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DOI: 10.2478/awutm-2025-0009

Annals of West University of Timisoara
Mathematics and Computer Science
61 (2025), 120–130

Solution to an open problem on a logarithmic integral and derived results

Clément Coine and Christophe Chesneau

Abstract This article solves an open problem that was previously stated by providing an exact evaluation of a logarithmic integral. Furthermore, the result is generalized by introducing a new adjustable parameter. Based on this extension, other integral formulas are derived. Applications are presented for integral inequalities of Hölder and Hardy-Hilbert types, in which the introduced parameter plays a pivotal role in establishing the bounds under suitable integrability conditions.

AMS Subject Classification (2020) 33B15; 33B20
Keywords integrals; logarithm; integration by parts; change of variables; Hölder-type integral inequalities; Hardy-Hilbert-type integral inequalities

1 Introduction

Many integral formulas are essential tools in mathematics, physics, and engineering. They are classically used to solve problems involving areas and volumes, as well as more complex physical phenomena. A key reference on the topic is the book [4]. Despite the vast existing body of work on this subject, the search for new integral formulas remains an active area of research. Such formulas can lead to more elegant solutions, simplify intricate calculations, and reveal new mathematical relationships. Recent developments in this field are documented in [1–3, 6–9].

In particular, an open problem is posed in [3], specifically stated as [3, Open Problem 3]. We reproduce it below, word for word. *From a convergence point of view, the following integral is well defined:*

$$\mathfrak{D}(\alpha) = \int_0^{+\infty} \frac{1}{\sqrt{x}} \log\left(1 + \frac{1}{x^\alpha}\right) dx,$$

with $\alpha > 1/2$. *We have found in Proposition 2.8 that* $\mathfrak{D}(2) = 2\pi\sqrt{2} \approx 8.8857$. *With a numerical software, we can calculate approximate values, such as* $\mathfrak{D}(1) \approx 6.28319$, $\mathfrak{D}(1.5) \approx 7.2552$, *or* $\mathfrak{D}(2.5) \approx 10.6896$, *among others. However, if it exists, the exact formula behind* $\mathfrak{D}(\alpha)$ *is, to our knowledge, an open problem.*

On this basis, this article has three aims: first, we solve and extend the open problem by introducing a new adjustable parameter. More precisely, we establish that

$$\mathfrak{D}(\alpha, \beta) = \int_0^{+\infty} \frac{1}{x^\alpha} \log\left(1 + \frac{1}{x^\beta}\right) dx = \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]},$$

under some conditions on α and β. Next, we use this generalization to derive additional new integral formulas. Finally, we establish innovative integral inequalities of the Hölder and Hardy-Hilbert types. In particular, we investigate an upper bound for the following double integral:

$$\int_0^{+\infty} \int_0^{+\infty} \log\left(1 + \frac{1}{x^\beta y^\beta}\right) f(x)g(y)dxdy,$$

where f and g are functions that satisfy certain integrability conditions. The newly introduced parameter α will play a role in these conditions and the obtained upper bound. This article is self-contained, providing all necessary details or referencing precise sources.

The rest of the article is composed of three sections. The main integral formulas are presented in Section 2. Applications to integral inequalities are given in Section 3. Section 4 formulates a conclusion.

2 Main integral formulas

The main results, including the solution to the open problem, and related proofs are contained in this section.

2.1 Solution to an open problem

The proposition below presents a logarithmic integral formula depending on two parameters.

Proposition 2.1. *Let* $\alpha \in (0, 1)$ *and* $\beta > 1 - \alpha$. *Then we have*

$$\int_0^{+\infty} \frac{1}{x^\alpha} \log\left(1 + \frac{1}{x^\beta}\right) dx = \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}.$$

Proof. Taking into account that $\alpha \in (0, 1)$ and $\beta > 1 - \alpha > 0$, an integration by parts gives

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{x^\alpha} \log\left(1+\frac{1}{x^\beta}\right) dx \\
&= \left[\frac{1}{1-\alpha} x^{1-\alpha} \log\left(1+\frac{1}{x^\beta}\right)\right]_{x\to 0}^{x\to+\infty} - \int_0^{+\infty} \frac{1}{1-\alpha} x^{1-\alpha} \frac{(-\beta/x^{\beta+1})}{1+1/x^\beta} dx \\
&= 0 + \frac{\beta}{1-\alpha} \int_0^{+\infty} \frac{x^{-\alpha}}{1+x^\beta} dx.
\end{aligned}$$

Applying the formula in [4, Entry 3.2412] with $\mu = 1-\alpha$ and $\nu = \beta$, taking into account that $\alpha \in (0,1)$ and $\beta > 1-\alpha$, we obtain

$$\int_0^{+\infty} \frac{x^{-\alpha}}{1+x^\beta} dx = \int_0^{+\infty} \frac{x^{(1-\alpha)-1}}{1+x^\beta} dx = \frac{\pi}{\beta \sin[(1-\alpha)\pi/\beta]}.$$

As a result, we have

$$\int_0^{+\infty} \frac{1}{x^\alpha} \log\left(1+\frac{1}{x^\beta}\right) dx = \frac{\beta}{1-\alpha} \times \frac{\pi}{\beta \sin[(1-\alpha)\pi/\beta]} = \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}.$$

The desired expression is obtained, ending the proof. □

As outlined in the Introduction, this proposition provides an answer to the open problem described in [3, Open Problem 3], which corresponds to the case $\alpha = 1/2$. We thus generalize it thanks to the activation of α. This offers some perspectives that we explore in the rest of the article.

2.2 Integral formulas

The result below concerns an integral formula that depends on two logarithmic terms and two parameters. The proof is based on Proposition 2.1.

Proposition 2.2. *Let $\alpha \in (0,1)$ and $\beta > 1-\alpha$. Then we have*

$$\begin{aligned}
&\int_0^{+\infty} \frac{\log(x)}{x^\alpha} \log\left(1+\frac{1}{x^\beta}\right) dx \\
&= \frac{\pi}{(1-\alpha)^2 \beta \sin[(1-\alpha)\pi/\beta]} \left[\pi(\alpha-1)\frac{\cos[(1-\alpha)\pi/\beta]}{\sin[(1-\alpha)\pi/\beta]} - \beta\right].
\end{aligned}$$

Proof. It follows from Proposition 2.1 that

$$\int_0^{+\infty} \frac{1}{x^\alpha} \log\left(1+\frac{1}{x^\beta}\right) dx = \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]},$$

with $\alpha \in (0,1)$ and $\beta > 1-\alpha > 0$. Differentiating both sides with respect to α, we get

$$\frac{\partial}{\partial \alpha}\left[\int_0^{+\infty} \frac{1}{x^\alpha} \log\left(1+\frac{1}{x^\beta}\right) dx\right] = \frac{\partial}{\partial \alpha}\left[\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\right].$$

For the left-hand side term, applying the Leibniz integral rule, we get

$$\frac{\partial}{\partial\alpha}\left[\int_0^{+\infty}\frac{1}{x^\alpha}\log\left(1+\frac{1}{x^\beta}\right)dx\right]=\int_0^{+\infty}\frac{\partial}{\partial\alpha}\left[\frac{1}{x^\alpha}\log\left(1+\frac{1}{x^\beta}\right)\right]dx$$
$$=-\int_0^{+\infty}\frac{\log(x)}{x^\alpha}\log\left(1+\frac{1}{x^\beta}\right)dx.$$

For the right-hand side term, a direct calculation gives

$$\frac{\partial}{\partial\alpha}\left[\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\right]$$
$$=-\frac{\pi}{(1-\alpha)^2\beta\sin[(1-\alpha)\pi/\beta]}\left[\pi(\alpha-1)\frac{\cos[(1-\alpha)\pi/\beta]}{\sin[(1-\alpha)\pi/\beta]}-\beta\right].$$

We thus deduce that

$$\int_0^{+\infty}\frac{\log(x)}{x^\alpha}\log\left(1+\frac{1}{x^\beta}\right)dx$$
$$=-\left\{-\frac{\pi}{(1-\alpha)^2\beta\sin[(1-\alpha)\pi/\beta]}\left[\pi(\alpha-1)\frac{\cos[(1-\alpha)\pi/\beta]}{\sin[(1-\alpha)\pi/\beta]}-\beta\right]\right\}$$
$$=\frac{\pi}{(1-\alpha)^2\beta\sin[(1-\alpha)\pi/\beta]}\left[\pi(\alpha-1)\frac{\cos[(1-\alpha)\pi/\beta]}{\sin[(1-\alpha)\pi/\beta]}-\beta\right].$$

This concludes the proof. □

The proposition below can be seen as a variation of Proposition 2.2 that does not include ratio terms.

Proposition 2.3. *Let $\alpha\in(0,1)$ and $\beta>1-\alpha$. Then we have*

$$\int_0^{+\infty}x^{\alpha-2}\log(x)\log\left(1+x^\beta\right)dx$$
$$=\frac{\pi}{(1-\alpha)^2\beta\sin[(1-\alpha)\pi/\beta]}\left[\pi(1-\alpha)\frac{\cos[(1-\alpha)\pi/\beta]}{\sin[(1-\alpha)\pi/\beta]}+\beta\right].$$

Proof. It follows from Proposition 2.2 that

$$\int_0^{+\infty}\frac{\log(x)}{x^\alpha}\log\left(1+\frac{1}{x^\beta}\right)dx$$
$$=\frac{\pi}{(1-\alpha)^2\beta\sin[(1-\alpha)\pi/\beta]}\left[\pi(\alpha-1)\frac{\cos[(1-\alpha)\pi/\beta]}{\sin[(1-\alpha)\pi/\beta]}-\beta\right],$$

with $\alpha\in(0,1)$ and $\beta>1-\alpha>0$. Making the change of variables $x=1/y$, the left-hand side term can be expressed as

$$\int_0^{+\infty}\frac{\log(x)}{x^\alpha}\log\left(1+\frac{1}{x^\beta}\right)dx$$
$$=\int_{+\infty}^0\frac{\log(1/y)}{(1/y)^\alpha}\log\left[1+\frac{1}{(1/y)^\beta}\right]\left(-\frac{1}{y^2}dy\right)$$
$$=-\int_0^{+\infty}y^{\alpha-2}\log(y)\log\left(1+y^\beta\right)dy.$$

We thus have

$$\begin{aligned}
&\int_0^{+\infty} x^{\alpha-2}\log(x)\log\left(1+x^{\beta}\right)dx \\
&= -\left\{\frac{\pi}{(1-\alpha)^2\beta\sin[(1-\alpha)\pi/\beta]}\left[\pi(\alpha-1)\frac{\cos[(1-\alpha)\pi/\beta]}{\sin[(1-\alpha)\pi/\beta]}-\beta\right]\right\} \\
&= \frac{\pi}{(1-\alpha)^2\beta\sin[(1-\alpha)\pi/\beta]}\left[\pi(1-\alpha)\frac{\cos[(1-\alpha)\pi/\beta]}{\sin[(1-\alpha)\pi/\beta]}+\beta\right].
\end{aligned}$$

The desired expression is obtained, ending the proof. □

The result below details a known formula that can be derived from our findings.

Proposition 2.4. *Proposition 2.1 can be used to obtain a variation of the formula in [4, 4.2545], as described below. Let $\alpha \in (0,1)$ and $\beta > 1-\alpha$. Then we have*

$$\int_0^{+\infty} \frac{\log(x)}{x^{\alpha}(1+x^{\beta})}dx = -\frac{\pi^2\cos[(1-\alpha)\pi/\beta]}{\beta^2\sin^2[(1-\alpha)\pi/\beta]}.$$

Proof. It follows from Proposition 2.1 that

$$\int_0^{+\infty} \frac{1}{x^{\alpha}}\log\left(1+\frac{1}{x^{\beta}}\right)dx = \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]},$$

with $\alpha \in (0,1)$ and $\beta > 1-\alpha > 0$. Differentiating both sides with respect to β, we get

$$\frac{\partial}{\partial\beta}\left[\int_0^{+\infty} \frac{1}{x^{\alpha}}\log\left(1+\frac{1}{x^{\beta}}\right)dx\right] = \frac{\partial}{\partial\beta}\left[\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\right].$$

For the left-hand side term, applying the Leibniz integral rule, we get

$$\begin{aligned}
&\frac{\partial}{\partial\beta}\left[\int_0^{+\infty} \frac{1}{x^{\alpha}}\log\left(1+\frac{1}{x^{\beta}}\right)dx\right] = \int_0^{+\infty} \frac{\partial}{\partial\beta}\left[\frac{1}{x^{\alpha}}\log\left(1+\frac{1}{x^{\beta}}\right)\right]dx \\
&= -\int_0^{+\infty} \frac{\log(x)}{x^{\alpha}(1+x^{\beta})}dx.
\end{aligned}$$

For the right-hand side term, a direct calculation gives

$$\frac{\partial}{\partial\beta}\left[\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\right] = \frac{\pi^2\cos[(1-\alpha)\pi/\beta]}{\beta^2\sin^2[(1-\alpha)\pi/\beta]}.$$

We thus deduce that

$$\int_0^{+\infty} \frac{\log(x)}{x^{\alpha}(1+x^{\beta})}dx = -\frac{\pi^2\cos[(1-\alpha)\pi/\beta]}{\beta^2\sin^2[(1-\alpha)\pi/\beta]}.$$

The desired expression is obtained. This concludes the proof. □

The derived formulas can be applied in a wide variety of mathematical scenarios. In the rest of the article, we emphasize integral inequalities of different types.

3 Integral inequalities

3.1 A Hölder-type integral inequality

A comprehensive Hölder-type integral inequality based on Proposition 2.1 is described in the result below.

Proposition 3.1. *Let* $\alpha \in (0,1)$, $\beta > 1-\alpha$, $p > 1$, $q = p/(p-1)$ *and* $f : [0,+\infty) \mapsto [0,+\infty)$ *such that*

$$\int_0^{+\infty} x^{\alpha(q-1)} f^q(x)dx < +\infty.$$

Then we have

$$\int_0^{+\infty} \log^{1/p}\left(1+\frac{1}{x^\beta}\right) f(x)dx \leq \left[\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\right]^{1/p} \left[\int_0^{+\infty} x^{\alpha(q-1)} f^q(x)dx\right]^{1/q}.$$

Proof. A suitable decomposition of the integrand, the Hölder integral inequality, Proposition 2.1 and the definition of q give

$$\begin{aligned}
&\int_0^{+\infty} \log^{1/p}\left(1+\frac{1}{x^\beta}\right) f(x)dx = \int_0^{+\infty} x^{-\alpha/p} \log^{1/p}\left(1+\frac{1}{x^\beta}\right) \times x^{\alpha/p} f(x)dx \\
&\leq \left[\int_0^{+\infty} \frac{1}{x^\alpha} \log\left(1+\frac{1}{x^\beta}\right) dx\right]^{1/p} \left[\int_0^{+\infty} x^{\alpha q/p} f^q(x)dx\right]^{1/q} \\
&= \left[\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\right]^{1/p} \left[\int_0^{+\infty} x^{\alpha(q-1)} f^q(x)dx\right]^{1/q}.
\end{aligned}$$

This ends the proof. □

Note that, using the well-known logarithmic inequality $\log(1+z) \leq z$ for $z > -1$, we obviously have

$$\int_0^{+\infty} \log^{1/p}\left(1+\frac{1}{x^\beta}\right) f(x)dx \leq \int_0^{+\infty} \frac{1}{x^{\beta/p}} f(x)dx,$$

provided that the integral of the upper bound converges. Proposition 3.1 is more technical, but offers greater flexibility in the integrability condition, mainly thanks to α.

3.2 A Hardy-Hilbert-type integral inequality

Hardy-Hilbert-type integral inequalities involving logarithmic transformations are a classical topic in analysis. Foundational treatments can be found in the key books [5,10,11]. The result below presents a new approach based on the formula in Proposition 2.1.

Proposition 3.2. *Let* $\alpha \in (0,1)$, $\beta > 1-\alpha$, $p > 1$, $q = p/(p-1)$ *and* $f, g : [0,+\infty) \mapsto [0,+\infty)$ *such that*

$$\int_0^{+\infty} x^{\alpha p-1} f^p(x)dx < +\infty, \quad \int_0^{+\infty} y^{\alpha q-1} g^q(y)dy < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)g(y)dxdy$$
$$\leq \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\left[\int_0^{+\infty}x^{\alpha p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{\alpha q-1}g^q(y)dy\right]^{1/q}.$$

Proof. A suitable decomposition of the integrand and the Hölder integral inequality give

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(xy)^\alpha}\log\left(1+\frac{1}{x^\beta y^\beta}\right)x^\alpha f(x)y^\alpha g(y)dx\\
&=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(xy)^{\alpha/p}}\log^{1/p}\left(1+\frac{1}{x^\beta y^\beta}\right)x^\alpha f(x)\frac{1}{(xy)^{\alpha/q}}\log^{1/q}\left(1+\frac{1}{x^\beta y^\beta}\right)y^\alpha g(y)dxdy\\
&\leq \mathcal{A}^{1/p}\mathcal{B}^{1/q},
\end{aligned} \tag{3.1}$$

where

$$\mathcal{A}=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(xy)^\alpha}\log\left(1+\frac{1}{x^\beta y^\beta}\right)x^{\alpha p}f^p(x)dxdy$$

and

$$\mathcal{B}=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(xy)^\alpha}\log\left(1+\frac{1}{x^\beta y^\beta}\right)y^{\alpha q}g^q(y)dxdy.$$

It follows from the Fubini-Tonelli integral theorem, the change of variables $w=xy$ with respect to y and Proposition 2.1 that

$$\begin{aligned}
\mathcal{A}&=\int_0^{+\infty}x^{\alpha p-1}f^p(x)\left[\int_0^{+\infty}\frac{1}{(xy)^\alpha}\log\left(1+\frac{1}{x^\beta y^\beta}\right)xdy\right]dx\\
&=\int_0^{+\infty}x^{\alpha p-1}f^p(x)\left[\int_0^{+\infty}\frac{1}{w^\alpha}\log\left(1+\frac{1}{w^\beta}\right)dw\right]dx\\
&=\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\int_0^{+\infty}x^{\alpha p-1}f^p(x)dx.
\end{aligned} \tag{3.2}$$

Similarly, but with the change of variables $z=xy$ with respect to x, we get

$$\begin{aligned}
\mathcal{B}&=\int_0^{+\infty}y^{\alpha q-1}g^q(y)\left[\int_0^{+\infty}\frac{1}{(xy)^\alpha}\log\left(1+\frac{1}{x^\beta y^\beta}\right)ydx\right]dx\\
&=\int_0^{+\infty}y^{\alpha q-1}g^q(y)\left[\int_0^{+\infty}\frac{1}{z^\alpha}\log\left(1+\frac{1}{z^\beta}\right)dz\right]dy\\
&=\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\int_0^{+\infty}y^{\alpha q-1}g^q(y)dy.
\end{aligned} \tag{3.3}$$

Joining Equations (3.1), (3.2) and (3.3), we find that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)g(y)dxdy\\
&\leq\left[\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\int_0^{+\infty}x^{\alpha p-1}f^p(x)dx\right]^{1/p}\\
&\quad\times\left[\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\int_0^{+\infty}y^{\alpha q-1}g^q(y)dy\right]^{1/q}\\
&=\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\left[\int_0^{+\infty}x^{\alpha p-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}y^{\alpha q-1}g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof. □

To the best of our knowledge, this result is new in the literature.

Note that, using the well-known logarithmic inequality $\log(1+z)\leq z$ for $z>-1$, we get

$$\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)g(y)dxdy\leq\left[\int_0^{+\infty}\frac{1}{x^\beta}f(x)dx\right]\left[\int_0^{+\infty}\frac{1}{y^\beta}g(y)dy\right],$$

provided that the integrals of the upper bound converge. Proposition 3.2 offers a different perspective, with greater flexibility in the integrability condition, mainly thanks to α.

3.3 A variation with a single function

A consequence of Proposition 3.2 is the result below, which has the characteristic of depending on only one function.

Proposition 3.3. *Let $\alpha\in(0,1)$, $\beta>1-\alpha$, $p>1$, $q=p/(p-1)$ and $f:[0,+\infty)\mapsto[0,+\infty)$ such that*

$$\int_0^{+\infty}x^{\alpha p-1}f^p(x)dx<+\infty.$$

Then we have

$$\begin{aligned}
&\int_0^{+\infty}y^{-(\alpha q-1)(p-1)}\left[\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)dx\right]^p dy\\
&\leq\left\{\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\right\}^p\int_0^{+\infty}x^{\alpha p-1}f^p(x)dx.
\end{aligned}$$

Proof. We can write

$$\begin{aligned}
\mathcal{C}&=\int_0^{+\infty}y^{-(\alpha q-1)(p-1)}\left[\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)dx\right]^p dy\\
&=\int_0^{+\infty}y^{-(\alpha q-1)(p-1)}\left[\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)dx\right]\\
&\quad\times\left[\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)dx\right]^{p-1}dy\\
&=\int_0^{+\infty}\int_0^{+\infty}\log\left(1+\frac{1}{x^\beta y^\beta}\right)f(x)g_\diamond(y)dxdy,
\end{aligned}\tag{3.4}$$

where

$$g_\diamond(y) = \left[y^{-(\alpha q-1)} \int_0^{+\infty} \log\left(1+\frac{1}{x^\beta y^\beta}\right) f(x)dx\right]^{p-1}.$$

Applying Proposition 3.2 to the functions f and $g_\diamond$, we get

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \log\left(1+\frac{1}{x^\beta y^\beta}\right) f(x)g_\diamond(y)dxdy \\ &\le \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]} \left[\int_0^{+\infty} x^{\alpha p-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{\alpha q-1} g_\diamond^q(y)dy\right]^{1/q}. \end{aligned} \tag{3.5}$$

The rest of the proof consists in expressing the last integral, and to simplify $\mathcal{C}$ raised at a certain power. Using the definition of $g_\diamond$ and $q = p/(p-1)$, we have

$$\begin{aligned} &\int_0^{+\infty} y^{\alpha q-1} g_\diamond^q(y)dy \\ &= \int_0^{+\infty} y^{\alpha q-1} \left[y^{-(\alpha q-1)} \int_0^{+\infty} \log\left(1+\frac{1}{x^\beta y^\beta}\right) f(x)dx\right]^{q(p-1)} dy \\ &= \int_0^{+\infty} y^{\alpha q-1} \left[y^{-(\alpha q-1)} \int_0^{+\infty} \log\left(1+\frac{1}{x^\beta y^\beta}\right) f(x)dx\right]^{p} dy \\ &= \int_0^{+\infty} y^{-(\alpha q-1)(p-1)} \left[\int_0^{+\infty} \log\left(1+\frac{1}{x^\beta y^\beta}\right) f(x)dx\right]^{p} dy \\ &= \mathcal{C}. \end{aligned} \tag{3.6}$$

Joining Equations (3.4), (3.5) and (3.6), we obtain

$$\mathcal{C} \le \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]} \left[\int_0^{+\infty} x^{\alpha p-1} f^p(x)dx\right]^{1/p} \mathcal{C}^{1/q}.$$

Using $q = p/(p-1)$, this is equivalent to

$$\mathcal{C}^{1/p} \le \frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]} \left[\int_0^{+\infty} x^{\alpha p-1} f^p(x)dx\right]^{1/p},$$

which implies that

$$\begin{aligned} &\int_0^{+\infty} y^{-(\alpha q-1)(p-1)} \left[\int_0^{+\infty} \log\left(1+\frac{1}{x^\beta y^\beta}\right) f(x)dx\right]^{p} dy \\ &\le \left\{\frac{\pi}{(1-\alpha)\sin[(1-\alpha)\pi/\beta]}\right\}^p \int_0^{+\infty} x^{\alpha p-1} f^p(x)dx. \end{aligned}$$

This concludes the proof. □

4 Conclusion

In this article, we solved an open problem by providing an exact evaluation of a logarithmic integral. By introducing a new adjustable parameter, we were able to extend this result to a more general framework and derive further integral formulas. We then applied these findings to establish new Hölder and Hardy-Hilbert-type integral inequalities, in which the parameter has a critical influence on the resulting bounds. This provides a more in-depth analytical understanding of the behavior of these inequalities. From a practical standpoint, it opens up new possibilities for enhancing the bounds of functional and harmonic analysis.

Acknowledgement

The authors would like to thank the reviewer for his constructive comments on the article.

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Clément Coine
Normandie Université, UNICAEN, CNRS, LMNO, 14000 Caen, France.
E-mail: `clement.coine@unicaen.fr`

Christophe Chesneau
Normandie Université, UNICAEN, CNRS, LMNO, 14000 Caen, France.
E-mail: `christophe.chesneau@gmail.com`

Received: 28.05.2025
Revised: 12.07.2025
Accepted: 18.07.2025