

The Cardano-golden ratio and the associated curves

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Abstract The aim of this paper is to introduce and study the cubic real polynomials P having as Cardano resolvent exactly the quadratic equation providing the well-known golden ratio Φ . One obtains that these polynomials form a 1-parameter family and the unique positive root of the depressed case is called *Cardano-golden ratio*. We generalize this cubic depressed polynomial to arbitrary grade $n \geq 3$. Also for this depressed polynomial a cubic curve is naturally associated. A regular curve and a regular surface in $\mathbb{R}^3$, both called *golden*, are defined and studied from the point of view of differential geometry.

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1 Introduction

It is commonly recognized that Gerolamo Cardano, by using informations from Scipione del Ferro and Niccolo Fontana Tartaglia, published the first formula for solving cubic equations in his book *Ars Magna* from 1545. In this approach a given monic (and reduced) cubic polynomial P_d is solved through a monic quadratic polynomial $CR(P_d)$ which we call *the Cardano resolvent* of P_d ; see, for example, the excellent paper [1] or the book [5].

The starting point of this note is to consider the quadratic polynomial $CR(P_d)$ as being exactly the polynomial P_{golden}^2 defining the golden ratio (or mean) $\Phi = \frac{1}{2}(\sqrt{5} + 1)$. We call P_d (as well as the general cubic polynomials P with P_d as depressed form) as being *Cardano-golden polynomial*. We express completely this family obtaining its 1-parameter dependence. We introduce also the cubic curve $\mathcal{C}(golden) : y^2 = P_d(x)$ with P_d the Cardano-golden depressed polynomial; we found $\mathcal{C}(golden)$ as elliptic curve on the database <https://www.lmfdb.org/>.

In the last section we define a regular curve and a regular surface in the Euclidean space $\mathbb{R}^3$, both called *golden* due to the relationship with the general Cardano-golden

polynomial. These objects are studied from the point of view of differential geometry: for the golden curve we compute its curvature and torsion while for the golden surface we compute its Gaussian curvature.

2 The Cardano-golden cubic polynomial and a generalization

Fix a natural number $n \in \mathbb{N}^*$. The general setting of this work is provided by the n -dimensional real linear space of monic polynomials of degree n :

$$\mathbb{R}_n^{\text{monic}}[x] := \{P(x) = x^n + a_1x^{n-1} + \dots + a_n; a_1, \dots, a_n \in \mathbb{R}\}.$$

Our study focuses on a fixed cubic polynomial $P \in \mathbb{R}_3^{\text{monic}}[x]$:

$$P(x) := x^3 + Ax^2 + Bx + C \quad (2.1)$$

which with the well-known Tschirnhausen transformation $x = y - \frac{A}{3}$ has the *reduced* form (here the subscript d means *depressed*):

$$P_d(y) := y^3 + py + q, \quad p = B - \frac{A^2}{3} \leq B, \quad q = C - \frac{AB}{3} + \frac{2A^3}{27}. \quad (2.2)$$

We recall very briefly the study of P_d . Being a cubic polynomial it has at least a real root y_0 . We consider now an associated quadratic polynomial:

$$P_0(u) = u^2 - y_0u - \frac{p}{3} \in \mathbb{R}_2^{\text{monic}}[u], \quad (2.3)$$

and hence its roots $\alpha, \beta \in \mathbb{C}$ satisfy:

$$\alpha + \beta = y_0, \quad \alpha\beta = -\frac{p}{3}. \quad (2.4)$$

From $P_d(y_0) = 0$ it follows immediately:

$$\alpha^3 + \beta^3 = -q, \quad \alpha^3\beta^3 = -\frac{p^3}{27}, \quad (2.5)$$

which means that α^3 and β^3 are the (complex) roots of the quadratic polynomial

$$CR(P_d)(z) := z^2 + qz - \frac{p^3}{27}. \quad (2.6)$$

Let us call the polynomial $CR(P_d) \in \mathbb{R}_2^{\text{monic}}[z]$ as being the *Cardano resolvent* of P_d since its roots:

$$z_{\pm} := -\frac{q}{2} \pm \sqrt{\frac{\Delta}{4 \cdot 27}}, \quad \Delta := 4p^3 + 27q^2 \quad (2.7)$$

give the well-known Cardano formula of y_0 :

$$y_0 = \sqrt[3]{z_-} + \sqrt[3]{z_+}. \quad (2.8)$$

Remark 2.1. There exists also a *Lagrange resolvent* of P_d : $Z^2 + 27qZ - 27p^3 = 0$, but the map $Z = 27z$ transforms this resolvent into the Cardano resolvent.

A famous quadratic polynomial is ([4]):

$$P_{\text{golden}}^2(X) = X^2 - X - 1 \quad (2.9)$$

since its positive root is exactly *the golden ratio*:

$$\Phi := \frac{1 + \sqrt{5}}{2} \simeq 1.618. \quad (2.10)$$

Inspired by this fact we introduce:

Definition 2.1. *The polynomial P (or more precisely P_d) is called Cardano-golden polynomial if its Cardano resolvent is*

$$CR(P_d) = P_{\text{golden}}^2. \quad (2.11)$$

We have directly the relations:

$$q = -1, \quad p^3 = 27, \quad (2.12)$$

which yield:

Proposition 2.1. *i) The Cardano-golden depressed polynomial*

$$P_d(x) = x^3 + 3x - 1 \quad (2.13)$$

has one real root denoted Φ^C and two complex roots:

$$\Phi^C = \sqrt[3]{\Phi} - \frac{1}{\sqrt[3]{\Phi}} \simeq 0.32219 > 0, \quad z_{\pm} = -0.1611 \pm 1.7544i \in \mathbb{C} \setminus \mathbb{R}. \quad (2.14)$$

ii) The general class of Cardano-golden cubic polynomials P is a 1-parameter family provided by:

$$P_A(x) = x^3 + Ax^2 + \frac{A^2 + 9}{3}x + \frac{A^3}{27} + A - 1 = \left(x + \frac{A}{3}\right)^3 + 3x + A - 1, \quad A \in \mathbb{R}. \quad (2.15)$$

Definition 2.2. *Let us call Φ^C as Cardano-golden ratio.*

Remark 2.2. *i) The continued fraction expression of Φ^C is:*

$$\Phi^C = [0; 3, 9, 1, 1, 1, 2, 1, 2, 3, 6, \dots]. \quad (2.16)$$

The following matrix belongs to the Lie group $SL(3, \mathbb{Z})$ and has the characteristic polynomial $-P_d$:

$$\Gamma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{pmatrix}.$$

ii) A general cubic curve:

$$\mathcal{C} : y^2 = P_d(x) = x^3 + px + q \quad (2.17)$$

has the discriminant Δ defined in the relation (2.7); see also [2] or [3]. Hence we define the *Cardano-golden cubic curve*:

$$\mathcal{C}(\text{golden}) : y^2 = x^3 + 3x - 1, \quad \Delta(\text{golden}) = 27 \cdot 5 = 135 > 0 \quad (2.18)$$

which is the elliptic curve <https://www.lmfdb.org/EllipticCurve/Q/2160/x/2>.

Also, the expressions of P_{golden}^2 (from (2.9)) and P_d (from (2.13)) suggests to generalize them:

Definition 2.3. *If $n \geq 2$ then the n -golden polynomial is the monic polynomial:*

$$P_{golden}^n(x) := x^n + (-1)^{n-1} \frac{n(n-1)}{2} x - 1. \quad (2.19)$$

Let us study the low values of n :

- a) $P_{golden}^4(x) = x^4 - 6x - 1$ has two complex roots and two real roots of opposite signs; the strictly positive root is approximately 1.8696. Its discriminant is $\Delta = 256 + 27 \cdot 6^4 = 35248 = 2^4 \cdot 2203$.

The following matrix belongs to the Lie algebra $sl(4, \mathbb{Z})$ and has as characteristic polynomial exactly P_{golden}^4 :

$$\Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

- b) $P_{golden}^5(x) = x^5 + 10x - 1$ has four complex roots and a strictly positive root which is approximately 0.09999.

It is easy to prove that for an odd n the polynomial P_{golden}^n has a unique strictly positive root. Indeed, its derivative is

$$(P_{golden}^n)'(x) = n \left[x^{n-1} + (-1)^{n-1} \frac{n-1}{2} \right], \quad (2.20)$$

and hence for odd n the function $x \in \mathbb{R} \rightarrow P_{golden}^n(x) \in \mathbb{R}$ is strictly increasing. This fact together with the inequality:

$$P_{golden}^n(0) = -1 < 0 < \lim_{x \rightarrow +\infty} P_{golden}^n(x) = +\infty \quad (2.21)$$

gives the claimed conclusion.

If n is even, then the derivative polynomial $(P_{golden}^n)'$ has two roots:

$$x_{\pm}^n = \pm \sqrt[n-1]{\frac{n-1}{2}}, \quad (2.22)$$

and hence the unique strictly positive root of P_{golden}^n is strictly greater than x_+^n . Therefore, if n is even then the minimum of P_{golden}^n is:

$$P_{golden}^n(x_+^n) = -1 - \left(\frac{n}{2} + 1 \right) \frac{n-1}{2} x_+^n. \quad (2.23)$$

3 The golden space curve and the golden surface

The expression of the coefficients B and C suggests the map $A \in \mathbb{R} \rightarrow \mathbb{C}_{golden} : r(A) = (x(A), y(A), z(A)) = \left(A, \frac{A^2}{3} + 3, \frac{A^3}{27} + A - 1\right) \in \mathbb{R}^3$. This map parametrizes a space curve, which we can call *the golden 3D curve*, and is a skew (cubical) parabola. It belongs to the regular surface:

$$S(golden) : z = \frac{x}{3} \left(\frac{y}{3} - 1\right) + x - 1. \quad (3.1)$$

It follows that this surface can be called *the golden surface*.

The curvature and torsion functions of the golden 3D curve are:

$$k(A) = \frac{54\sqrt{162 - 9A^2 + A^4}}{(162 + 54A^2 + A^4)^{\frac{3}{2}}}, \quad \tau(A) = \frac{27}{162 - 9A^2 + A^4}. \quad (3.2)$$

Since $A^4 - 9A^2 + 162 = \frac{1}{4}(2A^2 - 9)^2 + \frac{567}{4}$, we get two inequalities:

$$k(A) \geq \frac{243\sqrt{7}}{(162 + 54A^2 + A^4)^{\frac{3}{2}}}, \quad 0 < \tau \leq \frac{4}{21}. \quad (3.3)$$

The Gaussian curvature of the golden surface is:

$$K(x, y) = \frac{-81}{[81 + x^2 + (y + 6)^2]^2} \in [-1/81, 0). \quad (3.4)$$

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