

# Extensions of two referenced forms of the Hilbert integral inequality

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**Abstract** This article demonstrates the broader flexibility of the techniques developed in previous studies, showing that they can be adapted to cover a wider range of Hilbert-type integral inequalities. By introducing a framework based on the incomplete gamma function, we derive two new generalizations that extend existing results while maintaining connections with classical inequalities. Several examples illustrate these extensions, offering new perspectives and expanding the scope of potential applications.

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**Keywords** Hilbert integral inequality; gamma function; incomplete gamma function

## 1 Introduction

Thanks to its elegance and wide range of applications, the Hilbert integral inequality occupies a central position in the field of mathematical analysis. Since its discovery, it has attracted considerable attention. Many researchers have revisited it, providing new proofs based on different techniques, while others have explored its potential by establishing refinements, extensions, and generalizations of various kinds. For a detailed account of its early history and foundational developments, we refer the reader to the classic monograph [4], which provides a comprehensive overview of its origins and initial impact.

As the mathematical basis of our article, a formal statement of the Hilbert integral inequality is given below.

**Theorem 1.1.** [4] Let  $p > 1$ ,  $q = p/(p-1)$ , and  $f, g : (0, +\infty) \rightarrow (0, +\infty)$  be two measurable functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y) dx dy \leq \frac{\pi}{\sin(\pi/p)} \left( \int_0^{+\infty} f^p(x) dx \right)^{1/p} \left( \int_0^{+\infty} g^q(y) dy \right)^{1/q},$$

provided that the integrals in the upper bound converge.

Modern developments on the topic can be found in [1–3, 5–12]. Among the recent contributions, particular attention should be given to the work of L. E. Azar [2], in which two new forms of the Hilbert integral inequality are established. The first of these provides a more precise refinement of the classical Hilbert integral inequality and is closely related to the Hardy integral inequality. The second introduces a differential form of the Hilbert integral inequality, offering a new perspective and broadening its range of applications.

A formal statement of the first form is given below, corresponding to [2, Theorem 3.1].

**Theorem 1.2.** [2, Theorem 3.1] Let  $p > 1$ ,  $q = p/(p-1)$ ,  $\lambda > 0$ ,  $f, g : (0, +\infty) \rightarrow (0, +\infty)$  be two measurable functions and  $F, G : (0, +\infty) \rightarrow (0, +\infty)$  be their respective primitives defined by

$$F(t) = \int_0^t f(x) dx, \quad G(t) = \int_0^t g(y) dy$$

for  $t > 0$ , provided that they exist. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y) dx dy \leq A \left( \int_0^{+\infty} x^{-\lambda-1} F^p(x) dx \right)^{1/p} \left( \int_0^{+\infty} y^{-\lambda-1} G^q(y) dy \right)^{1/q},$$

where

$$A = \frac{\Gamma\left(\frac{\lambda}{p} + 1\right) \Gamma\left(\frac{\lambda}{q} + 1\right)}{\Gamma(\lambda)} \quad (1.1)$$

and  $\Gamma(a)$  denotes the (complete) gamma function defined by  $\Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} dx$  for  $a > 0$ , provided that the integrals in the upper bound converge.

A formal statement of the second form is given below, corresponding to [2, Theorem 3.2].

**Theorem 1.3.** [2, Theorem 3.2] Let  $p > 1$ ,  $q = p/(p-1)$ ,  $\lambda > 0$ ,  $n$  be a non-negative integer such that  $n \leq \lambda/\max(p, q)$ ,  $f, g : (0, +\infty) \rightarrow (0, +\infty)$  be two measurable functions  $n$ -times differentiable, and  $f^{(n)}$  and  $g^{(n)}$  be their respective  $n$ -th derivatives. Then we have

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y) dx dy \\ & \leq B \left( \int_0^{+\infty} x^{p(n+1)-\lambda-1} (f^{(n)}(x))^p dx \right)^{1/p} \left( \int_0^{+\infty} y^{q(n+1)-\lambda-1} (g^{(n)}(y))^q dy \right)^{1/q}, \end{aligned}$$

where

$$B = \frac{\Gamma\left(\frac{\lambda}{p} - n\right) \Gamma\left(\frac{\lambda}{q} - n\right)}{\Gamma(\lambda)}, \quad (1.2)$$

provided that the integrals in the upper bound converge.

In this article, we highlight an important observation: the techniques developed in [2] are more flexible than they may initially appear. To demonstrate this, we consider more general Hilbert-type integral inequalities based on the following double integral:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{\Gamma_{low}(\lambda, \theta(x+y))}{(x+y)^\lambda} f(x)g(y) dx dy,$$

where the notations in Theorem 1.3 are adopted,  $\theta > 0$  and  $\Gamma_{low}(a, b)$  denotes the lower incomplete gamma function defined by  $\Gamma_{low}(a, b) = \int_0^b x^{a-1} e^{-x} dx$  for  $a, b > 0$ . Clearly, for any  $x, y > 0$ , we have

$$\lim_{\theta \rightarrow +\infty} \Gamma_{low}(\lambda, \theta(x+y)) = \Gamma(\lambda),$$

which is a constant that can be extracted from the integral. In this limiting case, we recover the framework considered in [2]. Building on this observation, and by adapting the proof techniques developed in [2], we establish two new general forms of the Hilbert integral inequality. The first extends [2, Theorem 3.1], while retaining its close connection with the Hardy integral inequality. The second generalizes [2, Theorem 3.2]. Several precise examples are given. These extensions provide fresh perspectives and significantly broaden the range of possible applications.

The remainder of the article is organized as follows: Section 2 introduces the first general form, while Section 3 presents the second. Finally, Section 4 provides the conclusion.

## 2 First general form

The first general form is stated in the theorem below, followed by the proof and some remarks. It is closely connected with the Hardy integral inequality and considers the primitives of the main functions.

**Theorem 2.1.** *Let  $p > 1$ ,  $q = p/(p-1)$ ,  $\lambda, \theta > 0$ ,  $f, g : (0, +\infty) \rightarrow (0, +\infty)$  be two measurable functions and  $F, G : (0, +\infty) \rightarrow (0, +\infty)$  be their respective primitives defined by*

$$F(t) = \int_0^t f(x) dx, \quad G(t) = \int_0^t g(y) dy$$

*for  $t > 0$ , provided that they exist. Then we have*

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{\Gamma_{low}(\lambda, \theta(x+y))}{(x+y)^\lambda} f(x)g(y) dx dy \\ & \leq C \left( \int_0^{+\infty} x^{-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{p} + 1, \theta x \right) F^p(x) dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{q} + 1, \theta y \right) G^q(y) dy \right)^{1/q}, \end{aligned}$$

where

$$C = \Gamma^{1/q} \left( \frac{\lambda}{p} + 1 \right) \Gamma^{1/p} \left( \frac{\lambda}{q} + 1 \right),$$

provided that the integrals in the upper bound converge.

*Proof.* Making the change of variables  $z = (x + y)t$  and identifying the lower incomplete gamma function, we get

$$\begin{aligned} \int_0^\theta t^{\lambda-1} e^{-(x+y)t} dt &= \int_0^{\theta(x+y)} \left( \frac{z}{x+y} \right)^{\lambda-1} e^{-z} \frac{z}{x+y} dz \\ &= \frac{1}{(x+y)^\lambda} \int_0^{\theta(x+y)} z^{\lambda-1} e^{-z} dz \\ &= \frac{\Gamma_{low}(\lambda, \theta(x+y))}{(x+y)^\lambda}. \end{aligned}$$

Using this, the Fubini-Tonelli integral theorem, and the identity  $1/p + 1/q = 1$ , and applying the Hölder integral inequality, we have

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \frac{\Gamma_{low}(\lambda, \theta(x+y))}{(x+y)^\lambda} f(x)g(y) dx dy \\ &= \int_0^{+\infty} \int_0^{+\infty} f(x)g(y) \left( \int_0^\theta t^{\lambda-1} e^{-(x+y)t} dt \right) dx dy \\ &= \int_0^\theta \left( t^{(\lambda-1)/p} \int_0^{+\infty} e^{-xt} f(x) dx \right) \left( t^{(\lambda-1)/q} \int_0^{+\infty} e^{-yt} g(y) dy \right) dt \\ &\leq \left( \int_0^\theta t^{\lambda-1} \left( \int_0^{+\infty} e^{-xt} f(x) dx \right)^p dt \right)^{1/p} \left( \int_0^\theta t^{\lambda-1} \left( \int_0^{+\infty} e^{-yt} g(y) dy \right)^q dt \right)^{1/q}. \end{aligned} \quad (2.1)$$

It follows from [2, Lemma 1.1] under appropriate parameter configurations that

$$\left( \int_0^{+\infty} e^{-xt} f(x) dx \right)^p \leq t^{1-\lambda/q} \Gamma^{p/q} \left( \frac{\lambda}{p} + 1 \right) \int_0^{+\infty} x^{-\lambda/q} e^{-tx} F^p(x) dx$$

and

$$\left( \int_0^{+\infty} e^{-yt} g(y) dy \right)^q \leq t^{1-\lambda/p} \Gamma^{q/p} \left( \frac{\lambda}{q} + 1 \right) \int_0^{+\infty} y^{-\lambda/p} e^{-ty} G^q(y) dy.$$

Substituting these two inequalities into Equation (2.1), and using the Fubini-Tonelli integral theorem and the identity  $1/p + 1/q = 1$ , we obtain

$$\begin{aligned} &\left( \int_0^\theta t^{\lambda-1} \left( \int_0^{+\infty} e^{-xt} f(x) dx \right)^p dt \right)^{1/p} \left( \int_0^\theta t^{\lambda-1} \left( \int_0^{+\infty} e^{-yt} g(y) dy \right)^q dt \right)^{1/q} \\ &\leq C \left( \int_0^{+\infty} x^{-\lambda/q} F^p(x) \left( \int_0^\theta t^{\lambda/p} e^{-xt} dt \right) dx \right)^{1/p} \\ &\quad \times \left( \int_0^{+\infty} y^{-\lambda/p} G^q(y) \left( \int_0^\theta t^{\lambda/q} e^{-yt} dt \right) dy \right)^{1/q}. \end{aligned} \quad (2.2)$$

Making the changes of variables  $u = xt$  and  $v = yt$ , we get

$$\int_0^\theta t^{\lambda/p} e^{-xt} dt = x^{-\lambda/p-1} \int_0^{\theta x} u^{\lambda/p} e^{-u} du = x^{-\lambda/p-1} \Gamma_{low} \left( \frac{\lambda}{p} + 1, \theta x \right)$$

and

$$\int_0^\theta t^{\lambda/q} e^{-yt} dt = y^{-\lambda/q-1} \int_0^{\theta y} v^{\lambda/q} e^{-v} dv = y^{-\lambda/q-1} \Gamma_{low} \left( \frac{\lambda}{q} + 1, \theta y \right).$$

Using this and the identity  $1/p + 1/q = 1$ , we obtain

$$\begin{aligned} & C \left( \int_0^{+\infty} x^{-\lambda/q} F^p(x) \left( \int_0^\theta t^{\lambda/p} e^{-xt} dt \right) dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{-\lambda/p} G^q(y) \left( \int_0^\theta t^{\lambda/q} e^{-yt} dt \right) dy \right)^{1/q} \\ & = C \left( \int_0^{+\infty} x^{-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{p} + 1, \theta x \right) F^p(x) dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{q} + 1, \theta y \right) G^q(y) dy \right)^{1/q}. \end{aligned} \quad (2.3)$$

Combining Equations (2.1), (2.2) and (2.3), we get

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{\Gamma_{low}(\lambda, \theta(x+y))}{(x+y)^\lambda} f(x)g(y) dx dy \\ & \leq C \left( \int_0^{+\infty} x^{-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{p} + 1, \theta x \right) F^p(x) dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{q} + 1, \theta y \right) G^q(y) dy \right)^{1/q}. \end{aligned}$$

This completes the proof.  $\square$

In particular, applying  $\theta \rightarrow +\infty$ , Theorem 2.1 gives

$$\begin{aligned} & \Gamma(\lambda) \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y) dx dy \\ & \leq C \Gamma^{1/p} \left( \frac{\lambda}{p} + 1 \right) \Gamma^{1/q} \left( \frac{\lambda}{q} + 1 \right) \left( \int_0^{+\infty} x^{-\lambda-1} F^p(x) dx \right)^{1/p} \left( \int_0^{+\infty} y^{-\lambda-1} G^q(y) dy \right)^{1/q}, \end{aligned}$$

so that

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y) dx dy \\ & \leq A \left( \int_0^{+\infty} x^{-\lambda-1} F^p(x) dx \right)^{1/p} \left( \int_0^{+\infty} y^{-\lambda-1} G^q(y) dy \right)^{1/q}, \end{aligned}$$

where

$$A = \frac{C \Gamma^{1/p} \left( \frac{\lambda}{p} + 1 \right) \Gamma^{1/q} \left( \frac{\lambda}{q} + 1 \right)}{\Gamma(\lambda)} = \frac{\Gamma \left( \frac{\lambda}{p} + 1 \right) \Gamma \left( \frac{\lambda}{q} + 1 \right)}{\Gamma(\lambda)},$$

as given by Equation (1.1), thereby recovering [2, Theorem 3.1]. Moreover, all intermediate values of  $\theta > 0$  in Theorem 2.1 yield new integral inequality results.

Two special cases of Theorem 2.1 are described in detail below.

- If we set  $\lambda = 1$ , since  $\Gamma_{low}(\lambda, b) = 1 - e^{-b}$  for  $b > 0$ , Theorem 2.1 gives

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{1 - e^{-\theta(x+y)}}{x+y} f(x)g(y) dx dy \\ & \leq C \left( \int_0^{+\infty} x^{-2} \Gamma_{low} \left( \frac{1}{p} + 1, \theta x \right) F^p(x) dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{-2} \Gamma_{low} \left( \frac{1}{q} + 1, \theta y \right) G^q(y) dy \right)^{1/q}, \end{aligned}$$

where

$$C = \Gamma^{1/q} \left( \frac{1}{p} + 1 \right) \Gamma^{1/p} \left( \frac{1}{q} + 1 \right).$$

- If we set  $\lambda = 2$ , since  $\Gamma_{low}(\lambda, b) = 1 - e^{-b}(1+b)$  for  $b > 0$ , Theorem 2.1 gives

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{1 - e^{-\theta(x+y)}(1 + \theta(x+y))}{(x+y)^2} f(x)g(y) dx dy \\ & \leq C \left( \int_0^{+\infty} x^{-3} \Gamma_{low} \left( \frac{2}{p} + 1, \theta x \right) F^p(x) dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{-3} \Gamma_{low} \left( \frac{2}{q} + 1, \theta y \right) G^q(y) dy \right)^{1/q}, \end{aligned}$$

where

$$C = \Gamma^{1/q} \left( \frac{2}{p} + 1 \right) \Gamma^{1/p} \left( \frac{2}{q} + 1 \right).$$

These are only two new integral inequalities among all the possible examples obtained by varying  $\lambda$ . The parameter  $\theta$  can also vary in a similar way.

### 3 Second general form

The second general form is stated in the theorem below, which is followed by the proof and some remarks. It can be described as a general differential form of the Hilbert integral inequality.

**Theorem 3.1.** *Let  $p > 1$ ,  $q = p/(p-1)$ ,  $\lambda, \theta > 0$ ,  $n$  be a non-negative integer such that  $n \leq \lambda/\max(p, q)$ ,  $f, g : (0, +\infty) \rightarrow (0, +\infty)$  be two measurable functions  $n$ -times differentiable, and  $f^{(n)}$  and  $g^{(n)}$  be their respective  $n$ -th derivatives. Then we have*

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{\Gamma_{low}(\lambda, \theta(x+y))}{(x+y)^\lambda} f(x)g(y) dx dy \\ & \leq D \left( \int_0^{+\infty} x^{p(n+1)-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{p} - n, \theta x \right) (f^{(n)}(x))^p dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{q(n+1)-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{q} - n, \theta y \right) (g^{(n)}(y))^q dy \right)^{1/q}, \end{aligned}$$

where

$$D = \Gamma^{1/q} \left( \frac{\lambda}{p} - n \right) \Gamma^{1/p} \left( \frac{\lambda}{q} - n \right),$$

provided that the integrals in the upper bound converge.

*Proof.* Proceeding as the beginning of the proof of Theorem 2.1, we have

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{\Gamma_{low}(\lambda, \theta(x+y))}{(x+y)^\lambda} f(x)g(y) dx dy \\ &= \int_0^{+\infty} \int_0^{+\infty} f(x)g(y) \left( \int_0^\theta t^{\lambda-1} e^{-(x+y)t} dt \right) dx dy \\ &= \int_0^\theta \left( t^{(\lambda-1)/p} \int_0^{+\infty} e^{-xt} f(x) dx \right) \left( t^{(\lambda-1)/q} \int_0^{+\infty} e^{-yt} g(y) dy \right) dt \\ &\leq \left( \int_0^\theta t^{\lambda-1} \left( \int_0^{+\infty} e^{-xt} f(x) dx \right)^p dt \right)^{1/p} \left( \int_0^\theta t^{\lambda-1} \left( \int_0^{+\infty} e^{-yt} g(y) dy \right)^q dt \right)^{1/q}. \end{aligned} \quad (3.1)$$

It follows from [2, Lemma 2.2] under appropriate parameter configurations that

$$\left( \int_0^{+\infty} e^{-xt} f(x) dx \right)^p \leq t^{-n-\lambda/q} \Gamma^{p/q} \left( \frac{\lambda}{p} - n \right) \int_0^{+\infty} x^{-(\lambda-p(n+1))/q} e^{-tx} (f^{(n)}(x))^p dx$$

and

$$\left( \int_0^{+\infty} e^{-yt} g(y) dy \right)^q \leq t^{-n-\lambda/p} \Gamma^{q/p} \left( \frac{\lambda}{q} - n \right) \int_0^{+\infty} y^{-(\lambda-q(n+1))/p} e^{-ty} (g^{(n)}(y))^q dy.$$

Substituting these two inequalities into Equation (3.1), using the Fubini-Tonelli integral theorem and the identity  $1/p + 1/q = 1$ , and making appropriate changes of variables (in a similar way that what has been done in the proof of Theorem 2.1), we obtain

$$\begin{aligned} & \left( \int_0^\theta t^{\lambda-1} \left( \int_0^{+\infty} e^{-xt} f(x) dx \right)^p dt \right)^{1/p} \left( \int_0^\theta t^{\lambda-1} \left( \int_0^{+\infty} e^{-yt} g(y) dy \right)^q dt \right)^{1/q} \\ &\leq D \left( \int_0^{+\infty} x^{-(\lambda-p(n+1))/q} (f^{(n)}(x))^p \left( \int_0^\theta t^{\lambda/p-n-1} e^{-xt} dt \right) dx \right)^{1/p} \\ &\quad \times \left( \int_0^{+\infty} y^{-(\lambda-q(n+1))/p} (g^{(n)}(y))^q \left( \int_0^\theta t^{\lambda/q-n-1} e^{-yt} dt \right) dy \right)^{1/q} \\ &= D \left( \int_0^{+\infty} x^{p(n+1)-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{p} - n, \theta x \right) (f^{(n)}(x))^p dx \right)^{1/p} \\ &\quad \times \left( \int_0^{+\infty} y^{q(n+1)-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{q} - n, \theta y \right) (g^{(n)}(y))^q dy \right)^{1/q}. \end{aligned} \quad (3.2)$$

Combining Equations (3.1) and (3.2), we obtain

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{\Gamma_{low}(\lambda, \theta(x+y))}{(x+y)^\lambda} f(x)g(y) dx dy \\ &\leq D \left( \int_0^{+\infty} x^{p(n+1)-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{p} - n, \theta x \right) (f^{(n)}(x))^p dx \right)^{1/p} \\ &\quad \times \left( \int_0^{+\infty} y^{q(n+1)-\lambda-1} \Gamma_{low} \left( \frac{\lambda}{q} - n, \theta y \right) (g^{(n)}(y))^q dy \right)^{1/q}. \end{aligned}$$

This completes the proof.  $\square$

In particular, applying  $\theta \rightarrow +\infty$ , Theorem 3.1 gives

$$\begin{aligned} & \Gamma(\lambda) \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y) dx dy \\ & \leq D \Gamma^{1/p} \left( \frac{\lambda}{p} - n \right) \Gamma^{1/q} \left( \frac{\lambda}{q} - n \right) \left( \int_0^{+\infty} x^{p(n+1)-\lambda-1} (f^{(n)}(x))^p dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{q(n+1)-\lambda-1} (g^{(n)}(y))^q dy \right)^{1/q}, \end{aligned}$$

so that

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y) dx dy \\ & \leq B \left( \int_0^{+\infty} x^{p(n+1)-\lambda-1} (f^{(n)}(x))^p dx \right)^{1/p} \left( \int_0^{+\infty} y^{q(n+1)-\lambda-1} (g^{(n)}(y))^q dy \right)^{1/q}, \end{aligned}$$

where

$$B = \frac{D \Gamma^{1/p} \left( \frac{\lambda}{p} - n \right) \Gamma^{1/q} \left( \frac{\lambda}{q} - n \right)}{\Gamma(\lambda)} = \frac{\Gamma \left( \frac{\lambda}{p} - n \right) \Gamma \left( \frac{\lambda}{q} - n \right)}{\Gamma(\lambda)},$$

as given by Equation (1.2), thereby recovering [2, Theorem 3.2]. Moreover, all intermediate values of  $\theta > 0$  in Theorem 3.1 lead to new integral inequality results.

Two special cases of Theorem 3.1 are described in detail below.

- If we set  $\lambda = 1$ , since  $\Gamma_{low}(\lambda, b) = 1 - e^{-b}$  for  $b > 0$ , Theorem 3.1 gives

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{1 - e^{-\theta(x+y)}}{x+y} f(x)g(y) dx dy \\ & \leq D \left( \int_0^{+\infty} x^{p(n+1)-2} \Gamma_{low} \left( \frac{1}{p} - n, \theta x \right) (f^{(n)}(x))^p dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{q(n+1)-2} \Gamma_{low} \left( \frac{1}{q} - n, \theta y \right) (g^{(n)}(y))^q dy \right)^{1/q}, \end{aligned}$$

where

$$D = \Gamma^{1/q} \left( \frac{1}{p} - n \right) \Gamma^{1/p} \left( \frac{1}{q} - n \right).$$

- If we set  $\lambda = 2$ , since  $\Gamma_{low}(\lambda, b) = 1 - e^{-b}(1+b)$  for  $b > 0$ , Theorem 3.1 gives

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{1 - e^{-\theta(x+y)}(1 + \theta(x+y))}{(x+y)^2} f(x)g(y) dx dy \\ & \leq D \left( \int_0^{+\infty} x^{p(n+1)-3} \Gamma_{low} \left( \frac{2}{p} - n, \theta x \right) (f^{(n)}(x))^p dx \right)^{1/p} \\ & \quad \times \left( \int_0^{+\infty} y^{q(n+1)-3} \Gamma_{low} \left( \frac{2}{q} - n, \theta y \right) (g^{(n)}(y))^q dy \right)^{1/q}, \end{aligned}$$

where

$$D = \Gamma^{1/q} \left( \frac{2}{p} - n \right) \Gamma^{1/p} \left( \frac{2}{q} - n \right).$$

These are just two new integral inequalities out of all the possible examples obtained by varying  $\lambda$ . We can also adjust the value of  $\theta$  to adapt to certain mathematical scenarios.

## 4 Conclusion

In conclusion, our results show that the techniques of [2] naturally extend to a broader class of Hilbert-type integral inequalities involving the lower incomplete gamma function. This enriches the theoretical framework and opens new avenues for applications. Further generalizations could be explored by replacing the kernel function with other special functions, and by investigating applications in functional analysis, operator theory, and related fields.

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