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New Hardy-Hilbert-type integral inequalities involving special inhomogeneous kernel functions

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Abstract In this article, new Hardy-Hilbert-type integral inequalities are established. Our main result is based on a special inhomogeneous two-parameter kernel function. It is of the ratio power form, and has the property of involving a product term which perturbs the standard homogeneity property. We then use this result to derive new weighted integral norm inequalities and other Hardy-Hilbert-type integral inequalities. They are also defined with inhomogeneous kernel functions, but with innovative power and logarithmic forms. Some of them are obtained by treating an adjustable parameter as a variable and integrating with respect to it, which remains an original technique of proof. The article concludes with an attempt to unify some new and old Hardy-Hilbert-type integral inequalities. Due to the mathematical complexity, the optimality of the final result remains an open question, giving some new perspectives to a classical topic.

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1 Introduction

The foundation of this article is based on the work of the English mathematician Godfrey Harold Hardy in [5], which introduced the Hardy-Hilbert integral inequality. The statement of this inequality is given below. Let $p > 1$, $q = p/(p-1)$, so that $1/p + 1/q = 1$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions (the interval $(0, +\infty)$ can be replaced by $[0, +\infty)$ without loss of generality). Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side must converge, i.e., $\int_0^{+\infty} f^p(x)dx < +\infty$ and $\int_0^{+\infty} g^q(y)dy < +\infty$. The Hardy-Hilbert integral inequality plays a fundamental role in several areas of mathematical analysis, particularly in operator theory, functional analysis and harmonic analysis. Since its inception, it has been the focus of extensive research, leading to numerous generalizations, refinements and applications. Significant contributions can be found in [1, 3, 4, 11–16], while a comprehensive survey is available in [2]. Further extensions to multidimensional settings are presented in [6–10].

Among the key results, a generalized one-parameter variant of the Hardy-Hilbert integral inequality established in [16] has influenced the motivation of this article. It is stated formally below. Let $p > 1$, $q = p/(p-1)$, $\lambda > 0$ and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then we have

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y)^\lambda} f(x)g(y)dxdy \\ &\leq B_{\text{eta}}\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left[\int_0^{+\infty} x^{p-\lambda-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q-\lambda-1} g^q(y)dy\right]^{1/q}, \end{aligned} \tag{1.1}$$

where the integrals on the right-hand side must converge, and $B_{\text{eta}}(a, b)$ is the standard beta function taken at $a, b > 0$, defined by $B_{\text{eta}}(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ or, equivalently,

$$B_{\text{eta}}(a, b) = \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt. \tag{1.2}$$

If we take $\lambda = 1$, using the well-known beta formula $B_{\text{eta}}(a, 1-a) = \pi/\sin(a\pi)$ for any $a \in (0, 1)$, this inequality simplifies to

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} x^{p-2} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q-2} g^q(y)dy\right]^{1/q}.$$

Due to the inclusion of the power functions x^{p-2} and y^{q-2}, this approach provides an alternative weighted integral norm upper bound to the one found in the Hardy-Hilbert integral inequality. Furthermore, we observe that the main double integral can be expressed in the following standard kernel function integral form:

$$\int_0^{+\infty} \int_0^{+\infty} k_{\text{ern}}(x, y) f(x)g(y)dxdy,$$

where $k_{\text{ern}}(x, y)$ is the kernel function defined by

$$k_{\text{ern}}(x, y) = \frac{1}{(x+y)^\lambda}. \tag{1.3}$$

This function is symmetric, i.e., for any $x, y > 0$, $k_{\text{ern}}(x, y) = k_{\text{ern}}(y, x)$, and homogeneous, i.e., it exists $\sigma \in \mathbb{R}$ such that, for any $\delta > 0$, the following equality holds: $k_{\text{ern}}(\delta x, \delta y) = \delta^\sigma k_{\text{ern}}(x, y)$. Here, we have $\sigma = -\lambda$. The inequality in Equation (1.1) achieves two important objectives: adaptability through the parameter λ and originality through the form of the upper bound obtained. This innovative approach has naturally inspired numerous studies, as reflected in its high bibliometric impact, with 148 citations as of March 2025.

In this article, we build on this framework while introducing a greater degree of generality and complexity. Our main focus is on establishing upper bounds for the following double integral:

$$\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{(x+y+\alpha xy)^{\beta}}f(x)g(y)dxdy,$$

where $\alpha \geq 0$ and $\beta > 0$. So we are dealing with the kernel function defined by $k_{\mathrm{ern}}(x,y) = 1/(x+y+\alpha xy)^{\beta}$. Drawing a parallel to the kernel function in Equation (1.3), the product term αxy disturbs the homogeneity of the sum term $x+y$ in the denominator. To the best of our knowledge, this inhomogeneous kernel function has not been studied in the literature, or at least not in full generality. Given the complexity of this framework, we analyze three distinct parameter cases in increasing order of difficulty: $\beta > 1$, $\beta = 1$ and $\beta \in (0,1)$. Once relevant upper bounds are established, we use them to derive new weighted integral norm inequalities and Hardy-Hilbert-type integral inequalities. Some of these results are obtained by minorizing $(x+y+\alpha xy)^{\beta}$. Others are proved by treating α as a variable and integrating with respect to it, which is an original approach in the field. This methodology leads to the introduction of innovative kernel functions, in particular inhomogeneous kernel power functions and inhomogeneous kernel logarithmic functions. To support this claim in the introduction, one of our special inequalities is

$$\begin{aligned}&\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{xy}\log\left(1+\nu\frac{xy}{x+y}\right)f(x)g(y)dxdy\\&\leq 2\pi\nu\left[\int_{0}^{+\infty}\frac{x^{p/2-1}}{\sqrt{1+\nu x}+1}f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty}\frac{y^{q/2-1}}{\sqrt{1+\nu y}+1}g^{q}(y)dy\right]^{1/q},\end{aligned}$$

where $\nu > 0$. The corresponding kernel function is

$$k_{\mathrm{ern}}(x,y) = \frac{1}{xy}\log\left(1+\nu\frac{xy}{x+y}\right),$$

so of the two-parameter logarithmic type and inhomogeneous. To the best of our knowledge, this particular form has not been encountered in the literature, like all the forms that will be presented.

Next, we attempt to develop a unified framework that integrates the techniques derived from the preceding results with those arising from Equation (1.1). We use the term "attempt" because, due to the mathematical complexity of the setting, we are unable to establish the same level of optimality as that achieved in [16]. Therefore, this framework should be seen as a complementary extension to our original integral inequalities.

The following sections complete the article: Section 2 is devoted to the statement of our main theorem, followed by a detailed proof and a discussion. Additional theorems on new weighted integral norm inequalities and Hardy-Hilbert-type integral inequalities are presented in Section 3, together with their detailed proofs. Finally, the article concludes with Section 4.

2 Main theorem

The result below is our main theorem. It establishes new two-parameter Hardy-Hilbert-type integral inequalities under different parameter configurations.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. For any $\alpha \geq 0$ and $\beta > 0$, we set*

$$\mathcal{S} = \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x + y + \alpha xy)^\beta} f(x)g(y)dxdy.$$

Then, depending on the values of β, the inequalities below hold.

1. *For any $\beta > 1$, we have*

$$\mathcal{S} \leq \frac{1}{\beta - 1} \left[\int_0^{+\infty} \frac{1}{x^{\beta-1}(1 + \alpha x)} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y^{\beta-1}(1 + \alpha y)} g^q(y)dy\right]^{1/q},$$

   where the integrals on the right-hand side must converge.

2. *For $\beta = 1$, we have*

$$\mathcal{S} \leq \pi \left[\int_0^{+\infty} \sqrt{\frac{x^{p-2}}{1 + \alpha x}} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \sqrt{\frac{y^{q-2}}{1 + \alpha y}} g^q(y)dy\right]^{1/q},$$

   where the integrals on the right-hand side must converge.

3. *For any $\beta \in (0, 1)$, we have*

$$\mathcal{S} \leq B_{eta}\left(\frac{\beta}{2}, \frac{\beta}{2}\right) \left[\int_0^{+\infty} \sqrt{\frac{x^{(2-\beta)p-2}}{(1 + \alpha x)^\beta}} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1 + \alpha y)^\beta}} g^q(y)dy\right]^{1/q},$$

   where the integrals on the right-hand side must converge and $B_{eta}(a, b)$ is the standard beta function at $a, b > 0$ defined in Equation (1.2). In fact, this inequality is valid for any $\beta > 0$, not just for $\beta \in (0, 1)$.

Proof. We now proceed to prove the inequalities stated in the three items individually.

1. We suppose that $\beta > 1$. A suitable product decomposition of the integrand associated with $\mathcal{S}$ considering the equality $1/p + 1/q = 1$, followed by the application of the Hölder integral inequality at p and q, gives

$$\begin{aligned}\mathcal{S} &= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x + y + \alpha xy)^{\beta/p}} f(x) \times \frac{1}{(x + y + \alpha xy)^{\beta/q}} g(y)dxdy \\ &\leq A^{1/p}B^{1/q},\end{aligned} \tag{2.1}$$

   where

$$A = \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x + y + \alpha xy)^\beta} f^p(x)dxdy$$

and

$$B=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+\alpha xy)^\beta}g^q(y)dxdy.$$

Let us determine the expressions of A and B one by one.

Since the integrand associated with A is non-negative, we can apply the Fubini-Tonelli integral theorem that allows the exchange of the order of integration. This, followed by standard integral developments considering $\beta>1$, gives

$$\begin{aligned}A&=\int_0^{+\infty}f^p(x)\left[\int_0^{+\infty}\frac{1}{(x+y+\alpha xy)^\beta}dy\right]dx\\&=\int_0^{+\infty}f^p(x)\left[\int_0^{+\infty}\frac{1}{[x+(1+\alpha x)y]^\beta}dy\right]dx\\&=\int_0^{+\infty}f^p(x)\left[\frac{1}{1-\beta}\times\frac{1}{1+\alpha x}\times\frac{1}{[x+(1+\alpha x)y]^{\beta-1}}\right]_{y=0}^{y\to+\infty}dx\\&=\frac{1}{\beta-1}\int_0^{+\infty}\frac{1}{x^{\beta-1}(1+\alpha x)}f^p(x)dx.\end{aligned}\tag{2.2}$$

Similarly, we have

$$\begin{aligned}B&=\int_0^{+\infty}g^q(y)\left[\int_0^{+\infty}\frac{1}{(x+y+\alpha xy)^\beta}dx\right]dy\\&=\int_0^{+\infty}g^q(y)\left[\int_0^{+\infty}\frac{1}{[y+(1+\alpha y)x]^\beta}dx\right]dy\\&=\int_0^{+\infty}g^q(y)\left[\frac{1}{1-\beta}\times\frac{1}{1+\alpha y}\times\frac{1}{[y+(1+\alpha y)x]^{\beta-1}}\right]_{x=0}^{x\to+\infty}dy\\&=\frac{1}{\beta-1}\int_0^{+\infty}\frac{1}{y^{\beta-1}(1+\alpha y)}g^q(y)dy.\end{aligned}\tag{2.3}$$

Putting Equations (2.1), (2.2) and (2.3) together and using the equality $1/p+1/q=1$, we obtain

$$\begin{aligned}\mathcal{S}&\le\left[\frac{1}{\beta-1}\int_0^{+\infty}\frac{1}{x^{\beta-1}(1+\alpha x)}f^p(x)dx\right]^{1/p}\left[\frac{1}{\beta-1}\int_0^{+\infty}\frac{1}{y^{\beta-1}(1+\alpha y)}g^q(y)dy\right]^{1/q}\\&=\frac{1}{\beta-1}\left[\int_0^{+\infty}\frac{1}{x^{\beta-1}(1+\alpha x)}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y^{\beta-1}(1+\alpha y)}g^q(y)dy\right]^{1/q}.\end{aligned}$$

The desired result is achieved.

2. We suppose that $\beta=1$, so that

$$\mathcal{S}=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y+\alpha xy}f(x)g(y)dxdy.$$

A suitable product decomposition of the integrand associated with $\mathcal{S}$ considering the equality $1/p+1/q=1$, followed by the application of the Hölder integral inequality at p and q, gives

$$\begin{aligned} \mathcal{S} &= \int_0^{+\infty}\int_0^{+\infty} \frac{x^{1/(2q)}y^{-1/(2p)}}{(x+y+\alpha xy)^{1/p}} f(x) \times \frac{x^{-1/(2q)}y^{1/(2p)}}{(x+y+\alpha xy)^{1/q}} g(y)dxdy \\ &\leq C^{1/p}D^{1/q}, \end{aligned} \tag{2.4}$$

where

$$C = \int_0^{+\infty}\int_0^{+\infty} \frac{x^{p/(2q)}y^{-1/2}}{x+y+\alpha xy} f^p(x)dxdy$$

and

$$D = \int_0^{+\infty}\int_0^{+\infty} \frac{x^{-1/2}y^{q/(2p)}}{x+y+\alpha xy} g^q(y)dxdy.$$

Let us determine the expressions of C and D one by one.

Since the integrand associated with C is non-negative, we can apply the Fubini-Tonelli integral theorem that allows the exchange of the order of integration. This, followed by the use of standard square root and arctangent primitives and $q = p/(p-1)$, gives

$$\begin{aligned} C &= \int_0^{+\infty} x^{p/(2q)} f^p(x) \left[\int_0^{+\infty} \frac{y^{-1/2}}{x+y+\alpha xy} dy\right] dx \\ &= \int_0^{+\infty} x^{p/(2q)-1} f^p(x) \left\{\int_0^{+\infty} \frac{y^{-1/2}}{1+[\sqrt{y(1+\alpha x)/x}]^2} dy\right\} dx \\ &= \int_0^{+\infty} x^{p/(2q)-1} f^p(x) \left[2\sqrt{\frac{x}{1+\alpha x}} \arctan\left[\sqrt{\frac{y(1+\alpha x)}{x}}\right]\right]_{y=0}^{y\to+\infty} dx \\ &= \int_0^{+\infty} x^{p/(2q)-1} f^p(x) \times 2\sqrt{\frac{x}{1+\alpha x}} \times \frac{\pi}{2} dx \\ &= \pi \int_0^{+\infty} \sqrt{\frac{x^{p-2}}{1+\alpha x}} f^p(x)dx. \end{aligned} \tag{2.5}$$

Similarly, we have

$$\begin{aligned} D &= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[\int_0^{+\infty} \frac{x^{-1/2}}{x+y+\alpha xy} dx\right] dy \\ &= \int_0^{+\infty} y^{q/(2p)-1} g^q(y) \left\{\int_0^{+\infty} \frac{x^{-1/2}}{1+[\sqrt{x(1+\alpha y)/y}]^2} dx\right\} dy \\ &= \int_0^{+\infty} y^{q/(2p)-1} g^q(y) \left[2\sqrt{\frac{y}{1+\alpha y}} \arctan\left[\sqrt{\frac{x(1+\alpha y)}{y}}\right]\right]_{x=0}^{x\to+\infty} dy \\ &= \int_0^{+\infty} y^{q/(2p)-1} g^q(y) \times 2\sqrt{\frac{y}{1+\alpha y}} \times \frac{\pi}{2} dy \\ &= \pi \int_0^{+\infty} \sqrt{\frac{y^{q-2}}{1+\alpha y}} g^q(y)dy. \end{aligned} \tag{2.6}$$

Putting Equations (2.4), (2.5) and (2.6) together and using the equality $1/p+1/q = 1$, we obtain

$$\begin{aligned}\mathcal{S} &\leq \left[\pi\int_0^{+\infty}\sqrt{\frac{x^{p-2}}{1+\alpha x}}f^p(x)dx\right]^{1/p}\left[\pi\int_0^{+\infty}\sqrt{\frac{y^{q-2}}{1+\alpha y}}g^q(y)dy\right]^{1/q}\\ &= \pi\left[\int_0^{+\infty}\sqrt{\frac{x^{p-2}}{1+\alpha x}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\sqrt{\frac{y^{q-2}}{1+\alpha y}}g^q(y)dy\right]^{1/q}.\end{aligned}$$

This is the desired result.

3. We suppose that $\beta \in (0,1)$. A suitable product decomposition of the integrand associated with $\mathcal{S}$ considering the equality $1/p+1/q = 1$, followed by the application of the Hölder integral inequality at p and q, gives

$$\begin{aligned}\mathcal{S} &= \int_0^{+\infty}\int_0^{+\infty}\frac{x^{-(\beta/2-1)/q}y^{(\beta/2-1)/p}}{(x+y+\alpha xy)^{\beta/p}}f(x)\times\frac{x^{(\beta/2-1)/q}y^{-(\beta/2-1)/p}}{(x+y+\alpha xy)^{\beta/q}}g(y)dxdy\\ &\leq E^{1/p}F^{1/q},\end{aligned} \tag{2.7}$$

where

$$E = \int_0^{+\infty}\int_0^{+\infty}\frac{x^{-(\beta/2-1)p/q}y^{\beta/2-1}}{(x+y+\alpha xy)^{\beta}}f^p(x)dxdy$$

and

$$F = \int_0^{+\infty}\int_0^{+\infty}\frac{x^{\beta/2-1}y^{-(\beta/2-1)q/p}}{(x+y+\alpha xy)^{\beta}}g^q(y)dxdy.$$

Let us determine the expressions of E and F one by one.

Since the integrand associated with E is non-negative, we can apply the Fubini-Tonelli integral theorem that allows the exchange of the order of integration. This, followed by the change of variables $z = y(1+\alpha x)/x$ and the use of the beta function, gives

$$\begin{aligned}E &= \int_0^{+\infty}x^{-(\beta/2-1)p/q}f^p(x)\left[\int_0^{+\infty}\frac{y^{\beta/2-1}}{(x+y+\alpha xy)^{\beta}}dy\right]dx\\ &= \int_0^{+\infty}x^{-(\beta/2-1)p/q-\beta}f^p(x)\left[\int_0^{+\infty}\frac{y^{\beta/2-1}}{[1+y(1+\alpha x)/x]^{\beta}}dy\right]dx\\ &= \int_0^{+\infty}x^{-(\beta/2-1)p/q-\beta}f^p(x)\left(\frac{x}{1+\alpha x}\right)^{\beta/2}\left[\int_0^{+\infty}\frac{z^{\beta/2-1}}{(1+z)^{\beta}}dz\right]dx\\ &= \int_0^{+\infty}\frac{x^{(1-\beta/2)p-1}}{(1+\alpha x)^{\beta/2}}f^p(x)\times B_{\text{eta}}\left(\frac{\beta}{2},\frac{\beta}{2}\right)dx\\ &= B_{\text{eta}}\left(\frac{\beta}{2},\frac{\beta}{2}\right)\int_0^{+\infty}\sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^{\beta}}}f^p(x)dx.\end{aligned} \tag{2.8}$$

Similarly, but with the change of variables $w = x(1+\alpha y)/y$, we have

$$\begin{aligned} F &= \int_0^{+\infty} y^{-(\beta/2-1)q/p} g^q(y) \left[\int_0^{+\infty} \frac{x^{\beta/2-1}}{(x+y+\alpha xy)^\beta} dx \right] dy \\ &= \int_0^{+\infty} y^{-(\beta/2-1)q/p-\beta} g^q(y) \left[\int_0^{+\infty} \frac{x^{\beta/2-1}}{[1+x(1+\alpha y)/y]^\beta} dx \right] dy \\ &= \int_0^{+\infty} y^{-(\beta/2-1)q/p-\beta} g^q(y) \left(\frac{y}{1+\alpha y} \right)^{\beta/2} \left[\int_0^{+\infty} \frac{w^{\beta/2-1}}{(1+w)^\beta} dw \right] dy \qquad (2.9) \\ &= \int_0^{+\infty} \frac{y^{(1-\beta/2)q-1}}{(1+\alpha y)^{\beta/2}} g^q(y) \times B_{\mathrm{eta}}\left(\frac{\beta}{2}, \frac{\beta}{2} \right) dy \\ &= B_{\mathrm{eta}}\left(\frac{\beta}{2}, \frac{\beta}{2} \right) \int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^\beta}} g^q(y) dy. \end{aligned}$$

Putting Equations (2.7), (2.8) and (2.9) together and using the equality $1/p+1/q = 1$, we obtain

$$\begin{aligned} \mathcal{S} \leq & \left[B_{\mathrm{eta}}\left(\frac{\beta}{2}, \frac{\beta}{2} \right) \int_0^{+\infty} \sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^\beta}} f^p(x) dx \right]^{1/p} \\ & \times \left[B_{\mathrm{eta}}\left(\frac{\beta}{2}, \frac{\beta}{2} \right) \int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^\beta}} g^q(y) dy \right]^{1/q} \\ =& B_{\mathrm{eta}}\left(\frac{\beta}{2}, \frac{\beta}{2} \right) \left[\int_0^{+\infty} \sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^\beta}} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^\beta}} g^q(y) dy \right]^{1/q}. \end{aligned}$$

The desired result is achieved.

This concludes the proof. □

In Theorem 2.1, the main double integral is based on the kernel function defined by

$$k_{\mathrm{ern}}(x,y) = \frac{1}{(x+y+\alpha xy)^\beta}.$$

This function is clearly symmetric. It is also inhomogeneous for $\alpha > 0$ because, for any $x, y > 0$ and $\delta > 0$, we have

$$k_{\mathrm{ern}}(\delta x, \delta y) = \frac{1}{[\delta x + \delta y + \alpha(\delta x)(\delta y)]^\beta} = \frac{1}{\delta^\beta} \times \frac{1}{(x+y+\alpha \delta xy)^\beta} \neq \frac{1}{\delta^\beta} k_{\mathrm{ern}}(x,y).$$

So the new parameter α is the main cause of this inhomogeneity. The inequality in Theorem 2.1 thus belongs to the class of inhomogeneous Hardy-Hilbert-type integral inequalities.

As indicated in the statement, the item 3 of Theorem 2.1 is valid for any $\beta > 0$. In particular, if we take $\beta = 1$, since $B_{\text{eta}}(1/2, 1/2) = \pi$, then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y+\alpha xy} f(x)g(y)dxdy \\ \leq \pi \left[\int_0^{+\infty} \sqrt{\frac{x^{p-2}}{1+\alpha x}} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \sqrt{\frac{y^{q-2}}{1+\alpha y}} g^q(y)dy\right]^{1/q}.$$

It simplifies to the result of the item 2.

Still concerning the item 3, if we take $\alpha = 0$ and $p = 2$, so that $q = 2$, we also have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^\beta} f(x)g(y)dxdy \\ \leq B_{\text{eta}}\left(\frac{\beta}{2}, \frac{\beta}{2}\right) \sqrt{\int_0^{+\infty} x^{1-\beta} f^2(x)dx} \sqrt{\int_0^{+\infty} y^{1-\beta} g^2(y)dy}.$$

The same is obtained with Equation (1.1) by taking $\lambda = \beta$ and $p = 2$. However, there is no correspondence between these two results for $p \neq 2$. This point will be clarified in a forthcoming unified theorem, i.e., Theorem 3.6.

3 Additional theorems

3.1 Weighted integral norm inequalities

Theorem 2.1 can be used to derive new weighted integral norm inequalities, as presented in the result below.

Theorem 3.1. *Let $p > 1$, $q = p/(p-1)$, and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function. For any $\alpha \geq 0$ and $\beta > 0$, we set*

$$\mathfrak{T}(y) = \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)dx.$$

Then, depending on the values of β, the inequalities below hold.

1. *For any $\beta > 1$, we have*

$$\int_0^{+\infty} \left[y^{\beta-1}(1+\alpha y)\right]^{p-1} [\mathfrak{T}(y)]^p \, dy \leq \frac{1}{(\beta-1)^p} \int_0^{+\infty} \frac{1}{x^{\beta-1}(1+\alpha x)} f^p(x)dx,$$

where the integral on the right-hand side must converge.

2. *For $\beta = 1$, we have*

$$\int_0^{+\infty} \left(\frac{1+\alpha y}{y^{p-2}}\right)^{(p-1)/2} [\mathfrak{T}(y)]^p \, dy \leq \pi^p \int_0^{+\infty} \sqrt{\frac{x^{p-2}}{1+\alpha x}} f^p(x)dx,$$

where the integral on the right-hand side must converge.

3. *For any $\beta \in (0,1)$, we have*

$$\int_0^{+\infty} \left[\frac{(1+\alpha y)^\beta}{y^{(2-\beta)p-2}}\right]^{(p-1)/2} [\Upsilon(y)]^p \, dy \leq B_{eta}^p\left(\frac{\beta}{2}, \frac{\beta}{2}\right) \int_0^{+\infty} \sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^\beta}} f^p(x)dx,$$

where the integral on the right-hand side must converge and $B_{eta}(a,b)$ is the standard beta function at $a, b > 0$ defined in Equation (1.2). In fact, this inequality is valid for any $\beta > 0$, not just for $\beta \in (0,1)$.

Proof. We now proceed to prove the inequalities stated in the three items individually.

1. Let us set

$$G = \int_0^{+\infty} \left[y^{\beta-1}(1+\alpha y)\right]^{p-1} [\Upsilon(y)]^p \, dy.$$

A suitable product decomposition of the integrand and the Fubini-Tonelli integral theorem give

$$\begin{aligned} G &= \int_0^{+\infty} \left[y^{\beta-1}(1+\alpha y)\right]^{p-1} [\Upsilon(y)]^{p-1} \Upsilon(y) dy \\ &= \int_0^{+\infty} \left[y^{\beta-1}(1+\alpha y)\Upsilon(y)\right]^{p-1} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x) dx dy \\ &= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x) g_\star(y) dx dy, \end{aligned} \tag{3.1}$$

where

$$g_\star(y) = \left[y^{\beta-1}(1+\alpha y)\Upsilon(y)\right]^{p-1}.$$

Applying the item 1 of Theorem 2.1 to the functions f and $g_\star$, we get

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x) g_\star(y) dx dy \\ &\leq \frac{1}{\beta-1}\left[\int_0^{+\infty} \frac{1}{x^{\beta-1}(1+\alpha x)} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \frac{1}{y^{\beta-1}(1+\alpha y)} g_\star^q(y)dy\right]^{1/q}. \end{aligned} \tag{3.2}$$

Since $q = p/(p-1)$, we have

$$\begin{aligned} \int_0^{+\infty} \frac{1}{y^{\beta-1}(1+\alpha y)} g_\star^q(y)dy &= \int_0^{+\infty} \frac{1}{y^{\beta-1}(1+\alpha y)} \left[y^{\beta-1}(1+\alpha y)\Upsilon(y)\right]^{q(p-1)} dy \\ &= \int_0^{+\infty} \frac{1}{y^{\beta-1}(1+\alpha y)} \left[y^{\beta-1}(1+\alpha y)\Upsilon(y)\right]^{p} dy \\ &= \int_0^{+\infty} \left[y^{\beta-1}(1+\alpha y)\right]^{p-1} [\Upsilon(y)]^p \, dy = G. \end{aligned} \tag{3.3}$$

Putting Equations (3.1), (3.2) and (3.3) together, we obtain

$$G \leq \frac{1}{\beta-1}\left[\int_0^{+\infty} \frac{1}{x^{\beta-1}(1+\alpha x)} f^p(x)dx\right]^{1/p} G^{1/q},$$

so that, using the equality $1/p = 1 - 1/q$,

$$G^{1/p} = G^{1-1/q} \leq \frac{1}{\beta - 1}\left[\int_0^{+\infty} \frac{1}{x^{\beta-1}(1+\alpha x)} f^p(x)dx\right]^{1/p}$$

and, raising both sides to the exponent p,

$$\int_0^{+\infty} \left[y^{\beta-1}(1+\alpha y)\right]^{p-1} [\mathfrak{T}(y)]^p \, dy = G \leq \frac{1}{(\beta-1)^p} \int_0^{+\infty} \frac{1}{x^{\beta-1}(1+\alpha x)} f^p(x)dx.$$

2. Let us set

$$H = \int_0^{+\infty} \left(\frac{1+\alpha y}{y^{p-2}}\right)^{(p-1)/2} [\mathfrak{T}(y)]^p \, dy.$$

A suitable product decomposition of the integrand and the Fubini-Tonelli integral theorem give

$$\begin{aligned} H &= \int_0^{+\infty} \left(\frac{1+\alpha y}{y^{p-2}}\right)^{(p-1)/2} [\mathfrak{T}(y)]^{p-1} \mathfrak{T}(y)dy \\ &= \int_0^{+\infty} \left[\sqrt{\frac{1+\alpha y}{y^{p-2}}}\mathfrak{T}(y)\right]^{p-1} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)dxdy \\ &= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)g_\curlywedge(y)dxdy, \end{aligned} \tag{3.4}$$

where

$$g_\curlywedge(y) = \left[\sqrt{\frac{1+\alpha y}{y^{p-2}}}\mathfrak{T}(y)\right]^{p-1}.$$

Applying the item 2 of Theorem 2.1 to the functions f and $g_\curlywedge$, we get

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)g_\curlywedge(y)dxdy \\ &\leq \pi \left[\int_0^{+\infty} \sqrt{\frac{x^{p-2}}{1+\alpha x}} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \sqrt{\frac{y^{q-2}}{1+\alpha y}} g_\curlywedge^q(y)dy\right]^{1/q}. \end{aligned} \tag{3.5}$$

Since $q = p/(p-1)$, we have

$$\begin{aligned} \int_0^{+\infty} \sqrt{\frac{y^{q-2}}{1+\alpha y}} g_\curlywedge^q(y)dy &= \int_0^{+\infty} \sqrt{\frac{y^{q-2}}{1+\alpha y}} \left[\sqrt{\frac{1+\alpha y}{y^{p-2}}}\mathfrak{T}(y)\right]^{q(p-1)} dy \\ &= \int_0^{+\infty} \sqrt{\frac{y^{q-2}}{1+\alpha y}} \left[\sqrt{\frac{1+\alpha y}{y^{p-2}}}\mathfrak{T}(y)\right]^{p} dy \\ &= \int_0^{+\infty} \left(\frac{1+\alpha y}{y^{p-2}}\right)^{(p-1)/2} [\mathfrak{T}(y)]^p \, dy = H. \end{aligned} \tag{3.6}$$

Putting Equations (3.4), (3.5) and (3.6) together, we obtain

$$H \leq \pi \left[\int_0^{+\infty} \sqrt{\frac{x^{p-2}}{1+\alpha x}} f^p(x)dx\right]^{1/p} H^{1/q},$$

so that, using the equality $1/p = 1 - 1/q$,

$$H^{1/p} = H^{1-1/q} \leq \pi \left[\int_0^{+\infty} \sqrt{\frac{x^{p-2}}{1+\alpha x}} f^p(x) dx \right]^{1/p}$$

and, raising both sides to the exponent p,

$$\int_0^{+\infty} \left(\frac{1+\alpha y}{y^{p-2}} \right)^{(p-1)/2} [\mathcal{T}(y)]^p \, dy = H \leq \pi^p \int_0^{+\infty} \sqrt{\frac{x^{p-2}}{1+\alpha x}} f^p(x) dx.$$

3. Let us set

$$I = \int_0^{+\infty} \left[\frac{(1+\alpha y)^\beta}{y^{(2-\beta)p-2}} \right]^{(p-1)/2} [\mathcal{T}(y)]^p \, dy.$$

A suitable product decomposition of the integrand and the Fubini-Tonelli integral theorem give

$$\begin{aligned} I &= \int_0^{+\infty} \left[\frac{(1+\alpha y)^\beta}{y^{(2-\beta)p-2}} \right]^{(p-1)/2} [\mathcal{T}(y)]^{p-1} \mathcal{T}(y) dy \\ &= \int_0^{+\infty} \left[\sqrt{\frac{(1+\alpha y)^\beta}{y^{(2-\beta)p-2}}} \mathcal{T}(y) \right]^{p-1} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x) dx dy \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x) g_\triangleright(y) dx dy, \end{aligned} \tag{3.7}$$

where

$$g_\triangleright(y) = \left[\sqrt{\frac{(1+\alpha y)^\beta}{y^{(2-\beta)p-2}}} \mathcal{T}(y) \right]^{p-1}.$$

Applying the item 3 of Theorem 2.1 to the functions f and $g_\triangleright$, we get

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x) g_\triangleright(y) dx dy \\ &\leq B_{\mathrm{eta}} \left(\frac{\beta}{2}, \frac{\beta}{2} \right) \left[\int_0^{+\infty} \sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^\beta}} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^\beta}} g_\triangleright^q(y) dy \right]^{1/q}. \end{aligned} \tag{3.8}$$

Since $q = p/(p-1)$, we have

$$\begin{aligned} \int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^\beta}} g_\triangleright^q(y) dy &= \int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^\beta}} \left[\sqrt{\frac{(1+\alpha y)^\beta}{y^{(2-\beta)p-2}}} \mathcal{T}(y) \right]^{q(p-1)} dy \\ &= \int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^\beta}} \left[\sqrt{\frac{(1+\alpha y)^\beta}{y^{(2-\beta)p-2}}} \mathcal{T}(y) \right]^{p} dy \\ &= \int_0^{+\infty} \left[\frac{(1+\alpha y)^\beta}{y^{(2-\beta)p-2}} \right]^{(p-1)/2} [\mathcal{T}(y)]^p \, dy = I. \end{aligned} \tag{3.9}$$

Putting Equations (3.7), (3.8) and (3.9) together, we obtain

$$I \leq B_{\text{eta}}\left(\frac{\beta}{2}, \frac{\beta}{2}\right)\left[\int_0^{+\infty}\sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^{\beta}}}f^p(x)dx\right]^{1/p} I^{1/q},$$

so that, using the equality $1/p = 1 - 1/q$,

$$I^{1/p} = I^{1-1/q} \leq B_{\text{eta}}\left(\frac{\beta}{2}, \frac{\beta}{2}\right)\left[\int_0^{+\infty}\sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^{\beta}}}f^p(x)dx\right]^{1/p}$$

and, raising both sides to the exponent p,

$$\int_0^{+\infty}\left[\frac{(1+\alpha y)^{\beta}}{y^{(2-\beta)p-2}}\right]^{(p-1)/2}[\mathcal{T}(y)]^p\, dy = I \leq B_{\text{eta}}^p\left(\frac{\beta}{2}, \frac{\beta}{2}\right)\int_0^{+\infty}\sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^{\beta}}}f^p(x)dx.$$

This concludes the proof. □

In fact, as a direct application of the Hölder integral inequality, it can be proved that the items 1 in Theorems 2.1 and 3.1 are equivalent, the items 2 in Theorems 2.1 and 3.1 are equivalent, and the items 3 in Theorems 2.1 and 3.1 are equivalent. Theorem 3.1 can thus be seen as a one-function version of Theorem 2.1. It is also useful for studying the integral operator described by $\mathcal{T}$.

3.2 Consideration of new inhomogeneous kernel functions

Some Hardy-Hilbert-type integral inequalities based on new inhomogeneous kernel functions can be derived from Theorem 2.1 and power inequalities involving the parameter β. This is developed in the result below.

Theorem 3.2. *Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. For any $\gamma \geq 0$ and $\beta > 0$, we set*

$$\mathcal{U} = \int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^{\beta} + \gamma(xy)^{\beta}}f(x)g(y)dxdy.$$

Then, depending on the values of β, the inequalities below hold.

1. *For any $\beta > 1$, we have*

$$\mathcal{U} \leq \frac{2^{\beta-1}}{\beta-1}\left[\int_0^{+\infty}\frac{1}{x^{\beta-1}(1+\gamma^{1/\beta}x)}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{1}{y^{\beta-1}(1+\gamma^{1/\beta}y)}g^q(y)dy\right]^{1/q},$$

   where the integrals on the right-hand side must converge.

2. *For any $\beta \in (0, 1)$, we have*

$$\mathcal{U} \leq B_{eta}\left(\frac{\beta}{2}, \frac{\beta}{2}\right)\left[\int_0^{+\infty}\sqrt{\frac{x^{(2-\beta)p-2}}{(1+\gamma^{1/\beta}x)^{\beta}}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\sqrt{\frac{y^{(2-\beta)q-2}}{(1+\gamma^{1/\beta}y)^{\beta}}}g^q(y)dy\right]^{1/q},$$

   where the integrals on the right-hand side must converge and $B_{eta}(a, b)$ is the standard beta function at $a, b > 0$ defined in Equation (1.2). In fact, this inequality is valid for any $\beta > 0$, not just for $\beta \in (0, 1)$.

Proof. We now proceed to prove the inequalities stated in the three items individually.

1. We suppose that $\beta > 1$. Then a standard convexity-power inequality gives
$$(x+y+\gamma^{1/\beta}xy)^{\beta} \leq 2^{\beta-1}[(x+y)^{\beta}+(\gamma^{1/\beta}xy)^{\beta}] = 2^{\beta-1}[(x+y)^{\beta}+\gamma(xy)^{\beta}],$$
so that
$$\frac{1}{(x+y)^{\beta}+\gamma(xy)^{\beta}} \leq \frac{2^{\beta-1}}{(x+y+\gamma^{1/\beta}xy)^{\beta}}.$$
Using this and the item 1 of Theorem 2.1 applied to $\alpha = \gamma^{1/\beta}$, we obtain
$$\begin{aligned}
\mathfrak{U} &= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^{\beta}+\gamma(xy)^{\beta}} f(x)g(y)dxdy \\
&\leq 2^{\beta-1}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\gamma^{1/\beta}xy)^{\beta}} f(x)g(y)dxdy \\
&\leq \frac{2^{\beta-1}}{\beta-1}\left[\int_0^{+\infty} \frac{1}{x^{\beta-1}(1+\gamma^{1/\beta}x)} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} \frac{1}{y^{\beta-1}(1+\gamma^{1/\beta}y)} g^q(y)dy\right]^{1/q}.
\end{aligned}$$
This is the desired inequality.

2. We now suppose that $\beta \in (0,1)$. Then a standard convexity-power inequality gives
$$(x+y+\gamma^{1/\beta}xy)^{\beta} \leq (x+y)^{\beta}+(\gamma^{1/\beta}xy)^{\beta} = (x+y)^{\beta}+\gamma(xy)^{\beta},$$
so that
$$\frac{1}{(x+y)^{\beta}+\gamma(xy)^{\beta}} \leq \frac{1}{(x+y+\gamma^{1/\beta}xy)^{\beta}}.$$
Using this and the item 3 of Theorem 2.1 applied to $\alpha = \gamma^{1/\beta}$, we obtain
$$\begin{aligned}
\mathfrak{U} &= \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y)^{\beta}+\gamma(xy)^{\beta}} f(x)g(y)dxdy \\
&\leq \int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\gamma^{1/\beta}xy)^{\beta}} f(x)g(y)dxdy \\
&\leq B_{\text{eta}}\left(\frac{\beta}{2},\frac{\beta}{2}\right)\left[\int_0^{+\infty} \sqrt{\frac{x^{(2-\beta)p-2}}{(1+\gamma^{1/\beta}x)^{\beta}}} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} \sqrt{\frac{y^{(2-\beta)q-2}}{(1+\gamma^{1/\beta}y)^{\beta}}} g^q(y)dy\right]^{1/q}.
\end{aligned}$$
The desired inequality is achieved.

This concludes the proof. □

The kernel function associated with the main double integral is thus
$$k_{\text{ern}}(x,y) = \frac{1}{(x+y)^{\beta}+\gamma(xy)^{\beta}}.$$
It is clearly symmetric and inhomogeneous. To the best of our knowledge, this kernel function and the inequalities in Theorem 3.2 have not been studied in the literature.

The theorem below presents a new Hardy-Hilbert-type integral inequality based on an inhomogeneous two-parameter power kernel function. The proof relies on Theorem 2.1, treating α as a variable and integrating it over a suitable interval.

Theorem 3.3. *Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then, for any $\beta > 1$, $\mu \geq 0$ and $\nu > 0$ such that $\nu > \mu$, we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{xy}\left[\frac{1}{(x+y+\mu xy)^{\beta-1}} - \frac{1}{(x+y+\nu xy)^{\beta-1}}\right] f(x)g(y)dxdy$$
$$\leq \left[\int_0^{+\infty} \frac{1}{x^\beta}\log\left(\frac{1+\nu x}{1+\mu x}\right) f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} \frac{1}{y^\beta}\log\left(\frac{1+\nu y}{1+\mu y}\right) g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side must converge.

Proof. It follows from the item 1 of Theorem 2.1 that, for any $\beta > 1$,

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)g(y)dxdy$$
$$\leq \frac{1}{\beta-1}\left[\int_0^{+\infty} \frac{1}{x^{\beta-1}(1+\alpha x)} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} \frac{1}{y^{\beta-1}(1+\alpha y)} g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$. Treating α as a variable and integrating it over the interval (μ, ν), we have

$$J \leq \frac{1}{\beta-1}K, \tag{3.10}$$

where

$$J = \int_\mu^\nu \left\{\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)g(y)dxdy\right\} d\alpha$$

and

$$K = \int_\mu^\nu \left\{\left[\int_0^{+\infty} \frac{1}{x^{\beta-1}(1+\alpha x)} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} \frac{1}{y^{\beta-1}(1+\alpha y)} g^q(y)dy\right]^{1/q}\right\} d\alpha.$$

Let us study J and K one by one.

Using the Fubini-Tonelli integral theorem and a standard power primitive, we get

$$J = \int_0^{+\infty}\int_0^{+\infty} \left\{\int_\mu^\nu \frac{1}{(x+y+\alpha xy)^\beta} d\alpha\right\} f(x)g(y)dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty} \left[\frac{1}{1-\beta} \times \frac{1}{xy} \times \frac{1}{(x+y+\alpha xy)^{\beta-1}}\right]_{\alpha=\mu}^{\alpha=\nu} f(x)g(y)dxdy$$
$$= \frac{1}{\beta-1}\int_0^{+\infty}\int_0^{+\infty} \frac{1}{xy}\left[\frac{1}{(x+y+\mu xy)^{\beta-1}} - \frac{1}{(x+y+\nu xy)^{\beta-1}}\right] f(x)g(y)dxdy. \tag{3.11}$$

Using the Hölder integral inequality at p and q, the Fubini-Tonelli integral theorem and standard logarithmic primitives, we get

$$\begin{aligned}
K &\leq \left\{\int_{\mu}^{\nu}\left[\int_{0}^{+\infty}\frac{1}{x^{\beta-1}(1+\alpha x)}f^{p}(x)dx\right]^{p/p}d\alpha\right\}^{1/p} \\
&\quad\times\left\{\int_{\mu}^{\nu}\left[\int_{0}^{+\infty}\frac{1}{y^{\beta-1}(1+\alpha y)}g^{q}(y)dy\right]^{q/q}d\alpha\right\}^{1/q} \\
&=\left[\int_{0}^{+\infty}\frac{1}{x^{\beta-1}}\left(\int_{\mu}^{\nu}\frac{1}{1+\alpha x}d\alpha\right)f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty}\frac{1}{y^{\beta-1}}\left(\int_{\mu}^{\nu}\frac{1}{1+\alpha y}d\alpha\right)g^{q}(y)dy\right]^{1/q} \\
&=\left\{\int_{0}^{+\infty}\frac{1}{x^{\beta-1}}\left[\frac{1}{x}\log(1+\alpha x)\right]_{\alpha=\mu}^{\alpha=\nu}f^{p}(x)dx\right\}^{1/p} \\
&\quad\times\left\{\int_{0}^{+\infty}\frac{1}{y^{\beta-1}}\left[\frac{1}{y}\log(1+\alpha y)\right]_{\alpha=\mu}^{\alpha=\nu}g^{q}(y)dy\right\}^{1/q} \\
&=\left[\int_{0}^{+\infty}\frac{1}{x^{\beta}}\log\left(\frac{1+\nu x}{1+\mu x}\right)f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty}\frac{1}{y^{\beta}}\log\left(\frac{1+\nu y}{1+\mu y}\right)g^{q}(y)dy\right]^{1/q}.
\end{aligned} \tag{3.12}$$

Putting Equations (3.10), (3.11) and (3.12) together, we obtain

$$\begin{aligned}
&\frac{1}{\beta-1}\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{xy}\left[\frac{1}{(x+y+\mu xy)^{\beta-1}}-\frac{1}{(x+y+\nu xy)^{\beta-1}}\right]f(x)g(y)dxdy \\
&\leq\frac{1}{\beta-1}\left[\int_{0}^{+\infty}\frac{1}{x^{\beta}}\log\left(\frac{1+\nu x}{1+\mu x}\right)f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty}\frac{1}{y^{\beta}}\log\left(\frac{1+\nu y}{1+\mu y}\right)g^{q}(y)dy\right]^{1/q},
\end{aligned}$$

so that, since $\beta > 1$,

$$\begin{aligned}
&\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{xy}\left[\frac{1}{(x+y+\mu xy)^{\beta-1}}-\frac{1}{(x+y+\nu xy)^{\beta-1}}\right]f(x)g(y)dxdy \\
&\leq\left[\int_{0}^{+\infty}\frac{1}{x^{\beta}}\log\left(\frac{1+\nu x}{1+\mu x}\right)f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty}\frac{1}{y^{\beta}}\log\left(\frac{1+\nu y}{1+\mu y}\right)g^{q}(y)dy\right]^{1/q}.
\end{aligned}$$

This concludes the proof. □

The kernel function associated with the main double integral is thus

$$k_{\mathrm{ern}}(x,y)=\frac{1}{xy}\left[\frac{1}{(x+y+\mu xy)^{\beta-1}}-\frac{1}{(x+y+\nu xy)^{\beta-1}}\right].$$

It is clearly symmetric and inhomogeneous.

If we take $\mu = 0$, then the new inequality simplifies to

$$\begin{aligned}
&\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{xy}\left[\frac{1}{(x+y)^{\beta-1}}-\frac{1}{(x+y+\nu xy)^{\beta-1}}\right]f(x)g(y)dxdy \\
&\leq\left[\int_{0}^{+\infty}\frac{1}{x^{\beta}}\log(1+\nu x)f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty}\frac{1}{y^{\beta}}\log(1+\nu y)g^{q}(y)dy\right]^{1/q}.
\end{aligned}$$

The presence of logarithmically weighted integral norms in the upper bound has a certain originality.

On a similar mathematical basis, the theorem below presents a new Hardy-Hilbert-type integral inequality based on an inhomogeneous two-parameter logarithmic kernel function.

Theorem 3.4. *Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Then, for any $\mu \geq 0$ and $\nu > 0$ such that $\nu > \mu$, we have*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{xy}\log\left(\frac{x+y+\nu xy}{x+y+\mu xy}\right) f(x)g(y)dxdy$$
$$\leq 2\pi(\nu-\mu)\left[\int_0^{+\infty}\frac{x^{p/2-1}}{\sqrt{1+\nu x}+\sqrt{1+\mu x}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{y^{q/2-1}}{\sqrt{1+\nu y}+\sqrt{1+\mu y}}g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side must converge.

Proof. It follows from the item 2 of Theorem 2.1 that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y+\alpha xy}f(x)g(y)dxdy$$
$$\leq \pi\left[\int_0^{+\infty}\sqrt{\frac{x^{p-2}}{1+\alpha x}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\sqrt{\frac{y^{q-2}}{1+\alpha y}}g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$. Treating α as a variable and integrating it over the interval (μ, ν), we have

$$L \leq \pi M, \tag{3.13}$$

where

$$L = \int_\mu^\nu \left\{\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y+\alpha xy}f(x)g(y)dxdy\right\}d\alpha$$

and

$$M = \int_\mu^\nu\left\{\left[\int_0^{+\infty}\sqrt{\frac{x^{p-2}}{1+\alpha x}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\sqrt{\frac{y^{q-2}}{1+\alpha y}}g^q(y)dy\right]^{1/q}\right\}d\alpha.$$

Let us study L and M one by one.

Using the Fubini-Tonelli integral theorem and a standard logarithmic primitive, we have

$$\begin{aligned} L &= \int_0^{+\infty}\int_0^{+\infty}\left\{\int_\mu^\nu \frac{1}{x+y+\alpha xy}d\alpha\right\}f(x)g(y)dxdy \\ &= \int_0^{+\infty}\int_0^{+\infty}\left[\frac{1}{xy}\log(x+y+\alpha xy)\right]_{\alpha=\mu}^{\alpha=\nu}f(x)g(y)dxdy \\ &= \int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\log\left(\frac{x+y+\nu xy}{x+y+\mu xy}\right)f(x)g(y)dxdy. \end{aligned} \tag{3.14}$$

Using the Hölder integral inequality at p and q, the Fubini-Tonelli integral theorem, standard square root primitives, natural conjugate and the equality $1/p + 1/q = 1$, we get

$$\begin{aligned}
M \leq & \left\{ \int_{\mu}^{\nu} \left[\int_{0}^{+\infty} \sqrt{\frac{x^{p-2}}{1+\alpha x}} f^{p}(x)dx \right]^{p/p} d\alpha \right\}^{1/p} \\
& \times \left\{ \int_{\mu}^{\nu} \left[\int_{0}^{+\infty} \sqrt{\frac{y^{q-2}}{1+\alpha y}} g^{q}(y)dy \right]^{q/q} d\alpha \right\}^{1/q} \\
= & \left\{ \int_{0}^{+\infty} x^{p/2-1} \left[\int_{\mu}^{\nu} \frac{1}{\sqrt{1+\alpha x}} d\alpha \right] f^{p}(x)dx \right\}^{1/p} \\
& \times \left\{ \int_{0}^{+\infty} y^{q/2-1} \left[\int_{\mu}^{\nu} \frac{1}{\sqrt{1+\alpha y}} d\alpha \right] g^{q}(y)dy \right\}^{1/q} \\
= & \left\{ \int_{0}^{+\infty} x^{p/2-1} \left[\frac{2}{x}\sqrt{1+\alpha x} \right]_{\alpha=\mu}^{\alpha=\nu} f^{p}(x)dx \right\}^{1/p} \\
& \times \left\{ \int_{0}^{+\infty} y^{q/2-1} \left[\frac{2}{y}\sqrt{1+\alpha y} \right]_{\alpha=\mu}^{\alpha=\nu} g^{q}(y)dy \right\}^{1/q} \\
= & 2\left[\int_{0}^{+\infty} x^{p/2-2} \left[\sqrt{1+\nu x} - \sqrt{1+\mu x} \right] f^{p}(x)dx \right]^{1/p} \\
& \times \left[\int_{0}^{+\infty} y^{p/2-2} \left[\sqrt{1+\nu y} - \sqrt{1+\mu y} \right] g^{q}(y)dy \right]^{1/q} \\
= & 2\left[\int_{0}^{+\infty} x^{p/2-2} \frac{(1+\nu x)-(1+\mu x)}{\sqrt{1+\nu x}+\sqrt{1+\mu x}} f^{p}(x)dx \right]^{1/p} \\
& \times \left[\int_{0}^{+\infty} y^{p/2-2} \frac{(1+\nu y)-(1+\mu u)}{\sqrt{1+\nu y}+\sqrt{1+\mu y}} g^{q}(y)dy \right]^{1/q} \\
= & 2(\nu-\mu)\left[\int_{0}^{+\infty} \frac{x^{p/2-1}}{\sqrt{1+\nu x}+\sqrt{1+\mu x}} f^{p}(x)dx \right]^{1/p} \\
& \times \left[\int_{0}^{+\infty} \frac{y^{q/2-1}}{\sqrt{1+\nu y}+\sqrt{1+\mu y}} g^{q}(y)dy \right]^{1/q}.
\end{aligned} \tag{3.15}$$

Putting Equations (3.13), (3.14) and (3.15) together, we obtain

$$\begin{aligned}
& \int_{0}^{+\infty} \int_{0}^{+\infty} \frac{1}{xy} \log\left(\frac{x+y+\nu xy}{x+y+\mu xy} \right) f(x)g(y)dxdy \\
& \leq 2\pi(\nu-\mu)\left[\int_{0}^{+\infty} \frac{x^{p/2-1}}{\sqrt{1+\nu x}+\sqrt{1+\mu x}} f^{p}(x)dx \right]^{1/p} \left[\int_{0}^{+\infty} \frac{y^{q/2-1}}{\sqrt{1+\nu y}+\sqrt{1+\mu y}} g^{q}(y)dy \right]^{1/q}.
\end{aligned}$$

This concludes the proof. □

The kernel function associated with the main double integral is thus

$$k_{\mathrm{ern}}(x,y)=\frac{1}{xy}\log\left(\frac{x+y+\nu xy}{x+y+\mu xy}\right).$$

It is obviously symmetric and inhomogeneous.

If we take $\mu=0$, then the new inequality simplifies to

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\log\left(1+\nu\frac{xy}{x+y}\right)f(x)g(y)dxdy\\&\leq 2\pi\nu\left[\int_0^{+\infty}\frac{x^{p/2-1}}{\sqrt{1+\nu x}+1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{y^{q/2-1}}{\sqrt{1+\nu y}+1}g^q(y)dy\right]^{1/q}.\end{aligned}$$

Again we emphasize the originality of the form, with a logarithmic term that includes the ratio-product-sum term $xy/(x+y)$.

Another variant of the Hardy-Hilbert-type integral inequality is proposed in the theorem below, using an inhomogeneous two-parameter power kernel function.

Theorem 3.5. *Let $p>1$, $q=p/(p-1)$, and $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions. Then, for any $\beta\in(0,1)$, $\mu\geq 0$ and $\nu>0$ such that $\nu>\mu$, we have*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right]f(x)g(y)dxdy\\&\leq\frac{2(1-\beta)}{2-\beta}B_{eta}\left(\frac{\beta}{2},\frac{\beta}{2}\right)\\&\quad\times\left\{\int_0^{+\infty}x^{(1-\beta/2)p-2}\left[(1+\nu x)^{1-\beta/2}-(1+\mu x)^{1-\beta/2}\right]f^p(x)dx\right\}^{1/p}\\&\quad\times\left\{\int_0^{+\infty}y^{(1-\beta/2)q-2}\left[(1+\nu y)^{1-\beta/2}-(1+\mu y)^{1-\beta/2}\right]g^q(y)dy\right\}^{1/q},\end{aligned}$$

where the integrals on the right-hand side must converge and $B_{eta}(a,b)$ is the standard beta function at $a,b>0$ defined in Equation (1.2).

Proof. It follows from the item 3 of Theorem 2.1 that, for any $\beta\in(0,1)$,

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+\alpha xy)^{\beta}}f(x)g(y)dxdy\\&\leq B_{\mathrm{eta}}\left(\frac{\beta}{2},\frac{\beta}{2}\right)\left[\int_0^{+\infty}\sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^{\beta}}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^{\beta}}}g^q(y)dy\right]^{1/q},\end{aligned}$$

with $\alpha>0$. Treating α as a variable and integrating it over the interval (μ,ν), we have

$$N\leq B_{\mathrm{eta}}\left(\frac{\beta}{2},\frac{\beta}{2}\right)O, \tag{3.16}$$

where

$$N=\int_{\mu}^{\nu}\left\{\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{(x+y+\alpha xy)^{\beta}}f(x)g(y)dxdy\right\}d\alpha$$

and

$$O=\int_{\mu}^{\nu}\left\{\left[\int_{0}^{+\infty}\sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^{\beta}}}f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty}\sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^{\beta}}}g^{q}(y)dy\right]^{1/q}\right\}d\alpha.$$

Let us study N and O one by one.

Using the Fubini-Tonelli integral theorem and a standard power primitive, we get

$$\begin{aligned}N&=\int_{0}^{+\infty}\int_{0}^{+\infty}\left\{\int_{\mu}^{\nu}\frac{1}{(x+y+\alpha xy)^{\beta}}d\alpha\right\}f(x)g(y)dxdy\\&=\int_{0}^{+\infty}\int_{0}^{+\infty}\left[\frac{1}{1-\beta}\times\frac{1}{xy}\times(x+y+\alpha xy)^{1-\beta}\right]_{\alpha=\mu}^{\alpha=\nu}f(x)g(y)dxdy\\&=\frac{1}{1-\beta}\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{xy}\left[(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right]f(x)g(y)dxdy.\end{aligned}\tag{3.17}$$

Using the Hölder integral inequality at p and q, the Fubini-Tonelli integral theorem, standard power primitives and the equality $1/p+1/q=1$, we get

$$\begin{aligned}O&\le\left\{\int_{\mu}^{\nu}\left[\int_{0}^{+\infty}\sqrt{\frac{x^{(2-\beta)p-2}}{(1+\alpha x)^{\beta}}}f^{p}(x)dx\right]^{p/p}d\alpha\right\}^{1/p}\\&\quad\times\left\{\int_{\mu}^{\nu}\left[\int_{0}^{+\infty}\sqrt{\frac{y^{(2-\beta)q-2}}{(1+\alpha y)^{\beta}}}g^{q}(y)dy\right]^{q/q}d\alpha\right\}^{1/q}\\&=\left\{\int_{0}^{+\infty}x^{(1-\beta/2)p-1}\left[\int_{\mu}^{\nu}\frac{1}{(1+\alpha x)^{\beta/2}}d\alpha\right]f^{p}(x)dx\right\}^{1/p}\\&\quad\times\left\{\int_{0}^{+\infty}y^{(1-\beta/2)q-1}\left[\int_{\mu}^{\nu}\frac{1}{(1+\alpha y)^{\beta/2}}d\alpha\right]g^{q}(y)dy\right\}^{1/q}\\&=\left\{\int_{0}^{+\infty}x^{(1-\beta/2)p-1}\left[\frac{1}{x}\times\frac{2}{2-\beta}\times(1+\alpha x)^{1-\beta/2}\right]_{\alpha=\mu}^{\alpha=\nu}f^{p}(x)dx\right\}^{1/p}\\&\quad\times\left\{\int_{0}^{+\infty}y^{(1-\beta/2)q-1}\left[\frac{1}{y}\times\frac{2}{2-\beta}\times(1+\alpha y)^{1-\beta/2}\right]_{\alpha=\mu}^{\alpha=\nu}g^{q}(y)dy\right\}^{1/q}\\&=\frac{2}{2-\beta}\left\{\int_{0}^{+\infty}x^{(1-\beta/2)p-2}\left[(1+\nu x)^{1-\beta/2}-(1+\mu x)^{1-\beta/2}\right]f^{p}(x)dx\right\}^{1/p}\\&\quad\times\left\{\int_{0}^{+\infty}y^{(1-\beta/2)q-2}\left[(1+\nu y)^{1-\beta/2}-(1+\mu y)^{1-\beta/2}\right]g^{q}(y)dy\right\}^{1/q}.\end{aligned}\tag{3.18}$$

Putting Equations (3.16), (3.17) and (3.18) together, we get

$$\begin{aligned}
&\frac{1}{1-\beta}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right]f(x)g(y)dxdy\\
&\leq \frac{2}{2-\beta}B_{\text{eta}}\left(\frac{\beta}{2},\frac{\beta}{2}\right)\\
&\quad\times\left\{\int_0^{+\infty}x^{(1-\beta/2)p-2}\left[(1+\nu x)^{1-\beta/2}-(1+\mu x)^{1-\beta/2}\right]f^p(x)dx\right\}^{1/p}\\
&\quad\times\left\{\int_0^{+\infty}y^{(1-\beta/2)q-2}\left[(1+\nu y)^{1-\beta/2}-(1+\mu y)^{1-\beta/2}\right]g^q(y)dy\right\}^{1/q},
\end{aligned}$$

so that, since $\beta\in(0,1)$,

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right]f(x)g(y)dxdy\\
&\leq \frac{2(1-\beta)}{2-\beta}B_{\text{eta}}\left(\frac{\beta}{2},\frac{\beta}{2}\right)\\
&\quad\times\left\{\int_0^{+\infty}x^{(1-\beta/2)p-2}\left[(1+\nu x)^{1-\beta/2}-(1+\mu x)^{1-\beta/2}\right]f^p(x)dx\right\}^{1/p}\\
&\quad\times\left\{\int_0^{+\infty}y^{(1-\beta/2)q-2}\left[(1+\nu y)^{1-\beta/2}-(1+\mu y)^{1-\beta/2}\right]g^q(y)dy\right\}^{1/q}.
\end{aligned}$$

This concludes the proof. □

The kernel function associated with the main double integral is thus

$$k_{\text{ern}}(x,y)=\frac{1}{xy}\left[(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right].$$

It is obviously symmetric and inhomogeneous.

If we take $\mu=0$, then the new inequality simplifies to

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[(x+y+\nu xy)^{1-\beta}-(x+y)^{1-\beta}\right]f(x)g(y)dxdy\\
&\leq \frac{2(1-\beta)}{2-\beta}B_{\text{eta}}\left(\frac{\beta}{2},\frac{\beta}{2}\right)\\
&\quad\times\left\{\int_0^{+\infty}x^{(1-\beta/2)p-2}\left[(1+\nu x)^{1-\beta/2}-1\right]f^p(x)dx\right\}^{1/p}\\
&\quad\times\left\{\int_0^{+\infty}y^{(1-\beta/2)q-2}\left[(1+\nu y)^{1-\beta/2}-1\right]g^q(y)dy\right\}^{1/q}.
\end{aligned}$$

This Hardy-Hilbert integral type is interesting because of the two adjustable parameters and the original form of the kernel function.

3.3 Attempt of a unified framework

The theorem below unifies some cases of Theorem 2.1 and the result in [16], as recalled in Equation (1.1). The proof revisits that of Theorem 2.1, with the aim of obtaining "$B_{\text{eta}}(\beta/p,\beta/q)$" as the factor constant.

Theorem 3.6. *Let $p>1$, $q=p/(p-1)$, and $f,g:(0,+\infty)\mapsto(0,+\infty)$ be two functions. Then, for any $\alpha\geq 0$ and $\beta>0$, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+\alpha xy)^{\beta}}f(x)g(y)dxdy$$
$$\leq B_{\text{eta}}\left(\frac{\beta}{p},\frac{\beta}{q}\right)\left[\int_0^{+\infty}\frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}}g^q(y)dy\right]^{1/q},$$

where the integrals on the right-hand side must converge and $B_{\text{eta}}(a,b)$ is the standard beta function at $a,b>0$ defined in Equation (1.2).

Proof. A suitable product decomposition of the integrand considering the equality $1/p+1/q=1$, followed by the application of the Hölder integral inequality at p and q, gives

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+\alpha xy)^{\beta}}f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{-(\beta/p-1)/q}y^{(\beta/q-1)/p}}{(x+y+\alpha xy)^{\beta/p}}f(x)\times\frac{x^{(\beta/p-1)/q}y^{-(\beta/q-1)/p}}{(x+y+\alpha xy)^{\beta/q}}g(y)dxdy\\
&\leq P^{1/p}Q^{1/q},
\end{aligned}\tag{3.19}$$

where

$$P=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{-(\beta/p-1)p/q}y^{\beta/q-1}}{(x+y+\alpha xy)^{\beta}}f^p(x)dxdy$$

and

$$Q=\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\beta/p-1}y^{-(\beta/q-1)q/p}}{(x+y+\alpha xy)^{\beta}}g^q(y)dxdy.$$

Let us determine the expressions of P and Q one by one.

Since the integrand associated with P is non-negative, we can apply the Fubini-Tonelli integral theorem that allows the exchange of the order of integration. This, followed by the change of variables $z=y(1+\alpha x)/x$, the equality $1/p+1/q=1$ and the use of the beta function with the property $B_{\text{eta}}(a,b)=B_{\text{eta}}(b,a)$ for any $a,b>0$, gives

$$\begin{aligned}
P&=\int_0^{+\infty}x^{-(\beta/p-1)p/q}f^p(x)\left[\int_0^{+\infty}\frac{y^{\beta/q-1}}{(x+y+\alpha xy)^{\beta}}dy\right]dx\\
&=\int_0^{+\infty}x^{-(\beta/p-1)p/q-\beta}f^p(x)\left[\int_0^{+\infty}\frac{y^{\beta/q-1}}{[1+y(1+\alpha x)/x]^{\beta}}dy\right]dx\\
&=\int_0^{+\infty}x^{-(\beta/p-1)p/q-\beta}f^p(x)\left(\frac{x}{1+\alpha x}\right)^{\beta/q}\left[\int_0^{+\infty}\frac{z^{\beta/q-1}}{(1+z)^{\beta/q+\beta/p}}dz\right]dx\\
&=\int_0^{+\infty}\frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}}f^p(x)\times B_{\text{eta}}\left(\frac{\beta}{q},\frac{\beta}{p}\right)dx\\
&=B_{\text{eta}}\left(\frac{\beta}{p},\frac{\beta}{q}\right)\int_0^{+\infty}\frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}}f^p(x)dx.
\end{aligned}\tag{3.20}$$

Similarly, but with the changes of variables $w = x(1 + \alpha y)/y$, we have

$$\begin{aligned}
Q &= \int_0^{+\infty} y^{-(\beta/q-1)q/p} g^q(y) \left[\int_0^{+\infty} \frac{x^{\beta/p-1}}{(x+y+\alpha xy)^\beta} dx\right] dy \\
&= \int_0^{+\infty} y^{-(\beta/q-1)q/p-\beta} g^q(y) \left[\int_0^{+\infty} \frac{x^{\beta/p-1}}{[1+x(1+\alpha y)/y]^\beta} dx\right] dy \\
&= \int_0^{+\infty} y^{-(\beta/q-1)q/p-\beta} g^q(y) \left(\frac{y}{1+\alpha y}\right)^{\beta/p} \left[\int_0^{+\infty} \frac{w^{\beta/p-1}}{(1+w)^\beta} dw\right] dy \\
&= \int_0^{+\infty} \frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}} g^q(y) \times B_{\mathrm{eta}}\left(\frac{\beta}{p}, \frac{\beta}{q}\right) dy \\
&= B_{\mathrm{eta}}\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \int_0^{+\infty} \frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}} g^q(y) dy.
\end{aligned} \tag{3.21}$$

Putting Equations (3.19), (3.20) and (3.21) together and using the equality $1/p+1/q = 1$, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)g(y)dxdy \\
&\leq \left[B_{\mathrm{eta}}\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \int_0^{+\infty} \frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}} f^p(x)dx\right]^{1/p} \\
&\quad \times \left[B_{\mathrm{eta}}\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \int_0^{+\infty} \frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}} g^q(y)dy\right]^{1/q} \\
&= B_{\mathrm{eta}}\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \left[\int_0^{+\infty} \frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This concludes the proof. $\square$

In particular, if we take $\beta = 1$, using a well-known beta formula, we have $B_{\mathrm{eta}}(1/p, 1/q) = B_{\mathrm{eta}}(1/p, 1 - 1/p) = \pi/\sin(\pi/p)$, so that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y+\alpha xy} f(x)g(y)dxdy \\
&\leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} \frac{x^{p-2}}{(1+\alpha x)^{1/q}} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \frac{y^{q-2}}{(1+\alpha y)^{1/p}} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

To the best of our knowledge, this is a new Hardy-Hilbert integral inequality. If we take $\alpha = 0$, this conforms to the result in [16].

A limitation of this theorem lies in the complexity and generality of the framework, which is an obstacle to determining the optimality of the constant factor. This remains a challenging mathematical problem that we leave for future investigation. Nevertheless, two additional results derived from these theorems are presented below.

The theorem below provides a new two-parameter weighted integral inequality.

Theorem 3.7. *Let $p > 1$, $q = p/(p-1)$, and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function. Then, for any $\alpha \geq 0$ and $\beta > 0$, we have*

$$\int_0^{+\infty} \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}}\right]^{p-1} \left[\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x)dx\right]^p dy$$
$$\leq B_{eta}^p\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \int_0^{+\infty} \frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}} f^p(x)dx,$$

where the integral on the right-hand side must converge and $B_{eta}(a,b)$ is the standard beta function at $a, b > 0$ defined in Equation (1.2).

Proof. Let us set

$$R = \int_0^{+\infty} \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}}\right]^{p-1} \left[\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x)dx\right]^p dy.$$

A suitable product decomposition of the integrand and the Fubini-Tonelli integral theorem give

$$\begin{aligned} R &= \int_0^{+\infty} \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}}\right]^{p-1} \left[\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x)dx\right]^{p-1} \\ &\quad \times \left[\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x)dx\right] dy \\ &= \int_0^{+\infty} \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x)dx\right]^{p-1} \\ &\quad \times \left[\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x)dx\right] dy \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x) g_{\wedge}(y) dxdy, \end{aligned} \tag{3.22}$$

where

$$g_{\wedge}(y) = \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x)dx\right]^{p-1}.$$

It follows from Theorem 3.6 applied to the functions f and $g_{\wedge}$ that

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^{\beta}} f(x) g_{\wedge}(y) dxdy \\ &\leq B_{\mathrm{eta}}\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \left[\int_0^{+\infty} \frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} \frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}} g_{\wedge}^q(y)dy\right]^{1/q}. \end{aligned} \tag{3.23}$$

Since $q = p/(p-1)$, we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}} g_\wedge^q(y)dy \\
&= \int_0^{+\infty} \frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}} \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)dx\right]^{q(p-1)} dy \\
&= \int_0^{+\infty} \frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}} \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}} \int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)dx\right]^{p} dy \\
&= \int_0^{+\infty} \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}}\right]^{p-1} \left[\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)dx\right]^{p} dy = R.
\end{aligned} \tag{3.24}$$

Putting Equations (3.22), (3.23) and (3.24) together, we obtain

$$R \leq B_{\text{eta}}\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \left[\int_0^{+\infty} \frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}} f^p(x)dx\right]^{1/p} R^{1/q},$$

so that, using $1/p = 1 - 1/q$,

$$R^{1/p} = R^{1-1/q} \leq B_{\text{eta}}\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \left[\int_0^{+\infty} \frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}} f^p(x)dx\right]^{1/p}$$

and, raising both sides to the exponent p,

$$\begin{aligned}
&\int_0^{+\infty} \left[\frac{(1+\alpha y)^{\beta/p}}{y^{q-\beta-1}}\right]^{p-1} \left[\int_0^{+\infty} \frac{1}{(x+y+\alpha xy)^\beta} f(x)dx\right]^{p} dy = R \\
&\leq B_{\text{eta}}^p\left(\frac{\beta}{p}, \frac{\beta}{q}\right) \int_0^{+\infty} \frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}} f^p(x)dx.
\end{aligned}$$

This ends the proof. □

In particular, if we take $\beta = 1$, using $B_{\text{eta}}(1/p, 1/q) = \pi/\sin(\pi/p)$, we obtain

$$\begin{aligned}
&\int_0^{+\infty} \left[\frac{(1+\alpha y)^{1/p}}{y^{q-2}}\right]^{p-1} \left[\int_0^{+\infty} \frac{1}{x+y+\alpha xy} f(x)dx\right]^{p} dy \\
&\leq \frac{\pi^p}{\sin^p(\pi/p)} \int_0^{+\infty} \frac{x^{p-2}}{(1+\alpha x)^{1/q}} f^p(x)dx.
\end{aligned}$$

Another variant of the Hardy-Hilbert-type integral inequality is proposed in the theorem below, using an inhomogeneous two-parameter power kernel function.

Theorem 3.8. *Let $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions. Let $\beta > 0$ and $\mu \geq 0$ and $\nu > 0$ such that $\nu > \mu$. Then, depending on the values of β, the inequalities below hold.*

1. *For any* $\beta \in (0,+\infty)\backslash\{1,p,q\}$*, we have*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left|(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right| f(x)g(y)dxdy\\&\leq|1-\beta|\left(\frac{q}{|q-\beta|}\right)^{1/p}\left(\frac{p}{|p-\beta|}\right)^{1/q}B_{eta}\left(\frac{\beta}{p},\frac{\beta}{q}\right)\\&\quad\times\left[\int_0^{+\infty}x^{p-\beta-2}\left|(1+\nu x)^{1-\beta/q}-(1+\mu x)^{1-\beta/q}\right| f^p(x)dx\right]^{1/p}\\&\quad\times\left[\int_0^{+\infty}y^{q-\beta-2}\left|(1+\nu y)^{1-\beta/p}-(1+\mu y)^{1-\beta/p}\right| g^q(y)dy\right]^{1/q},\end{aligned}$$

where the integrals on the right-hand side must converge and $B_{eta}(a,b)$ *is the standard beta function at* $a,b>0$ *defined in Equation* (1.2).

2. *For* $\beta=1$*, we have*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\log\left(\frac{x+y+\nu xy}{x+y+\mu xy}\right)f(x)g(y)dxdy\\&\leq\left(\frac{q}{q-1}\right)^{1/p}\left(\frac{p}{p-1}\right)^{1/q}\frac{\pi}{\sin(\pi/p)}\\&\quad\times\left[\int_0^{+\infty}x^{p-3}\left[(1+\nu x)^{1/p}-(1+\mu x)^{1/p}\right]f^p(x)dx\right]^{1/p}\\&\quad\times\left[\int_0^{+\infty}y^{q-3}\left[(1+\nu y)^{1/q}-(1+\mu y)^{1/q}\right]g^q(y)dy\right]^{1/q}.\end{aligned}$$

3. *For any* $\beta=p$ *with* $p\neq 2$*, we have*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[(x+y+\mu xy)^{1-p}-(x+y+\nu xy)^{1-p}\right]f(x)g(y)dxdy\\&\leq\frac{1}{|2-p|^{1/p}}\left[\int_0^{+\infty}x^{-2}\left|(1+\nu x)^{2-p}-(1+\mu x)^{2-p}\right| f^p(x)dx\right]^{1/p}\\&\quad\times\left[\int_0^{+\infty}y^{q-p-2}\log\left(\frac{1+\nu y}{1+\mu y}\right)g^q(y)dy\right]^{1/q}.\end{aligned}$$

4. *For any* $\beta=q$ *with* $q\neq 2$*, we have*

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[(x+y+\mu xy)^{1-q}-(x+y+\nu xy)^{1-q}\right]f(x)g(y)dxdy\\&\leq\frac{1}{|2-q|^{1/q}}\left[\int_0^{+\infty}x^{p-q-2}\log\left(\frac{1+\nu x}{1+\mu x}\right)f^p(x)dx\right]^{1/p}\\&\quad\times\left[\int_0^{+\infty}y^{-2}\left|(1+\nu y)^{2-q}-(1+\mu y)^{2-q}\right| g^q(y)dy\right]^{1/q}.\end{aligned}$$

5. *For any $\beta = p$ with $p = 2$, so $q = 2$, we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+\mu xy)(x+y+\nu xy)}f(x)g(y)dxdy$$
$$\leq \frac{1}{\nu-\mu}\sqrt{\int_0^{+\infty}x^{-2}\log\left(\frac{1+\nu x}{1+\mu x}\right)f^2(x)dx}\sqrt{\int_0^{+\infty}y^{-2}\log\left(\frac{1+\nu y}{1+\mu y}\right)g^2(y)dy.}$$

Proof.

1. For any $\beta \in (0,+\infty)\backslash\{1,p,q\}$, it follows from Theorem 3.6 that, for any $\beta > 0$,

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+\alpha xy)^{\beta}}f(x)g(y)dxdy$$
$$\leq B_{\text{eta}}\left(\frac{\beta}{p},\frac{\beta}{q}\right)\left[\int_0^{+\infty}\frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}}g^q(y)dy\right]^{1/q},$$

with $\alpha > 0$. Treating α as a variable and integrating α over the interval (μ,ν), we have

$$S \leq B_{\text{eta}}\left(\frac{\beta}{p},\frac{\beta}{q}\right)T, \tag{3.25}$$

where

$$S = \int_\mu^\nu\left\{\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y+\alpha xy)^{\beta}}f(x)g(y)dxdy\right\}d\alpha$$

and

$$T = \int_\mu^\nu\left\{\left[\int_0^{+\infty}\frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}\frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}}g^q(y)dy\right]^{1/q}\right\}d\alpha.$$

Let us determine the expressions of S and T one by one.

Using the Fubini-Tonelli integral theorem, a standard power primitive and the appropriate use of the absolute values, we have

$$\begin{aligned}
S &= \int_0^{+\infty}\int_0^{+\infty}\left\{\int_\mu^\nu\frac{1}{(x+y+\alpha xy)^{\beta}}d\alpha\right\}f(x)g(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty}\left[\frac{1}{1-\beta}\times\frac{1}{xy}\times(x+y+\alpha xy)^{1-\beta}\right]_{\alpha=\mu}^{\alpha=\nu}f(x)g(y)dxdy\\
&= \frac{1}{1-\beta}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left[(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right]f(x)g(y)dxdy\\
&= \frac{1}{|1-\beta|}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left|(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right|f(x)g(y)dxdy.
\end{aligned} \tag{3.26}$$

Using the Hölder integral inequality at p and q, the Fubini-Tonelli integral theorem, standard power primitives, the use of absolute values and the equality $1/p+1/q=1$, we get

$$\begin{aligned}
T &\leq \left\{\int_{\mu}^{\nu}\left[\int_{0}^{+\infty}\frac{x^{p-\beta-1}}{(1+\alpha x)^{\beta/q}}f^{p}(x)dx\right]^{p/p}d\alpha\right\}^{1/p}\\
&\quad\times\left\{\int_{\mu}^{\nu}\left[\int_{0}^{+\infty}\frac{y^{q-\beta-1}}{(1+\alpha y)^{\beta/p}}g^{q}(y)dy\right]^{q/q}d\alpha\right\}^{1/q}\\
&=\left\{\int_{0}^{+\infty}x^{p-\beta-1}\left[\int_{\mu}^{\nu}\frac{1}{(1+\alpha x)^{\beta/q}}d\alpha\right]f^{p}(x)dx\right\}^{1/p}\\
&\quad\times\left\{\int_{0}^{+\infty}y^{q-\beta-1}\left[\int_{\mu}^{\nu}\frac{1}{(1+\alpha y)^{\beta/p}}d\alpha\right]g^{q}(y)dy\right\}^{1/q}\\
&=\left\{\int_{0}^{+\infty}x^{p-\beta-1}\left[\frac{1}{x}\times\frac{q}{q-\beta}\times(1+\alpha x)^{1-\beta/q}\right]_{\alpha=\mu}^{\alpha=\nu}f^{p}(x)dx\right\}^{1/p}\\
&\quad\times\left\{\int_{0}^{+\infty}y^{q-\beta-1}\left[\frac{1}{y}\times\frac{p}{p-\beta}\times(1+\alpha y)^{1-\beta/p}\right]_{\alpha=\mu}^{\alpha=\nu}g^{q}(y)dy\right\}^{1/q} \qquad (3.27)\\
&=\left[\frac{q}{|q-\beta|}\int_{0}^{+\infty}x^{p-\beta-2}\left|(1+\nu x)^{1-\beta/q}-(1+\mu x)^{1-\beta/q}\right|f^{p}(x)dx\right]^{1/p}\\
&\quad\times\left[\frac{p}{|p-\beta|}\int_{0}^{+\infty}y^{q-\beta-2}\left|(1+\nu y)^{1-\beta/p}-(1+\mu y)^{1-\beta/p}\right|g^{q}(y)dy\right]^{1/q}\\
&=\left(\frac{q}{|q-\beta|}\right)^{1/p}\left(\frac{p}{|p-\beta|}\right)^{1/q}\\
&\quad\times\left[\int_{0}^{+\infty}x^{p-\beta-2}\left|(1+\nu x)^{1-\beta/q}-(1+\mu x)^{1-\beta/q}\right|f^{p}(x)dx\right]^{1/p}\\
&\quad\times\left[\int_{0}^{+\infty}y^{q-\beta-2}\left|(1+\nu y)^{1-\beta/p}-(1+\mu y)^{1-\beta/p}\right|g^{q}(y)dy\right]^{1/q}.
\end{aligned}$$

Putting Equations (3.25), (3.26) and (3.27) together, we obtain

$$\begin{aligned}
&\frac{1}{|1-\beta|}\int_{0}^{+\infty}\int_{0}^{+\infty}\frac{1}{xy}\left|(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right|f(x)g(y)dxdy\\
&\leq\left(\frac{q}{|q-\beta|}\right)^{1/p}\left(\frac{p}{|p-\beta|}\right)^{1/q}B_{\text{eta}}\left(\frac{\beta}{p},\frac{\beta}{q}\right)\\
&\quad\times\left[\int_{0}^{+\infty}x^{p-\beta-2}\left|(1+\nu x)^{1-\beta/q}-(1+\mu x)^{1-\beta/q}\right|f^{p}(x)dx\right]^{1/p}\\
&\quad\times\left[\int_{0}^{+\infty}y^{q-\beta-2}\left|(1+\nu y)^{1-\beta/p}-(1+\mu y)^{1-\beta/p}\right|g^{q}(y)dy\right]^{1/q},
\end{aligned}$$

so that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\left|(x+y+\nu xy)^{1-\beta}-(x+y+\mu xy)^{1-\beta}\right|f(x)g(y)dxdy$$
$$\leq |1-\beta|\left(\frac{q}{|q-\beta|}\right)^{1/p}\left(\frac{p}{|p-\beta|}\right)^{1/q}B_{\text{eta}}\left(\frac{\beta}{p},\frac{\beta}{q}\right)$$
$$\times\left[\int_0^{+\infty}x^{p-\beta-2}\left|(1+\nu x)^{1-\beta/q}-(1+\mu x)^{1-\beta/q}\right|f^p(x)dx\right]^{1/p}$$
$$\times\left[\int_0^{+\infty}y^{q-\beta-2}\left|(1+\nu y)^{1-\beta/p}-(1+\mu y)^{1-\beta/p}\right|g^q(y)dy\right]^{1/q}.$$

This is the desired result.

For the other three items, for the sake of redundancy, we will only comment on the points that explain the expressions of the given inequalities.

2. For $\beta = 1$, adopting the notation of the previous item and using a standard logarithmic primitive, we get

$$S=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{xy}\log\left(\frac{x+y+\nu xy}{x+y+\mu xy}\right)f(x)g(y)dxdy.$$

The term T remains unchanged; it is sufficient to replace $\beta = 1$ in its expression. The same is true for the remaining constant.

3. For $\beta = p$ with $p \neq 2$, adopting the notation of the item 1, the term S remains unchanged. In the term T, only the integral with respect to y has to be recalculated with a standard logarithmic primitive. The remaining constant can be expressed as indicated, using $B_{\text{eta}}(1, p-1) = 1/(p-1)$, among other classical developments.

4. For $\beta = q$ with $q \neq 2$, we proceed as in the previous item, with the appropriate logarithmic change for the calculus on the variable x.

5. For any $\beta = p$ with $p = 2$, so $q = 2$, for the term S, we use the decomposition

$$(x+y+\mu xy)^{-1}-(x+y+\nu xy)^{-1}=(\nu-\mu)\frac{xy}{(x+y+\mu xy)(x+y+\nu xy)}.$$

For the term T, we proceed as in the items 3 and 4 for the upper bounds.

This ends the proof. □

This is a new collection of Hardy-Hilbert-type integral inequalities, thus extending the existing theory by the introduction of inhomogeneous kernel functions.

4 Conclusion

In this article, we have established Hardy-Hilbert-type integral inequalities that are innovative in form and generality. In particular, we have obtained relevant upper bounds for a class of double integrals involving a special inhomogeneous kernel function governed by two parameters, α and β. The introduction of α is the key point: it increases the flexibility of our results while changing the homogeneity of a classical kernel function. Through a detailed case-by-case analysis of β, we derived several weighted integral norm inequalities and Hardy-Hilbert-type integral inequalities. Finally, we propose a framework that unifies new and established results. We hope that this work will serve as a foundation for further research, inspiring deeper investigations of Hardy-Hilbert-type integral inequalities with inhomogeneous integral kernels and their applications in mathematical analysis. The discrete analogues of our results are also an interesting perspective that we leave for a future article.

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