\mathbb{X}_k = \left\{ \left\lfloor \left\langle \frac{a}{b^2} \right\rangle k \right\rfloor, \left\lfloor \left\langle \frac{a}{b^2} \right\rangle k + \frac{b-1}{b} \right\rfloor \right\}, \quad k \in \llbracket 0, b-1 \rrbracket,$$

the range can be written as

$$\text{Range}(f_{a/b}) = \bigcup_{k=0}^{b-1} \left( \left\lfloor \frac{a}{b^2} \right\rfloor k + \mathbb{X}_k \right). \quad (3.9)$$

**Step 3.** Set

$$\mathbb{J} = \bigcup_{k=0}^{b-1} \mathbb{X}_k.$$

Since  $b \in \mathbb{N} \setminus \{1\}$ , applying the two-step division of  $a$  in base  $b$  we have

$$a = b^2 s + bq + r, \quad (3.10)$$

where the unique coefficients are

$$s \in \mathbb{N}_0, \quad q \in \llbracket 0, b-1 \rrbracket, \quad r \in \llbracket 1, b-1 \rrbracket.$$

It follows from (3.10) that

$$s = \left\lfloor \frac{a}{b^2} \right\rfloor \quad \text{and} \quad \left\langle \frac{a}{b^2} \right\rangle = \frac{bq + r}{b^2}.$$

These equalities are relevant in the following two derivations of the expressions for  $\max \mathbb{J}$  and  $\mu_{a,b}$  in terms of  $q$  and  $r$ . Since both members of  $\mathbb{X}_k$  are nondecreasing in  $k$ , we have

$$\begin{aligned} \max \mathbb{J} &= \left\lfloor \left\langle \frac{a}{b^2} \right\rangle (b-1) + \frac{b-1}{b} \right\rfloor \\ &= \left\lfloor \frac{(bq + r)(b-1)}{b^2} + \frac{b-1}{b} \right\rfloor \\ &= \left\lfloor q + 1 + \frac{b(r-1-q) - r}{b^2} \right\rfloor \end{aligned}$$

$$= q + \begin{cases} 0, & 1 \leq r \leq q+1, \\ 1, & q+2 \leq r \leq b-1. \end{cases}$$

Similarly,

$$\mu_{a,b} = \left\lceil \frac{b^2 \left\langle \frac{a}{b^2} \right\rangle - 1}{b+1} \right\rceil = \left\lceil \frac{bq + r - 1}{b+1} \right\rceil = q + \left\lceil \frac{r - q - 1}{b+1} \right\rceil = q + \begin{cases} 0, & 1 \leq r \leq q+1, \\ 1, & q+2 \leq r \leq b-1. \end{cases}$$

Consequently  $\mu_{a,b} \in \llbracket 0, b-1 \rrbracket$  and  $\mu_{a,b} = \max \mathbb{J}$ .

Let  $j \in \llbracket 0, \mu_{a,b} \rrbracket$  and  $k \in \llbracket 0, b-1 \rrbracket$ . Then  $j \in \mathbb{X}_k$  if and only if  $k \in \mathbb{Y}_j$ . Indeed, the following equivalences hold

$$\begin{aligned} j \in \mathbb{X}_k &\Leftrightarrow j = \left\lfloor \left\langle \frac{a}{b^2} \right\rangle k \right\rfloor \quad \text{or} \quad j = \left\lfloor \left\langle \frac{a}{b^2} \right\rangle k + \frac{b-1}{b} \right\rfloor \\ &\Leftrightarrow j \leq \left\langle \frac{a}{b^2} \right\rangle k < j+1 \quad \text{or} \quad j \leq \left\langle \frac{a}{b^2} \right\rangle k + 1 - \frac{1}{b} < j+1 \\ &\Leftrightarrow j \leq \left\langle \frac{a}{b^2} \right\rangle k < j+1 \quad \text{or} \quad j-1 + \frac{1}{b} \leq \left\langle \frac{a}{b^2} \right\rangle k < j + \frac{1}{b} \\ &\Leftrightarrow j-1 + \frac{1}{b} \leq \left\langle \frac{a}{b^2} \right\rangle k < j+1 \\ &\Leftrightarrow k \in \mathbb{Y}_j. \end{aligned}$$

Furthermore, the following equivalences hold

$$\begin{aligned} x \in \bigcup_{k=0}^{b-1} \left( \left\lfloor \frac{a}{b^2} \right\rfloor k + \mathbb{X}_k \right) &\Leftrightarrow \exists k \in \llbracket 0, b-1 \rrbracket : x \in \left\lfloor \frac{a}{b^2} \right\rfloor k + \mathbb{X}_k \\ &\Leftrightarrow \exists k \in \llbracket 0, b-1 \rrbracket \exists j \in \llbracket 0, \mu_{a,b} \rrbracket : j \in \mathbb{X}_k \text{ and } x = \left\lfloor \frac{a}{b^2} \right\rfloor k + j \\ &\Leftrightarrow \exists j \in \llbracket 0, \mu_{a,b} \rrbracket \exists k \in \llbracket 0, b-1 \rrbracket : k \in \mathbb{Y}_j \text{ and } x = j + \left\lfloor \frac{a}{b^2} \right\rfloor k \\ &\Leftrightarrow \exists j \in \llbracket 0, \mu_{a,b} \rrbracket : x \in j + \left\lfloor \frac{a}{b^2} \right\rfloor \mathbb{Y}_j \\ &\Leftrightarrow x \in \bigcup_{j=0}^{\mu_{a,b}} \left( j + \left\lfloor \frac{a}{b^2} \right\rfloor \mathbb{Y}_j \right). \end{aligned}$$

Now (3.1) follows from (3.9).

**Step 4.** We prove  $\mathbb{Y}_j \neq \emptyset$  for all  $j \in \llbracket 0, \mu_{a,b} \rrbracket$ . Since  $b \in \mathbb{N} \setminus \{1\}$  and  $\gcd(a, b) = 1$ , we have  $\langle a/(b^2) \rangle \in (0, 1)$ . For simplicity we write  $\mu = \mu_{a,b}$ . For  $j = 0$ , we have  $0 \in \mathbb{Y}_0$ . We claim that for every  $j \in \llbracket 1, \mu \rrbracket$  we have

$$\kappa_j = \left\lceil \frac{j-1 + 1/b}{\langle a/(b^2) \rangle} \right\rceil \in \mathbb{Y}_j.$$

By the definition of the ceiling function, we have

$$\mu - 1 < \frac{b^2 \left\langle \frac{a}{b^2} \right\rangle - 1}{b+1},$$

hence

$$(\mu - 1)(b + 1) + 1 < b^2 \left\langle \frac{a}{b^2} \right\rangle = a - b^2 \left\lfloor \frac{a}{b^2} \right\rfloor.$$

Since the last strict inequality is stated for two integers, we have

$$(\mu - 1)(b + 1) + 2 \leq b^2 \left\langle \frac{a}{b^2} \right\rangle.$$

Multiplying both sides by  $b - 1 > 0$ , we obtain

$$(\mu - 1)(b^2 - 1) + 2(b - 1) \leq b^2(b - 1) \left\langle \frac{a}{b^2} \right\rangle,$$

or, equivalently,

$$b^2(\mu - 1) + b + b - 1 - \mu \leq b^2(b - 1) \left\langle \frac{a}{b^2} \right\rangle.$$

Since  $b - 1 - \mu \geq 0$  we have

$$b^2(\mu - 1) + b \leq b^2(b - 1) \left\langle \frac{a}{b^2} \right\rangle.$$

Consequently, for every  $j \in \llbracket 1, \mu \rrbracket$  we have

$$-b^2 + b \leq b^2(j - 1) + b \leq b^2(\mu - 1) + b \leq b^2(b - 1) \left\langle \frac{a}{b^2} \right\rangle.$$

Dividing by  $b^2 \langle a/(b^2) \rangle > 0$  and taking the ceiling of each term in the resulting inequalities shows that  $\kappa_j \in \llbracket 0, b - 1 \rrbracket$ . Since  $x \leq \lceil x \rceil$ , we have

$$j - 1 + 1/b \leq \left\langle \frac{a}{b^2} \right\rangle \kappa_j.$$

Since  $\lceil x \rceil < x + 1$  and  $j - 1 + 1/b + \langle a/(b^2) \rangle < j + 1$ , we have that

$$\left\langle \frac{a}{b^2} \right\rangle \kappa_j < j + 1.$$

Hence  $\kappa_j \in \mathbb{Y}_j$ .

**Step 5.** We prove that  $\text{Range}(f_{a/b}) = \{0\}$  if and only if  $b = 1$  or  $a = 1$ .

If  $b = 1$ ,  $\text{Range}(f_{a/b}) = \{0\}$  is clear. If  $a = 1$  and  $b \in \mathbb{N} \setminus \{1\}$ , then (3.1), with  $\mu_{a,b} = 0$  and  $\lfloor a/(b^2) \rfloor = 0$ , yields  $\text{Range}(f_{a/b}) = \{0\}$ .

For the converse, we prove the contrapositive: if  $a, b \in \mathbb{N} \setminus \{1\}$ , then  $\text{Range}(f_{a/b}) \setminus \{0\} \neq \emptyset$ . Assume  $a, b \in \mathbb{N} \setminus \{1\}$ . By (3.2), if  $\lfloor a/(b^2) \rfloor = 0$ , then  $\mu_{a,b} = \lceil (a - 1)/(b + 1) \rceil \geq 1$ , and  $(1 + \lfloor a/(b^2) \rfloor) \mathbb{Y}_1 \setminus \{0\} \neq \emptyset$ . If  $\lfloor a/(b^2) \rfloor \geq 1$ , then  $1 \in \mathbb{Y}_0$ , and  $(0 + \lfloor a/(b^2) \rfloor) \mathbb{Y}_0 \setminus \{0\} \neq \emptyset$ . The contrapositive follows.  $\square$

## 4 Earlier results and a previous conjecture

This section emphasizes the scope of Theorem 3.1 by revisiting the results for rational parameters proved in [1], as well as a question left open there. However, we restrict our discussion only to a selection of these results.

Note first that for  $a, b \in \mathbb{N}$  such that  $1 < a < b$ , if  $k \in \llbracket 0, b-1 \rrbracket$  then

$$0 \leq \frac{ak}{b^2} = \frac{a}{b} \cdot \frac{k}{b} < 1,$$

and

$$0 \leq \frac{ak}{b^2} + \frac{b-1}{b} \leq \frac{a(b-1)}{b^2} + \frac{b-1}{b} = \frac{b-1}{b} \cdot \frac{a+b}{b} < 2.$$

Moreover, a simple calculation shows that

$$\frac{a(b-1)}{b^2} + \frac{b-1}{b} - 1 = \frac{(b-1)a-b}{b^2} \geq \frac{2(b-1)-b}{b^2} \geq 0.$$

Therefore, if  $\gcd(a, b) = 1$  and  $1 < a < b$ , then  $\text{Range}(f_{a/b}) = \{0, 1\}$ , which recovers [1, Proposition 3.4].

Next, consider the supra-unitary rational (non-integer) case. Let  $b \in \mathbb{N} \setminus \{1\}$  and suppose that

$$a = sb^2 + b + 1, \quad s \in \mathbb{N}_0.$$

In this case, we have  $\mu_{a,b} = 1$ . Thus, by Theorem 3.1, we get

$$\text{Range}(f_{a/b}) = \bigcup_{j=0}^1 (j + s\mathbb{Y}_j).$$

Now, by the definition of the sets  $\mathbb{Y}_j$ , we have

$$\mathbb{Y}_0 = \left\{ k \in \llbracket 0, b-1 \rrbracket : \frac{-b^2+b}{b+1} \leq k < \frac{b^2}{b+1} \right\} = \llbracket 0, b-1 \rrbracket$$

as  $(-b^2+b)/(b+1) \leq 0$  and  $b-1 < b^2/(b+1)$ , and

$$\mathbb{Y}_1 = \left\{ k \in \llbracket 0, b-1 \rrbracket : \frac{b}{b+1} \leq k < \frac{2b^2}{b+1} \right\} = \llbracket 1, b-1 \rrbracket$$

as  $0 < b/(b+1) < 1$  and  $b-1 < 2b^2/(b+1)$ . In other words, we find that

$$\text{Range}(f_{sb+1+1/b}) = (s\llbracket 0, b-1 \rrbracket) \cup (1 + s\llbracket 1, b-1 \rrbracket),$$

which proves [1, Corollary 3.14].

Finally, a calculation similar to the one above with

$$a = sb^2 - c = (s-1)b^2 + b^2 - c, \quad s \in \mathbb{N},$$

where  $c \in \llbracket 1, b-1 \rrbracket$  with  $\gcd(b, c) = 1$ , yields  $\mu_{a,b} = b-1$  and

$$\text{Range}(f_{a/b}) = \bigcup_{j=0}^{b-1} (j + (s-1)\mathbb{Y}_j).$$

Furthermore, a straightforward calculation gives  $\mathbb{Y}_j = \{j, j+1\}$  for  $j \in \llbracket 0, b-2 \rrbracket$  and  $\mathbb{Y}_{b-1} = \{b-1\}$ . Thus, after a slight rewriting, we get

$$\text{Range}(f_{sb-c/b}) = (s\llbracket 0, b-1 \rrbracket) \cup (-1 + s\llbracket 1, b-1 \rrbracket),$$

which confirms the statement in [1, Conjecture 3.17].

## 5 Attaining the range cardinalities

The goal of this section is to prove that, for each fixed  $b \in \mathbb{N}$ , every integer in  $\llbracket 1, 2b - 1 \rrbracket$  is attained as the cardinality of  $\text{Range}(f_{a/b})$  for some  $a \in \mathbb{N}$  relatively prime to  $b$ . This is the content of the following theorem.

**Theorem 5.1.** *Let  $b \in \mathbb{N}$ . Then, as  $a$  ranges over all positive integers relatively prime to  $b$ , the cardinalities of the ranges  $\text{Range}(f_{a/b})$  are exactly the integers from 1 to  $2b - 1$ . That is,*

$$\left\{ |\text{Range}(f_{a/b})| : a \in \mathbb{N}, \gcd(a, b) = 1 \right\} = \llbracket 1, 2b - 1 \rrbracket.$$

*Proof.* If  $b = 1$ , then the result holds trivially; in this case we have

$$|\text{Range}(f_{a/b})| = |\{0\}| = 1.$$

For the remainder of the proof we assume  $b \in \mathbb{N} \setminus \{1\}$ . Recall first that in Step 2 of the proof of Theorem 3.1 we showed that if  $a, b \in \mathbb{N}$  and  $\gcd(a, b) = 1$ , then

$$\text{Range}(f_{a/b}) = \left\{ \left\lfloor \frac{ak}{b^2} \right\rfloor, \left\lfloor \frac{ak}{b^2} + \frac{b-1}{b} \right\rfloor : k \in \llbracket 0, b-1 \rrbracket \right\}. \quad (5.1)$$

Formula (5.1) lists two values for each  $k \in \llbracket 0, b-1 \rrbracket$ . However, for  $k = 0$  both listed values are 0. Therefore

$$|\text{Range}(f_{a/b})| \leq 2b - 1$$

for every  $a \in \mathbb{N}$  relatively prime to  $b$ . In particular, the value  $2b$  never occurs as a cardinality  $|\text{Range}(f_{a/b})|$ . Thus, it remains to show that every value  $t \in \llbracket 1, 2b - 1 \rrbracket$  is attained by  $|\text{Range}(f_{a/b})|$  for an appropriate choice of  $a$ . We do so by considering two cases:

$$t \in \llbracket 1, b \rrbracket \quad \text{and} \quad t \in \llbracket b+1, 2b-1 \rrbracket.$$

**Step 1.** Given  $t \in \llbracket 1, b \rrbracket$ , let  $a = b(t-1) + 1$ ; note that  $\gcd(a, b) = 1$ . It is immediate to see that  $\mu_{a,b} = t-1$  and  $\lfloor a/b^2 \rfloor = 0$ . By Theorem 3.1, we obtain that

$$\text{Range}(f_{a/b}) = \bigcup_{j=0}^{t-1} (j + 0\mathbb{Y}_j) = \llbracket 0, t-1 \rrbracket,$$

hence  $|\text{Range}(f_{a/b})| = t$ .

**Step 2.** Let  $t \in \llbracket b+1, 2b-1 \rrbracket$  be arbitrary. Set  $q = t - b \in \llbracket 1, b-1 \rrbracket$  and  $a = b^2 + bq + 1$ . Then  $\mu_{a,b} = q$  and  $\lfloor a/b^2 \rfloor = 1$ . Using again Theorem 3.1, we have

$$\text{Range}(f_{a/b}) = \bigcup_{j=0}^q (j + \mathbb{Y}_j), \quad (5.2)$$

where

$$\mathbb{Y}_j = \left\{ k \in \llbracket 0, b-1 \rrbracket : (j-1) + \frac{1}{b} \leq \frac{(bq+1)k}{b^2} < j+1 \right\}.$$

We claim that in this case we have  $\text{Range}(f_{a/b}) = \llbracket 0, q+b-1 \rrbracket$ , which in turn shows that  $|\text{Range}(f_{a/b})| = q + b = t$ .

The inclusion

$$\text{Range}(f_{a/b}) \subseteq \llbracket 0, b+q-1 \rrbracket$$

is immediate from (5.2), since each element in the union has the form  $j+k$ , with  $j \in \llbracket 0, q \rrbracket$  and  $k \in \mathbb{Y}_j \subseteq \llbracket 0, b-1 \rrbracket$ .

We prove the reverse inclusion directly. Let  $m \in \llbracket 0, b+q-1 \rrbracket$  be arbitrary. Define

$$j = \left\lfloor \frac{q(m+1)}{b+q} \right\rfloor \quad \text{and} \quad k = m - j. \quad (5.3)$$

We will prove that  $j \in \llbracket 0, q \rrbracket$  and  $k \in \mathbb{Y}_j$ , proving that  $m \in j + \mathbb{Y}_j \subseteq \text{Range}(f_{a/b})$ .

If  $m = 0$ , then  $j = 0$ , so  $k = 0 \in \mathbb{Y}_0$  and  $m \in 0 + \mathbb{Y}_0$ . In the rest of the proof we assume  $m \geq 1$ . We first check that  $j \in \llbracket 0, q \rrbracket$  and  $k \in \llbracket 0, b-1 \rrbracket$ . Since  $0 \leq m+1 \leq b+q$ , the definition of  $j$  gives  $0 \leq j \leq q$ . Since  $q < b \leq bm$ , we have

$$\frac{q(m+1)}{b+q} \leq m.$$

Hence  $j \leq m$ , and therefore  $k \geq 0$ .

Next we prove  $k \leq b-1$ . If  $m \leq b-1$ , this is clear. If  $m \geq b$ , then  $m-b+1$  is an integer and

$$\frac{q(m+1)}{b+q} - (m-b+1) = \frac{b(b+q-1-m)}{b+q} \geq 0,$$

since  $m \leq b+q-1$ . Thus

$$j = \left\lfloor \frac{q(m+1)}{b+q} \right\rfloor \geq m-b+1,$$

and so  $k = m - j \leq b-1$ . Therefore  $k \in \llbracket 0, b-1 \rrbracket$ .

Now we prove that  $k \in \mathbb{Y}_j$ . From (5.3),

$$j \leq \frac{q(m+1)}{b+q} < j+1. \quad (5.4)$$

Since  $m = j+k$ , the left-hand inequality in (5.4) gives

$$j(b+q) \leq q(j+k+1),$$

hence  $bj \leq qk + q$ , that is,  $bj - q \leq qk$ . Using  $q \leq b-1$  we have  $bj - b + 1 \leq qk$ . Multiplying by  $b > 0$ , and using  $k \geq 0$ , we obtain  $b^2j - b^2 + b \leq bqk + k$ . Dividing by  $b^2$ , this yields

$$j - 1 + \frac{1}{b} \leq \frac{(bq+1)k}{b^2}. \quad (5.5)$$

Similarly, since  $m = j+k$ , the right-hand inequality in (5.4) gives  $q(j+k+1) < (j+1)(b+q)$ , that is  $qk < b(j+1)$ . Since both sides are integers, this implies  $qk \leq b(j+1)-1$ . Multiplying by  $b$  yields  $bqk \leq b^2(j+1)-b$  and adding  $k \leq b-1$ , we get

$$bqk + k \leq b^2(j+1) - b + (b-1) = b^2(j+1) - 1 < b^2(j+1).$$

Therefore

$$\frac{(bq+1)k}{b^2} < j+1. \quad (5.6)$$

Combining (5.5) and (5.6), we get  $k \in \mathbb{Y}_j$ . Hence  $m = j+k \in j + \mathbb{Y}_j \subseteq \text{Range}(f_{a/b})$ . This proves  $\llbracket 0, b+q-1 \rrbracket \subseteq \text{Range}(f_{a/b})$ , and the theorem follows.  $\square$

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## References

- [1] Á. Bényi, B. Ćurgus, *Floor, ceiling, and the space between*, An. Univ. Vest Timiș., Ser. Mat.-Inform. **61** (2025), 131–156.
- [2] V. Bergelson, A. Leibman, *Distribution of values of bounded generalized polynomials*, Acta Math. **198**, no. 2 (2007), 155–230.
- [3] A. Dubickas, *Difference between two floor functions*, Monatsh. Math. **209**, no. 4 (2026), 613–635.
- [4] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed., Addison-Wesley, 1994.
- [5] J. C. Lagarias, T. Murayama, D. H. Richman, *Dilated floor functions that commute*, Amer. Math. Monthly **123**, no. 10 (2016), 1033–1038.

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