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Study of a multi-parameter three-dimensional Hardy-Hilbert type integral inequality

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Abstract This article is devoted to the study of a new three-dimensional Hardy-Hilbert integral inequality. It is innovative mainly for its generality, characterized by the presence of many adjustable parameters and complex power-sum interactions of the variables. Several new three-dimensional integral inequalities are derived from the main result, showing how it can be applied in different analytical frameworks. The proofs are presented in detail, ensuring a rigorous theoretical foundation.

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1 Introduction

The classical Hardy-Hilbert integral inequality is fundamental to modern mathematics. It provides a sharp upper bound for a double integral involving a particular ratio-product function. Applications are mainly found in analysis, where it plays a crucial role in the study of integral operators and function spaces. A statement of this inequality is given below. Let $p, q > 1$ such that $1/p + 1/q = 1$ and $f, g : [0, +\infty) \to [0, +\infty)$ be two functions. Then the following holds:

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \daleth \left[\int_0^{+\infty} f^p(x)dx \right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy \right]^{1/q},$$

where

$$\daleth = \frac{\pi}{\sin(\pi/p)},$$

provided that the integrals involved in the upper bound exist. The technical details of this result can be found in [11, 28]. It has been the subject of numerous two-dimensional generalizations and variants. A famous one-parameter extension was examined in [23], as given below. Let $p, q > 1$ such that $1/p + 1/q = 1$, $\lambda > 2 - \min(p, q)$ and $f, g : [0, +\infty) \to [0, +\infty)$ be two functions. Then the following holds:

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \leq \daleth \left[\int_0^{+\infty} x^{1-\lambda} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{1-\lambda} g^q(y)dy\right]^{1/q},$$

where

$$\daleth = \frac{1}{\Gamma(\lambda)}\Gamma\left(1 + \frac{\lambda - 2}{p}\right)\Gamma\left(1 + \frac{\lambda - 2}{q}\right)$$

and $\Gamma(\kappa) = \int_0^{+\infty} x^{\kappa-1}e^{-x}dx$ is the standard gamma function at $\kappa > 0$, provided that the integrals involved in the upper bound exist. The importance of the gamma function in defining the factor constant is a remarkable fact. Other two-dimensional extensions and variants of various kinds can be found in [1–4, 8, 9, 15, 20–22, 25–27]. The study of multidimensional Hardy-Hilbert type integral inequalities has also attracted attention. An important one-parameter three-dimensional extension was established in [24], as given below. Let $p, q, r > 1$ such that $1/p + 1/q + 1/r = 1$, $\lambda > 3 - \min(p, q, r)$, and $f, g, h : [0, +\infty) \to [0, +\infty)$ be three functions. Then the following holds:

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)h(z)}{(x+y+z)^\lambda} dxdydz \\ &\leq \digamma \left[\int_0^{+\infty} x^{2-\lambda} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{2-\lambda} g^q(y)dy\right]^{1/q} \left[\int_0^{+\infty} z^{2-\lambda} h^r(z)dz\right]^{1/r}, \end{aligned} \tag{1.1}$$

where

$$\digamma = \frac{1}{\Gamma(\lambda)}\Gamma\left(1 + \frac{\lambda - 3}{p}\right)\Gamma\left(1 + \frac{\lambda - 3}{q}\right)\Gamma\left(1 + \frac{\lambda - 3}{r}\right), \tag{1.2}$$

provided that the integrals involved in the upper bound exist. This inequality thus extends the setting of the standard Hardy-Hilbert integral inequality by an additional dimension and an adjustable parameter λ. Other three- or higher-dimensional studies on this topic can be found in [6, 12–14, 16–18, 24, 29, 30], complemented by the survey in [7].

In this article, we contribute to the development of three-dimensional Hardy-Hilbert type integral inequalities by reaching a higher level of sophistication. The main result consists in finding a valuable upper bound for a triple integral of the following form:

$$\text{“}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c f^\mu(x) g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dxdydz.\text{”}$$

The upper bound obtained depends on the gamma function and certain weighted integral norms of the main functions, i.e., f, g and h. The techniques employed make it as sharp as possible. The novelty of this result lies in the generality and complexity of the considered setting, which makes use of numerous adaptable parameters, i.e., α, β, γ, a, b, c, μ, ν, δ, s, t and u, and various power-sum interactions of the variables x, y and z, characterized by the ratio-product involving x^{α}, y^{β}, z^{γ}, $(x+y)^{a}$, $(y+z)^{b}$, $(x+z)^{c}$ and $(x+y+z)^{s+t+u}$. The proof relies on a suitable decomposition of the integrand, a well-configured application of the generalized Hölder integral inequality (see [5, 19]), and well-considered changes of variables. We then use this flexible result to establish several other new three-dimensional integral inequalities. These include an upper bound for a triple integral depending on two main functions, of the following form:

$$“\left[\int_{0}^{+\infty} x^{-(a-s+\gamma\omega+\beta\lambda+2/p)\theta_{q,r}} \times \left\{\int_{0}^{+\infty}\int_{0}^{+\infty} \frac{x^{\alpha}y^{\beta}z^{\gamma}(x+y)^{a}(y+z)^{b}(x+z)^{c}g^{\nu}(y)h^{\delta}(z)}{(x+y+z)^{s+t+u}}dydz\right\}^{\theta_{q,r}} dx\right]^{1/\theta_{q,r}},”$$

and the same for two other similar integrals. We also study another kind of three-dimensional Hardy-Hilbert type integral inequality based on a triple integral of the following form:

$$“\int_{0}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty} \frac{x^{\alpha+\phi}y^{\beta+\psi}z^{\gamma+\varphi}(x+y)^{a}(y+z)^{b}(x+z)^{c}F^{\mu}(x)G^{\nu}(y)H^{\delta}(z)}{(x+y+z)^{s+t+u}}dxdydz,”$$

where ϕ, ψ and φ are additional parameters and F, G and H are the primitives associated with f, g and h, respectively. The upper bound obtained is still defined with the gamma function and simple integral norms of the main functions, i.e., not weighted norms as for the main theorem. These results can be applied to various areas of mathematics, in particular to the study of integral operators, functional spaces and related optimization problems.

The rest of the article is organized as follows: The main result is stated and proved in Section 2. Section 3 contains the secondary results. A conclusion is given in Section 4.

2 Main theorem

The theorem below is our main result. The corresponding proof and a discussion follow.

Theorem 2.1. *Let $p,q,r>1$ such that $1/p+1/q+1/r=1$, $f,g,h:[0,+\infty)\to[0,+\infty)$ be three functions, and $\alpha,\beta,\gamma,a,b,c,\mu,\nu,\delta,s,t,u,\epsilon,\lambda,\omega\in\mathbb{R}$ such that $\gamma\omega p+1>0$, $(s-\gamma\omega)p-1>0$, $sp>0$, $\beta\lambda p+1>0$, $(s-a-\gamma\omega-\beta\lambda)p-2>0$, $(s-a-\gamma\omega)p-1>0$, $\alpha\epsilon q+1>0$, $(t-\alpha\epsilon)q-1>0$, $tq>0$, $\gamma(1-\omega)q+1>0$, $(t-b-\alpha\epsilon-\gamma(1-\omega))q-2>0$, $(t-b-\alpha\epsilon)q-1>0$, $\beta(1-\lambda)r+1>0$, $(u-\beta(1-\lambda))r-1>0$, $ur>0$, $\alpha(1-\epsilon)r+1>0$, $(u-c-\beta(1-\lambda)-\alpha(1-\epsilon))r-2>0$ and $(u-c-\beta(1-\lambda))r-1>0$. Then the following holds:*

$$\int_{0}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty} \frac{x^{\alpha}y^{\beta}z^{\gamma}(x+y)^{a}(y+z)^{b}(x+z)^{c}f^{\mu}(x)g^{\nu}(y)h^{\delta}(z)}{(x+y+z)^{s+t+u}}dxdydz$$
$$\leq \aleph\left[\int_{0}^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda)p+2}f^{\mu p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2}g^{\nu q}(y)dy\right]^{1/q}$$
$$\times\left[\int_{0}^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}h^{\delta r}(z)dz\right]^{1/r},$$

where

$$\begin{aligned}\aleph &= \left[\frac{\Gamma(\gamma\omega p+1)\Gamma((s-\gamma\omega)p-1)}{\Gamma(sp)}\right]^{1/p}\left[\frac{\Gamma(\beta\lambda p+1)\Gamma((s-a-\gamma\omega-\beta\lambda)p-2)}{\Gamma((s-a-\gamma\omega)p-1)}\right]^{1/p}\\ &\times\left[\frac{\Gamma(\alpha\epsilon q+1)\Gamma((t-\alpha\epsilon)q-1)}{\Gamma(tq)}\right]^{1/q}\\ &\times\left[\frac{\Gamma(\gamma(1-\omega)q+1)\Gamma((t-b-\alpha\epsilon-\gamma(1-\omega))q-2)}{\Gamma((t-b-\alpha\epsilon)q-1)}\right]^{1/q}\\ &\times\left[\frac{\Gamma(\beta(1-\lambda)r+1)\Gamma((u-\beta(1-\lambda))r-1)}{\Gamma(ur)}\right]^{1/r}\\ &\times\left[\frac{\Gamma(\alpha(1-\epsilon)r+1)\Gamma((u-c-\beta(1-\lambda)-\alpha(1-\epsilon))r-2)}{\Gamma((u-c-\beta(1-\lambda))r-1)}\right]^{1/r},\end{aligned} \tag{2.1}$$

provided that the integrals involved in the upper bound exist.

Proof. An appropriate decomposition of the integrand, followed by the generalized Hölder integral inequality, yields

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha}y^{\beta}z^{\gamma}(x+y)^a(y+z)^b(x+z)^c f^{\mu}(x)g^{\nu}(y)h^{\delta}(z)}{(x+y+z)^{s+t+u}}dxdydz\\ &=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{y^{\beta\lambda}z^{\gamma\omega}(x+y)^a f^{\mu}(x)}{(x+y+z)^s}\times\frac{x^{\alpha\epsilon}z^{\gamma(1-\omega)}(y+z)^b g^{\nu}(y)}{(x+y+z)^t}\\ &\quad\times\frac{x^{\alpha(1-\epsilon)}y^{\beta(1-\lambda)}(x+z)^c h^{\delta}(z)}{(x+y+z)^u}dxdydz\\ &\leq U^{1/p}V^{1/q}W^{1/r},\end{aligned} \tag{2.2}$$

where

$$U=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{y^{\beta\lambda p}z^{\gamma\omega p}(x+y)^{ap}f^{\mu p}(x)}{(x+y+z)^{sp}}dxdydz,$$

$$V=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha\epsilon q}z^{\gamma(1-\omega)q}(y+z)^{bq}g^{\nu q}(y)}{(x+y+z)^{tq}}dxdydz$$

and

$$W=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha(1-\epsilon)r}y^{\beta(1-\lambda)r}(x+z)^{cr}h^{\delta r}(z)}{(x+y+z)^{ur}}dxdydz.$$

Let us now examine the exact expressions of U, V and W, in turn.

For the term U, using the Fubini-Tonelli integral theorem, we can write

$$\begin{aligned}U&=\int_0^{+\infty}f^{\mu p}(x)\left\{\int_0^{+\infty}y^{\beta\lambda p}\left[\int_0^{+\infty}\frac{[z/(x+y)]^{\gamma\omega p}(x+y)^{(a-s+\gamma\omega)p+1}}{[1+z/(x+y)]^{sp}}\frac{1}{x+y}dz\right]dy\right\}dx\\ &=\int_0^{+\infty}x^{(a-s+\gamma\omega+\beta\lambda)p+2}f^{\mu p}(x)\left[\int_0^{+\infty}\frac{(y/x)^{\beta\lambda p}}{(1+y/x)^{(s-a-\gamma\omega)p-1}}\frac{1}{x}\Omega(x,y)dy\right]dx,\end{aligned}$$

where

$$\Omega(x,y)=\int_0^{+\infty}\frac{[z/(x+y)]^{\gamma\omega p}}{[1+z/(x+y)]^{sp}}\frac{1}{x+y}dz.$$

Let us determine $\Omega(x,y)$. To do this, the lemma below is required.

Lemma 2.2. *([10, 3.194.3, with $\beta=1$]) Let $a>-1$ and $b>a+1$. Then the following holds:*

$$\int_0^{+\infty}\frac{x^a}{(1+x)^b}dx=\frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)}.$$

Applying the change of variables $m=z/(x+y)$ with respect to z and using Lemma 2.2 with a well-configured parameterization, we obtain

$$\Omega(x,y)=\int_0^{+\infty}\frac{m^{\gamma\omega p}}{(1+m)^{sp}}dm=\frac{\Gamma(\gamma\omega p+1)\Gamma((s-\gamma\omega)p-1)}{\Gamma(sp)}.$$

We thus have

$$U=\frac{\Gamma(\gamma\omega p+1)\Gamma((s-\gamma\omega)p-1)}{\Gamma(sp)}\int_0^{+\infty}x^{(a-s+\gamma\omega+\beta\lambda)p+2}f^{\mu p}(x)\Xi(x)dx,$$

where

$$\Xi(x)=\int_0^{+\infty}\frac{(y/x)^{\beta\lambda p}}{(1+y/x)^{(s-a-\gamma\omega)p-1}}\frac{1}{x}dy.$$

Let us determine $\Xi(x)$. Applying the change of variables $n=y/x$ with respect to y and using Lemma 2.2 with a well-configured parameterization, we obtain

$$\Xi(x)=\int_0^{+\infty}\frac{n^{\beta\lambda p}}{(1+n)^{(s-a-\gamma\omega)p-1}}dn=\frac{\Gamma(\beta\lambda p+1)\Gamma((s-a-\gamma\omega-\beta\lambda)p-2)}{\Gamma((s-a-\gamma\omega)p-1)}.$$

We thus have

$$\begin{aligned}U=&\frac{\Gamma(\gamma\omega p+1)\Gamma((s-\gamma\omega)p-1)}{\Gamma(sp)}\times\frac{\Gamma(\beta\lambda p+1)\Gamma((s-a-\gamma\omega-\beta\lambda)p-2)}{\Gamma((s-a-\gamma\omega)p-1)}\\&\times\int_0^{+\infty}x^{(a-s+\gamma\omega+\beta\lambda)p+2}f^{\mu p}(x)dx.\end{aligned}\tag{2.3}$$

For the term V, we proceed in a similar way, paying particular attention to the parameters involved. Using the Fubini-Tonelli integral theorem, the following expression holds:

$$\begin{aligned}V&=\int_0^{+\infty}g^{\nu q}(y)\left\{\int_0^{+\infty}z^{\gamma(1-\omega)q}\left[\int_0^{+\infty}\frac{[x/(y+z)]^{\alpha\epsilon q}(y+z)^{(b-t+\alpha\epsilon)q+1}}{[1+x/(y+z)]^{tq}}\frac{1}{y+z}dx\right]dz\right\}dy\\&=\int_0^{+\infty}y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2}g^{\nu q}(y)\left[\int_0^{+\infty}\frac{(z/y)^{\gamma(1-\omega)q}}{(1+z/y)^{(t-b-\alpha\epsilon)q-1}}\frac{1}{y}\Upsilon(y,z)dz\right]dy,\end{aligned}$$

where

$$\Upsilon(y,z)=\int_0^{+\infty}\frac{[x/(y+z)]^{\alpha\epsilon q}}{[1+x/(y+z)]^{tq}}\frac{1}{y+z}dx.$$

Let us determine $\Upsilon(y,z)$. Applying the change of variables $m = x/(y+z)$ with respect to x and using Lemma 2.2 with a well-configured parameterization, we get

$$\Upsilon(y,z) = \int_0^{+\infty} \frac{m^{\alpha\epsilon q}}{(1+m)^{tq}} dm = \frac{\Gamma(\alpha\epsilon q+1)\Gamma((t-\alpha\epsilon)q-1)}{\Gamma(tq)}.$$

We thus have

$$V = \frac{\Gamma(\alpha\epsilon q+1)\Gamma((t-\alpha\epsilon)q-1)}{\Gamma(tq)} \int_0^{+\infty} y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2} g^{\nu q}(y)\Phi(y)dy,$$

where

$$\Phi(y) = \int_0^{+\infty} \frac{(z/y)^{\gamma(1-\omega)q}}{(1+z/y)^{(t-b-\alpha\epsilon)q-1}} \frac{1}{y} dz.$$

Let us determine $\Phi(y)$. Applying the change of variables $n = z/y$ with respect to z and using Lemma 2.2 with a well-configured parameterization, we obtain

$$\Phi(y) = \int_0^{+\infty} \frac{n^{\gamma(1-\omega)q}}{(1+n)^{(t-b-\alpha\epsilon)q-1}} dn = \frac{\Gamma(\gamma(1-\omega)q+1)\Gamma((t-b-\alpha\epsilon-\gamma(1-\omega))q-2)}{\Gamma((t-b-\alpha\epsilon)q-1)}.$$

We thus have

$$\begin{aligned} V =& \frac{\Gamma(\alpha\epsilon q+1)\Gamma((t-\alpha\epsilon)q-1)}{\Gamma(tq)} \times \frac{\Gamma(\gamma(1-\omega)q+1)\Gamma((t-b-\alpha\epsilon-\gamma(1-\omega))q-2)}{\Gamma((t-b-\alpha\epsilon)q-1)} \\ &\times \int_0^{+\infty} y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2} g^{\nu q}(y)dy. \end{aligned} \tag{2.4}$$

For the term W, we use the same mathematics. Using the Fubini-Tonelli integral theorem, we can write

$$\begin{aligned} W =& \\ &\int_0^{+\infty} h^{\delta r}(z) \left\{ \int_0^{+\infty} x^{\alpha(1-\epsilon)r} \left[\int_0^{+\infty} \frac{[y/(x+z)]^{\beta(1-\lambda)r}(x+z)^{(c-u+\beta(1-\lambda))r+1}}{[1+y/(x+z)]^{ur}} \frac{1}{x+z} dy \right] dx \right\} dz \\ =& \int_0^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^{\delta r}(z) \left[\int_0^{+\infty} \frac{(x/z)^{\alpha(1-\epsilon)r}}{(1+x/z)^{(u-c-\beta(1-\lambda))r-1}} \frac{1}{z} \Psi(x,z) dx \right] dz, \end{aligned}$$

where

$$\Psi(x,z) = \int_0^{+\infty} \frac{[y/(x+z)]^{\beta(1-\lambda)r}}{[1+y/(x+z)]^{ur}} \frac{1}{x+z} dy.$$

Let us determine $\Psi(x,z)$. Applying the change of variables $m = y/(x+z)$ with respect to y and using Lemma 2.2 with a well-configured parameterization, we obtain

$$\Psi(x,z) = \int_0^{+\infty} \frac{m^{\beta(1-\lambda)r}}{(1+m)^{ur}} dm = \frac{\Gamma(\beta(1-\lambda)r+1)\Gamma((u-\beta(1-\lambda))r-1)}{\Gamma(ur)}.$$

We thus have

$$W=\frac{\Gamma(\beta(1-\lambda)r+1)\Gamma((u-\beta(1-\lambda))r-1)}{\Gamma(ur)}\int_0^{+\infty}z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}h^{\delta r}(z)\Theta(z)dz,$$

where

$$\Theta(z)=\int_0^{+\infty}\frac{(x/z)^{\alpha(1-\epsilon)r}}{(1+x/z)^{(u-c-\beta(1-\lambda))r-1}}\frac{1}{z}dx.$$

Let us determine $\Theta(z)$. Applying the change of variables $n=x/z$ with respect to x and using Lemma 2.2 with a well-configured parameterization, we find that

$$\begin{aligned}\Theta(z)&=\int_0^{+\infty}\frac{n^{\alpha(1-\epsilon)r}}{(1+n)^{(u-c-\beta(1-\lambda))r-1}}dn\\&=\frac{\Gamma(\alpha(1-\epsilon)r+1)\Gamma((u-c-\beta(1-\lambda)-\alpha(1-\epsilon))r-2)}{\Gamma((u-c-\beta(1-\lambda))r-1)}.\end{aligned}$$

We thus have

$$\begin{aligned}W=&\frac{\Gamma(\beta(1-\lambda)r+1)\Gamma((u-\beta(1-\lambda))r-1)}{\Gamma(ur)}\\&\times\frac{\Gamma(\alpha(1-\epsilon)r+1)\Gamma((u-c-\beta(1-\lambda)-\alpha(1-\epsilon))r-2)}{\Gamma((u-c-\beta(1-\lambda))r-1)}\\&\times\int_0^{+\infty}z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}h^{\delta r}(z)dz.\end{aligned}\tag{2.5}$$

Combining Equations (2.2), (2.3), (2.4) and (2.5), we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha y^\beta z^\gamma(x+y)^a(y+z)^b(x+z)^c f^\mu(x)g^\nu(y)h^\delta(z)}{(x+y+z)^{s+t+u}}dxdydz\\&\le\left[\frac{\Gamma(\gamma\omega p+1)\Gamma((s-\gamma\omega)p-1)}{\Gamma(sp)}\right]^{1/p}\left[\frac{\Gamma(\beta\lambda p+1)\Gamma((s-a-\gamma\omega-\beta\lambda)p-2)}{\Gamma((s-a-\gamma\omega)p-1)}\right]^{1/p}\\&\quad\times\left[\frac{\Gamma(\alpha\epsilon q+1)\Gamma((t-\alpha\epsilon)q-1)}{\Gamma(tq)}\right]^{1/q}\\&\quad\times\left[\frac{\Gamma(\gamma(1-\omega)q+1)\Gamma((t-b-\alpha\epsilon-\gamma(1-\omega))q-2)}{\Gamma((t-b-\alpha\epsilon)q-1)}\right]^{1/q}\\&\quad\times\left[\frac{\Gamma(\beta(1-\lambda)r+1)\Gamma((u-\beta(1-\lambda))r-1)}{\Gamma(ur)}\right]^{1/r}\\&\quad\times\left[\frac{\Gamma(\alpha(1-\epsilon)r+1)\Gamma((u-c-\beta(1-\lambda)-\alpha(1-\epsilon))r-2)}{\Gamma((u-c-\beta(1-\lambda))r-1)}\right]^{1/r}\\&\quad\times\left[\int_0^{+\infty}x^{(a-s+\gamma\omega+\beta\lambda)p+2}f^{\mu p}(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2}g^{\nu q}(y)dy\right]^{1/q}\\&\quad\times\left[\int_0^{+\infty}z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}h^{\delta r}(z)dz\right]^{1/r}\end{aligned}$$

$$= \aleph \left[\int_0^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda)p+2} f^{\mu p}(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2} g^{\nu q}(y) dy \right]^{1/q}$$
$$\times \left[\int_0^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^{\delta r}(z) dz \right]^{1/r}.$$

This ends the proof of Theorem 2.1. □

To the best of our knowledge, this theorem provides one of the rare three-dimensional Hardy-Hilbert integral inequalities involving so many parameters and sophisticated power-sum interactions of variables. Note that the parameters ϵ, λ and ω are not involved in the definition of the main triple integral; they are additional parameters that can be adjusted in different mathematical scenarios. The counterpart of this overall complexity makes it difficult to discuss the optimality of the factor constant, i.e., $\aleph$. In the setting corresponding to that of Equation (1.1), it differs from the factor constant, i.e., $\digamma$ in Equation (1.2). We explain this difference by the techniques used in the proof, which are adapted to manage the complexity of the setting in the most appropriate way. They have been applied with a minimum number of intermediate inequalities, maximizing the sharpness of the factor constant.

3 Secondary propositions

The generality of Theorem 2.1 makes it adaptable to different mathematical scenarios. We support this claim with two secondary propositions, which show new three-dimensional integral inequalities.

3.1 A secondary proposition

The result below presents new kinds of three-dimensional integral inequalities depending on only two main functions, i.e., f and g, or f and h, or g and h. The proof is based on a well-adapted use of Theorem 2.1.

Proposition 3.1. *Let $p, q, r > 1$ such that $1/p + 1/q + 1/r = 1$, $f, g, h : [0, +\infty) \to [0, +\infty)$ be three functions, and $\alpha, \beta, \gamma, a, b, c, \mu, \nu, \delta, s, t, u, \epsilon, \lambda, \omega \in \mathbb{R}$ such that $\gamma\omega p + 1 > 0$, $(s - \gamma\omega)p - 1 > 0$, $sp > 0$, $\beta\lambda p + 1 > 0$, $(s - a - \gamma\omega - \beta\lambda)p - 2 > 0$, $(s - a - \gamma\omega)p - 1 > 0$, $\alpha\epsilon q + 1 > 0$, $(t - \alpha\epsilon)q - 1 > 0$, $tq > 0$, $\gamma(1-\omega)q + 1 > 0$, $(t - b - \alpha\epsilon - \gamma(1-\omega))q - 2 > 0$, $(t - b - \alpha\epsilon)q - 1 > 0$, $\beta(1-\lambda)r + 1 > 0$, $(u - \beta(1-\lambda))r - 1 > 0$, $ur > 0$, $\alpha(1-\epsilon)r + 1 > 0$, $(u - c - \beta(1-\lambda) - \alpha(1-\epsilon))r - 2 > 0$ and $(u - c - \beta(1-\lambda))r - 1 > 0$.*

- *Setting $\theta_{q,r} = qr/(q + r)$, the following holds:*

$$\left[\int_0^{+\infty} x^{-(a-s+\gamma\omega+\beta\lambda+2/p)\theta_{q,r}} \right.$$
$$\left. \times \left\{ \int_0^{+\infty} \int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz \right\}^{\theta_{q,r}} dx \right]^{1/\theta_{q,r}}$$
$$\leq \aleph \left[\int_0^{+\infty} y^{(b-t+\alpha\theta+\gamma(1-\omega))q+2} g^{\nu q}(y) dy \right]^{1/q} \left[\int_0^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\theta))r+2} h^{\delta r}(z) dz \right]^{1/r},$$

where $\aleph$ *is given in Equation* (2.1).

- *Setting* $\xi_{p,r} = pr/(p+r)$, *the following holds:*

$$\left[\int_0^{+\infty} y^{-(b-t+\alpha\epsilon+\gamma(1-\omega)+2/q)\xi_{p,r}} \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c f^\mu(x) h^\delta(z)}{(x+y+z)^{s+t+u}} dxdz\right\}^{\xi_{p,r}} dy\right]^{1/\xi_{p,r}}$$
$$\leq \aleph \left[\int_0^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda)p+2} f^{\mu p}(x)dx\right]^{1/p} \left[\int_0^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^{\delta r}(z)dz\right]^{1/r}.$$

- *Setting* $\zeta_{p,q} = pq/(p+q)$, *the following holds:*

$$\left[\int_0^{+\infty} z^{-(c-u+\beta(1-\lambda)+\alpha(1-\epsilon)+2/r)\zeta_{p,q}} \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c f^\mu(x) g^\nu(y)}{(x+y+z)^{s+t+u}} dxdy\right\}^{\zeta_{p,q}} dz\right]^{1/\zeta_{p,q}}$$
$$\leq \aleph \left[\int_0^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda)p+2} f^{\mu p}(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2} g^{\nu q}(y)dy\right]^{1/q}.$$

Proof. For the first point, let us set

$$Z = \int_0^{+\infty} x^{-(a-s+\gamma\omega+\beta\lambda+2/p)\theta_{q,r}} \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz\right\}^{\theta_{q,r}} dx.$$

Using the Fubini-Tonelli integral theorem, we can write

$$\begin{aligned} Z &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} x^{-(a-s+\gamma\omega+\beta\lambda+2/p)\theta_{q,r}} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} \\ &\quad \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz\right\}^{\theta_{q,r}-1} dxdydz \\ &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c f_\star^\mu(x) g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dxdydz, \end{aligned} \tag{3.1}$$

where

$$f_\star(x) = x^{-(a-s+\gamma\omega+\beta\lambda+2/p)\theta_{q,r}/\mu} \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz\right\}^{(\theta_{q,r}-1)/\mu}.$$

Applying Theorem 2.1 to $f = f_\star$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c f_\star^\mu(x) g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dxdydz \\
&\leq \aleph \left[\int_0^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda)p+2} f_\star^{\mu p}(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2} g^{\nu q}(y)dy\right]^{1/q} \\
&\quad \times \left[\int_0^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^{\delta r}(z)dz\right]^{1/r},
\end{aligned} \tag{3.2}$$

where $\aleph$ is given in Equation (2.1).

Let us examine the first integral term in the square brackets in the upper bound. We have

$$\begin{aligned}
&\int_0^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda)p+2} f_\star^{\mu p}(x)dx \\
&= \int_0^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda)p+2} x^{-(a-s+\gamma\omega+\beta\lambda+2/p)\theta_{q,r}p} \\
&\quad \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz\right\}^{(\theta_{q,r}-1)p} dx \\
&= \int_0^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda+2/p)(1-\theta_{q,r})p} \\
&\quad \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz\right\}^{(\theta_{q,r}-1)p} dx.
\end{aligned} \tag{3.3}$$

Since $1/p + 1/q + 1/r = 1$, we have $\theta_{q,r} = qr/(q+r) = 1/(1/q+1/r) = 1/(1-1/p)$, so that

$$(\theta_{q,r}-1)p = \left(\frac{1}{1-1/p} - 1\right)p = \frac{1/p}{1-1/p}p = \frac{1}{1-1/p} = \frac{1}{1/q+1/r} = \theta_{q,r}.$$

We therefore find that

$$\begin{aligned}
&\int_0^{+\infty} x^{(a-s+\gamma\omega+\beta\lambda+2/p)(1-\theta_{q,r})p} \\
&\quad \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz\right\}^{(\theta_{q,r}-1)p} dx \\
&= \int_0^{+\infty} x^{-(a-s+\gamma\omega+\beta\lambda+2/p)\theta_{q,r}} \\
&\quad \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz\right\}^{\theta_{q,r}} dx \\
&= Z.
\end{aligned} \tag{3.4}$$

Combining Equations (3.1), (3.2), (3.3) and (3.4), we get

$$\begin{aligned}
&Z \leq \aleph \times Z^{1/p} \\
&\quad \times \left[\int_0^{+\infty} y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2} g^{\nu q}(y)dy\right]^{1/q} \left[\int_0^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^{\delta r}(z)dz\right]^{1/r},
\end{aligned}$$

so that

$$Z^{1-1/p} \leq \aleph \left[\int_0^{+\infty} y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2} g^{\nu q}(y)dy\right]^{1/q} \left[\int_0^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^{\delta r}(z)dz\right]^{1/r}.$$

Using the expression of Z and the fact that $1-1/p = 1/q+1/r = 1/\theta_{q,r}$, we have

$$\Bigg[\int_0^{+\infty} x^{-(a-s+\gamma\omega+\beta\lambda+2/p)\theta_{q,r}} \times \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (x+y)^a (y+z)^b (x+z)^c g^\nu(y) h^\delta(z)}{(x+y+z)^{s+t+u}} dydz\right\}^{\theta_{q,r}} dx\Bigg]^{1/\theta_{q,r}}$$
$$\leq \aleph \left[\int_0^{+\infty} y^{(b-t+\alpha\theta+\gamma(1-\omega))q+2} g^{\nu q}(y)dy\right]^{1/q} \left[\int_0^{+\infty} z^{(c-u+\beta(1-\lambda)+\alpha(1-\theta))r+2} h^{\delta r}(z)dz\right]^{1/r}.$$

This is the desired inequality.

The inequalities of the second and third point can be proved in a similar way. For reasons of space and redundancy, the details are omitted.

This concludes the proof of Proposition 3.1. □

This proposition therefore deals with only two main functions, which is very different from Theorem 2.1. However, it has the quality of that theorem, i.e., a very general and adaptable three-dimensional integral setting, with various areas of application. It can be useful in the study of sophisticated integral operators, functional spaces and related optimization problems.

3.2 Another secondary proposition

The result below can be presented as a variant of Theorem 2.1, considering weighted primitives of the three main functions. A detailed proof follows. It combines Theorem 2.1 and the classical Hardy integral inequality (see [11]).

Proposition 3.2. *Let $p,q,r > 1$ such that $1/p+1/q+1/r = 1$, $f,g,h : [0,+\infty) \to [0,+\infty)$ be three functions, for any $x,y,z \geq 0$,*

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt, \quad H(z) = \int_0^z h(t)dt$$

be the primitives of f, g and h, respectively, and $\alpha,\beta,\gamma,a,b,c,\mu,\nu,\delta,s,t,u,\epsilon,\lambda,\omega,\phi,\psi,\varphi \in \mathbb{R}$ such that $\gamma\omega p+1 > 0$, $(s-\gamma\omega)p-1 > 0$, $sp > 0$, $\beta\lambda p+1 > 0$, $(s-a-\gamma\omega-\beta\lambda)p-2 > 0$, $(s-a-\gamma\omega)p-1 > 0$, $\alpha\epsilon q+1 > 0$, $(t-\alpha\epsilon)q-1 > 0$, $tq > 0$, $\gamma(1-\omega)q+1 > 0$, $(t-b-\alpha\epsilon-\gamma(1-\omega))q-2 > 0$, $(t-b-\alpha\epsilon)q-1 > 0$, $\beta(1-\lambda)r+1 > 0$, $(u-\beta(1-\lambda))r-1 > 0$, $ur > 0$, $\alpha(1-\epsilon)r+1 > 0$, $(u-c-\beta(1-\lambda)-\alpha(1-\epsilon))r-2 > 0$, $(u-c-\beta(1-\lambda))r-1 > 0$, $\mu p > 1$, $\nu q > 1$, $\delta r > 1$,

$$(a-s+\gamma\omega+\beta\lambda+\phi)p+2 = -\mu p,$$

$$(b-t+\alpha\epsilon+\gamma(1-\omega)+\psi)q+2 = -\nu q$$

and

$$(c-u+\beta(1-\lambda)+\alpha(1-\epsilon)+\varphi)r+2=-\delta r.$$

Then the following holds:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha+\phi}y^{\beta+\psi}z^{\gamma+\varphi}(x+y)^a(y+z)^b(x+z)^cF^{\mu}(x)G^{\nu}(y)H^{\delta}(z)}{(x+y+z)^{s+t+u}}dxdydz$$
$$\leq \beth\left[\int_0^{+\infty}f^{\mu p}(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^{\nu q}(y)dy\right]^{1/q}\left[\int_0^{+\infty}h^{\delta r}(z)dz\right]^{1/r},$$

where

$$\beth=\aleph\left(\frac{\mu p}{\mu p-1}\right)^{\mu}\left(\frac{\nu q}{\nu q-1}\right)^{\nu}\left(\frac{\delta r}{\delta r-1}\right)^{\delta},$$

and $\aleph$ *is given by Equation (2.1), provided that the integrals involved in the upper bound exist.*

Proof. The required parameter assumptions being satisfied, Theorem 2.1 can be applied to the functions $x^{\phi/\mu}F(x)$, $y^{\psi/\nu}G(y)$ and $z^{\varphi/\delta}H(z)$ instead of f, g and h, respectively. This gives

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha+\phi}y^{\beta+\psi}z^{\gamma+\varphi}(x+y)^a(y+z)^b(x+z)^cF^{\mu}(x)G^{\nu}(y)H^{\delta}(z)}{(x+y+z)^{s+t+u}}dxdydz\\
&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha}y^{\beta}z^{\gamma}(x+y)^a(y+z)^b(x+z)^c[x^{\phi/\mu}F(x)]^{\mu}[y^{\psi/\nu}G(y)][z^{\varphi/\delta}H(z)]^{\delta}}{(x+y+z)^{s+t+u}}dxdydz\\
&\leq\aleph\left[\int_0^{+\infty}x^{(a-s+\gamma\omega+\beta\lambda)p+2}x^{\phi p}F^{\mu p}(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{(b-t+\alpha\epsilon+\gamma(1-\omega))q+2}y^{\psi q}G^{\nu q}(y)dy\right]^{1/q}\\
&\quad\times\left[\int_0^{+\infty}z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}z^{\varphi r}H^{\delta r}(z)dz\right]^{1/r}\\
&=\aleph\left[\int_0^{+\infty}x^{(a-s+\gamma\omega+\beta\lambda+\phi)p+2}F^{\mu p}(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{(b-t+\alpha\epsilon+\gamma(1-\omega)+\psi)q+2}G^{\nu q}(y)dy\right]^{1/q}\\
&\quad\times\left[\int_0^{+\infty}z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon)+\varphi)r+2}H^{\delta r}(z)dz\right]^{1/r},
\end{aligned}\tag{3.5}$$

where $\aleph$ is given by Equation (2.1). Using the following parameter relationships: $(a-s+\gamma\omega+\beta\lambda+\phi)p+2=-\mu p$, $(b-t+\alpha\epsilon+\gamma(1-\omega)+\psi)q+2=-\nu q$ and $(c-u+\beta(1-\lambda)+\alpha(1-\epsilon)+\varphi)r+2=-\delta r$, we have

$$\begin{aligned}
&\left[\int_0^{+\infty}x^{(a-s+\gamma\omega+\beta\lambda+\phi)p+2}F^{\mu p}(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{(b-t+\alpha\epsilon+\gamma(1-\omega)+\psi)q+2}G^{\nu q}(y)dy\right]^{1/q}\\
&\quad\times\left[\int_0^{+\infty}z^{(c-u+\beta(1-\lambda)+\alpha(1-\epsilon)+\varphi)r+2}H^{\delta r}(z)dz\right]^{1/r}\\
&=\left\{\int_0^{+\infty}\left[\frac{F(x)}{x}\right]^{\mu p}dx\right\}^{1/p}\left\{\int_0^{+\infty}\left[\frac{G(y)}{y}\right]^{\nu q}dy\right\}^{1/q}\left\{\int_0^{+\infty}\left[\frac{H(z)}{z}\right]^{\delta r}dz\right\}^{1/r}.
\end{aligned}\tag{3.6}$$

Since $\mu p > 1$, $\nu q > 1$, $\delta r > 1$, the classical Hardy integral inequality applied to f, g and h at μp, νq and δr, respectively, gives

$$\int_0^{+\infty} \left[\frac{F(x)}{x}\right]^{\mu p} dx \leq \left(\frac{\mu p}{\mu p - 1}\right)^{\mu p} \int_0^{+\infty} f^{\mu p}(x)dx, \tag{3.7}$$

$$\int_0^{+\infty} \left[\frac{G(y)}{y}\right]^{\nu q} dy \leq \left(\frac{\nu q}{\nu q - 1}\right)^{\nu q} \int_0^{+\infty} g^{\nu q}(y)dy \tag{3.8}$$

and

$$\int_0^{+\infty} \left[\frac{H(z)}{z}\right]^{\delta r} dz \leq \left(\frac{\delta r}{\delta r - 1}\right)^{\delta r} \int_0^{+\infty} h^{\delta r}(z)dz, \tag{3.9}$$

respectively. Combining Equations (3.5), (3.6), (3.7), (3.8) and (3.9), we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha+\phi}y^{\beta+\psi}z^{\gamma+\varphi}(x+y)^a(y+z)^b(x+z)^c F^{\mu}(x)G^{\nu}(y)H^{\delta}(z)}{(x+y+z)^{s+t+u}} dxdydz \\
&\leq \beth \left[\int_0^{+\infty} f^{\mu p}(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^{\nu q}(y)dy\right]^{1/q} \left[\int_0^{+\infty} h^{\delta r}(z)dz\right]^{1/r},
\end{aligned}$$

where

$$\beth = \aleph \left(\frac{\mu p}{\mu p - 1}\right)^{\mu} \left(\frac{\nu q}{\nu q - 1}\right)^{\nu} \left(\frac{\delta r}{\delta r - 1}\right)^{\delta}.$$

This concludes the proof of Proposition 3.2. □

This proposition can therefore be seen as a primitive variant of Theorem 2.1. Unlike that theorem, the upper bound depends on the integral norms of the main functions, rather than on the weighted integral norm. Thanks to the large number of parameters involved, it is a very general and adaptable three-dimensional integral formulation with a wide range of applications.

4 Conclusion

In conclusion, this article introduces a new three-dimensional Hardy-Hilbert integral inequality. It is remarkable for its generality, characterized by numerous adjustable parameters and complex power-sum interactions of the variables. Several new inequalities are derived to illustrate its versatility. Detailed proofs establish the validity of the results. This work contributes to the study of multidimensional integral inequalities and opens up possibilities for further research. Among these, we may think of considering other types of interactions of the variables, such as operations involving "$1+xy$", "$1+xz$", "$1+yz$", "$z+xy$", "$x+yz$" or "$y+xz$" raised to certain powers, or four-dimensional versions of our results.

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