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# A note on cohomology and algebraic geometric codes on the curves over rings

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**Abstract** Let $A$ be a local Artinian ring with residue field $k(A)$. Let $X$ be a curve over $A$ and let be $X' = X \times_{spec\, A} spec\, k(A)$ the fiber of $X$ over $k(A)$. Consider $\mathcal{L}$ an invertible sheaf on $X$ and $\mathcal{L}' = \phi^\star \mathcal{L} \in Pic(X')$, where $\phi : X' \to X$ is the natural map. Let $C$ and $C'$ be the algebraic geometric codes constructed using the groups of cohomology $\Gamma(X, \mathcal{L})$ and $\Gamma(X', \mathcal{L}')$ respectively. In this note, we first give the complete relation between $\Gamma(X, \mathcal{L})$ and $\Gamma(X', \mathcal{L}')$ without any condition and finally, we provide relations between $C \otimes_A k(A)$ and $C'$ using exact sequences and dimensional theory. Therefore we extend, some results of Walker [18, 20] giving the characterization of WAG codes over rings.

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## 1 Introduction and Motivations

Goppa [6] introduced algebraic geometric codes. Using modular curves, the idea of Goppa was developed in [14] by Tsfasman et al. to solve the strongest Gilbert-Varshamov conjecture. So the advantages of algebraic geometry in studying codes was proved. Using properties of curves over rings studied in [8], J. Walker introduced in [18, 20] the notion of algebraic geometric codes over rings. In [19], J. Walker proved that the Nordstrom-Robinson codes are images under Gray map of a certain algebraic geometric code over $\mathbb{Z}_4$. In [15], [16] and [17], J. Voloch and J. Walker constructed Euclidean weights of codes from elliptic curves and from curves of higher genus over Witt rings. They also characterized trace-codes of algebraic geometric codes over Witt rings. In [1], K. G. Bartley and J. Walker presented some decoding algorithms of these codes. It is worth noting

that all the above results have been proved only for Strongly Algebraic Geometry codes (SAG) (see [12]). However, many open problems still hold. We focus here on the one stated by J. Walker in [18] (Page 67): What can be said for codes which are not SAG? For more precision, consider a line bundle $\mathcal{L}$ in a curve $X$ over a local Artinian ring $A$ and $\mathcal{L}'$ a pullback of $\mathcal{L}$ in a fiber $X' = X \times_{spec\, A} spec\, \mathbb{F}_q$ over $\mathbb{F}_q$ a residue field of $A$. Write $C$ and $C'$ the algebraic geometric codes of length $n$ defined with $X$ over $A$ and $X'$ over $\mathbb{F}_q$ respectively. Under the assumption that $2g-2 < deg\, \mathcal{L}$ or $H^1(X, \mathcal{L}) = 0$, Walker in [18, 20] proves that $\Gamma(X, \mathcal{L})$ is free $A-$module and $\Gamma(X', \mathcal{L}') = \Gamma(X, \mathcal{L}) \otimes \mathbb{F}_q$ (base change theorem on a curve over local Artinian ring with condition on the line bundle). As consequence she obtains that $C' = C \otimes \mathbb{F}_q$ and $C$ is a free code over $A$. That is the characterization of SAG over rings. Notice that the relation between $C$ and $C'$ is related to the link between the groups of cohomology $\Gamma(X, \mathcal{L})$ and $\Gamma(X', \mathcal{L}')$. This link has been one of the main problems of algebraic geometry, and up to now its has no full and complete solution. Indeed, following [9], one can see that $\Gamma(X, \mathcal{L}) \otimes_A k(A) \to \Gamma(X', \mathcal{L}')$ is an isomorphism if it is a surjection. Recalling also the above result obtained by Walker in [18,20] when $2g-2 < deg\, \mathcal{L}$ or $H^1(X, \mathcal{L}) = 0$, the natural questions follow: What can be said between $\Gamma(X, \mathcal{L})$ and $\Gamma(X', \mathcal{L}')$ if $\Gamma(X, \mathcal{L}) \otimes_A k(A) \to \Gamma(X', \mathcal{L}')$ is not surjective, or if $2g-2 \geq deg\, \mathcal{L}$? Furthermore the fundamental question is: What is the full and complete relation between $\Gamma(X, \mathcal{L})$ and $\Gamma(X', \mathcal{L}')$ without any condition? This relation is what we call the general base change theorem for a curve. Notice now that algebraic geometric codes are classified in 3 types: SAG codes $\subset$ AG codes $\subset$ WAG codes (see [12]). The aim of this paper is to give the general base change theorem for a curve and apply it to give the algebraic structures of WAG and AG codes when $H^1(X, \mathcal{L})$ is not necessarily 0.

## 2 Preliminaries

It is worth mentioning that for more details on curves and flat modules, one can refer to the works of J. Walker [18] and [20], Hartshorne [9], Blache [2], Gathmann [5], Grothendieck [7], Eisenbud [4] and Liu [11].

**Definition 2.1.** *[12] Let $X$ be a projective, nonsingular, absolutely irreducible curve defined over $\mathbb{F}_q$. The genus of $X$ is denoted by $g(X)$, or simply by $g$, if it is clear which curve is meant. Let $P_1, \ldots, P_n$ be $n$ distinct $\mathbb{F}_q$-rational points of $X$. We denote both the $n-$tuple $(P_1, \ldots, P_n)$ and the divisor $P_1 + \cdots + P_n$ by $D$ (the order of the $P_i$ is fixed). Let $G$ be a divisor on $X$ of degree $m$ with support disjoint from the support of $D$. Let $\mathbb{F}_q(X)$ be the function field of $X$ over $\mathbb{F}_q$ and $L(G) = \{f \in \mathbb{F}_q(X) | (f) \geq -G\} \cup \{0\}$. Define a map $ev : L(G) \to \mathbb{F}_q^n$ by $ev(f) = (f(P_1), \ldots, f(P_n))$. The image of $ev$ denoted $C(X, G, D)$ is called an Algebraic Geometric code over $\mathbb{F}_q$.*

In the sequel, since any Weil divisor is equivalent to a Cartier divisor for a nonsingular curve, we will write $\Gamma(X, \mathcal{L})$ instead of $L(G)$ to emphasize that it is a module of global sections of a curve $X$ with the sheaf $\mathcal{O}_X(G)$, $G$ being seen as a Cartier divisor on $X$.

**Definition 2.2.** *[12] A linear code $C$ of length $n$ over a finite field $\mathbb{F}_q$ is Weakly Algebraic-Geometric (WAG) if there exists a projective, nonsingular, absolutely irre-*

*ducible curve $X$ defined over $\mathbb{F}_q$ of genus $g$, and $n$ distinct rational points $P_1, \ldots, P_n$ on $X$ and a divisor $G$ with support disjoint from the support of $D$, where $D = P_1 + \cdots + P_n$ such that $C = C(X, G, D)$. The triple $(X, G, D)$ is called a Weakly Algebraic-Geometric representation (WAG representation). Furthermore*

- *An Algebraic-Geometric code (AG) is a WAG code $C(X, G, D)$ with $\deg G < n$.*
- *A Strongly Algebraic-Geometric code (SAG) is a WAG code $C(X, G, D)$ with $2g - 2 < \deg G < n$.*

The previous definition is also used for codes over a ring.

**Definition 2.3.** *[20] Let $A$ be a local finite Artinian ring with residue field $\mathbb{F}_q$ and $Y = spec\, A$. We say that "$X$ is a curve over the ring $A$" if $X$ is a smooth scheme over $Y$ of relative dimension one, which is connected irreducible projective. We write $X'$ for the fiber of $X$ over the unique maximal ideal of $A$. In other words $X' = X \times_{spec\, A} spec\, \mathbb{F}_q$. We assume that $X'$ is absolutely irreducible. Denote by $\phi$ the natural map $X' \to X$. For any line bundle $\mathcal{L}$ on $X$, write $\mathcal{L}' = \phi^*\mathcal{L} = \phi^{-1}\mathcal{L} \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{O}_{X'}$ the pullback of $\mathcal{L}$.*

Some properties of $X$ are given in [20] and [2]. Here we only mention those directly related to our results.

**Lemma 2.1.** *[20] Let $X$ be any scheme. Then*

1. *For any Cartier divisor $G$, $\mathcal{O}_X(G)$ is an invertible sheaf on $X$. The map $G \longmapsto \mathcal{O}_X(G)$ gives a one to one correspondence between Cartier divisor on $X$ and invertible subsheaves of $\mathcal{K}$.*
2. *$\mathcal{O}_X(G_1 - G_2) \simeq \mathcal{O}_X(G_1) \otimes \mathcal{O}_X(G_2)^{-1} = \mathcal{O}_X(G_1) \otimes \mathcal{O}_X(-G_2)$.*
3. *$G_1 \sim G_2$ if and only if $\mathcal{O}_X(G_1) \simeq \mathcal{O}_X(G_2)$ as sheaves.*

**Proposition 2.2.** *[20] If $X$ is a curve over $A$, then $CaCl(X) \simeq Pic(X)$.*

**Remark 2.1.** In the study of algebraic geometric codes over fields, one makes frequent use of the fact that closed points give Weil divisors. In order to study WAG codes over rings, we will need some sort of analog of this fact.

**Lemma 2.3** ($A$-point)**.** *[20] Let $X$ be a curve over $A$ and $Z$ a zero dimensional close subscheme of $X$. Let $i : Z \to X$ be an inclusion and $f : X \to Y$ the structure morphism. We call $Z$ an $A$-point of $X$ if the composition $f \circ i$ is an isomorphism of schemes.*

For more details on $A$-points, see [1, 2, 7, 15].

**Lemma 2.4.** *[20] Let $X$ be a curve over $A$ and $Z$ an $A$-point on $X$. Then there is a unique well-defined Cartier divisor (which we will also denote by $Z$) associated to $Z$.*

**Definition 2.4** (Genus, degree)**.** *Let $X$ be a curve over $A$ and $X'$ its fiber (curve) over $\mathbb{F}_q$. One defines the genus of $X$ to be the genus of $X'$. If $\mathcal{L}$ is a line bundle on $X$, write $\mathcal{L}' = \phi^*\mathcal{L}$. One defines a degree of $\mathcal{L}$ to be a degree of $\mathcal{L}'$.*

*For more precision, if $\mathcal{L} = \mathcal{O}_X(G)$, then there is a Weil divisor $D$ in $X'$ such that $\mathcal{L}' = \mathcal{O}_{X'}(D)$.*

Let $\mathcal{L}$ be an invertible sheaf on $X$ and $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$ be a set of disjoint $A$-points on $X$. Then for each $i$, $\gamma_i : \Gamma(Z_i, \mathcal{L}|_{Z_i}) \to A$ is an $A-$modules isomorphism, but non canonically.

Following the above definitions 2.1 and 2.2, Walker [20] proposed the following:

**Definition 2.5.** *Let $A, X, \mathcal{Z}, \mathcal{L}$ and $\gamma$ be as above. Define $C = C_A(X, \mathcal{L}, \mathcal{Z}, \gamma)$ to be the image of the composition ev:*

$$ev : \Gamma(X, \mathcal{L}) \to \oplus\Gamma(Z_i, \mathcal{L}|_{Z_i}) \to A^n,$$

*where $ev(f) = (f(Z_1), \ldots, f(Z_n))$. $C_A(X, \mathcal{L}, \mathcal{Z}, \gamma)$ is called algebraic geometric code over ring $A$.*

**Definition 2.6.** *Let $G$ be a Cartier divisor on $X$ and $P$ a closed point of $X$ which is a rational point of $X'$. We say $P$ is not in the support of $G$ if there is a neighborhood $U$ of $P$ such that the local equation for $G$ on $U$ is element of $\mathcal{O}_X(U)^\star$.*

**Proposition 2.5.** *[20] Let $A, X, \mathcal{Z}, \mathcal{L}$ and $\gamma$ be as above. Define the composition ev by*

$$ev : \Gamma(X, \mathcal{L}) \to \oplus\Gamma(Z_i, \mathcal{L}|_{Z_i}) \to A^n.$$

*Then $Ker\, ev = \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n))$.*

Firstly, it is worth mentioning that the following results are given under the assumption that $2g - 2 < deg\, \mathcal{L} < n$ and the central result comes from the link between $\Gamma(X, \mathcal{L})$ and $\Gamma(X', \mathcal{L}')$.

**Lemma 2.6.** *[20] Let $X$ be a curve over $A$. Then there are open affine subsets $U$ and $V$ of $X$ with $X = U \cup V$.*

**Lemma 2.7.** *[20] For any open affine subset $U \subset X$, write $U' = U \cap X' = U \times_{spec\, A} spec\, k(A)$. Then $U'$ is an affine open subset of $X'$ and $\Gamma(U', \mathcal{L}') = \Gamma(U, \mathcal{L}) \otimes_A k(A)$.*

**Theorem 2.8.** *[20] Let $A, X, \mathcal{Z}, \mathcal{L}$ and $\gamma$ be as above. Assume that $k(A) = \mathbb{F}_q$. Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of disjoints $\mathbb{F}_q$-rational points corresponding respectively to points $Z_1, \ldots, Z_n$ and let $\gamma' = \{\gamma'_1, \ldots, \gamma'_n\}$ be the system of isomorphism induced from the system $\gamma$ as follows: $\gamma'_i : \Gamma(P_i, \mathcal{L}'|_{P_i}) = \Gamma(Z_i, \mathcal{L}|_{Z_i}) \otimes_A \mathbb{F}_q \to \mathbb{F}_q$. Write $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$. Under the condition $2g - 2 < deg\, \mathcal{L} < n$, the following assertions hold:*

1. *$\Gamma(X', \mathcal{L}') \simeq \Gamma(X, \mathcal{L}) \otimes_A \mathbb{F}_q$ (Base change Theorem with conditions on line bundle).*
2. *$Ker\, ev = 0$.*
3. *$\Gamma(X, \mathcal{L})$ is free $A$-module of dimension $deg\, \mathcal{L} + 1 - g$.*
4. *$C$ is free $A$-module of dimension $deg\, \mathcal{L} + 1 - g$.*
5. *$C' = \pi(C)$, where $\pi$ is the canonical coordinatewise projection $A^n \to \mathbb{F}_q^n$, $C = C_A(X, \mathcal{Z}, \mathcal{L}, \gamma)$ and $C' = C_{\mathbb{F}_q}(X', \mathcal{P}, \mathcal{L}', \gamma')$.*

The following propositions are needed to establish Theorem 3.4.

**Proposition 2.9.** *1. Let $M, N$ and $L$ be three modules such that $0 \to N \to M \to L \to 0$ is exact sequence. If $L$ is a free module, then the sequence splits.*

*2. Every short exact sequence of vector spaces splits.*

**Proposition 2.10.** *Let $M_1, M_2$ and $M_3$ be three modules. If*

$$M_1 \xrightarrow{\beta} M_2 \xrightarrow{p} M_3 \to 0$$

*is composition of epimorphisms of modules, then one has the following exact sequence:*

$$0 \to Ker\,\beta \to Ker(p \circ \beta) \to Ker\,p \to 0.$$

# 3 Main Results

## 3.1 The general base change theorem for curve over a local Artinian ring

**Lemma 3.1.** *[4, 13] Let $A$ be a local Artinian ring, and $M$ a finite type $A-$module. The following conditions are equivalent:*

1. *$M$ is a projective $A$-module.*
2. *$M$ is a flat $A$-module.*
3. *$Tor_1(M, \cdot) = 0$.*
4. *$Tor_1(M, A/\mathcal{M}) = 0$ where $\mathcal{M}$ is the maximal ideal of $A$.*

But in many cases, the $A-$modules are not finitely generated. So when $A$ is a local Artinian ring, we have the following.

**Lemma 3.2.** *([10], Lemma* 10.101.2*) Let $M$ be a module over a local Artinian ring $A$. The following statements are equivalent:*

1. *$M$ is a flat $A-$module.*
2. *$M$ is a projective $A-$module.*
3. *$M$ is a free $A-$module.*

**Lemma 3.3.** *([10], Lemma* 10.101.6*) Let $M$ be a module over a local Artinian ring $A$. Let $I \subset A$ be a proper ideal of $A$. The following statements are equivalent:*

1. *$M$ is a flat $A-$module.*
2. *$M/IM$ is a flat $A/I-$module and $Tor_1(A/I, M) = 0$.*

Using the above lemmas, we can prove the main following result:

**Theorem 3.4.** *Let $X$ be a curve over $A$ and $\mathcal{L}$ a line bundle on $X$ . Let $X' = X \times_{spec\, A} spec\, k(A)$ and $\mathcal{L}' = \phi^*\mathcal{L} = \phi^{-1}\mathcal{L} \otimes_{\phi^{-1}\mathcal{O}_X} \mathcal{O}_{X'}$ where $\phi : X' \to X$ is the natural map. Define the degree of $\mathcal{L}$ to be $deg\, \mathcal{L} = deg\, \mathcal{L}'$. Without any conditions on $deg\, \mathcal{L}$, the modules $\Gamma(X, \mathcal{L})$ and $H^1(X, \mathcal{L})$ satisfy the following properties:*

1. $H^1(X', \mathcal{L}') = H^1(X, \mathcal{L}) \otimes_A k(A)$
2. $0 \longrightarrow \frac{\Gamma(X,\mathcal{L})\otimes_A k(A)}{Tor_2(H^1(X,\mathcal{L}),k(A))} \longrightarrow \Gamma(X', \mathcal{L}') \longrightarrow Tor_1(H^1(X, \mathcal{L}), k(A)) \to 0$ *is an exact sequence.*

*Proof.* Let $\mathfrak{U} = \{U, V\}$ be the covering of $X$ as in Lemma 2.6 Since $X = U \cup V$, the Cech groups of $\mathcal{L} \in Pic(X)$ are given by $C^0(\mathfrak{U}, \mathcal{L}) = \Gamma(U, \mathcal{L}) \oplus \Gamma(V, \mathcal{L})$ and $C^1(\mathfrak{U}, \mathcal{L}) = \Gamma(U \cap V, \mathcal{L})$. These groups are also flat $A-$modules because any invertible sheaf $\mathcal{L}$ is a flat sheaf. They induce the following exact sequence:

$$0 \to \Gamma \xrightarrow{i} C^0 \xrightarrow{d^0} C^1 \xrightarrow{t} H \to 0, \tag{3.1}$$

where $\Gamma = \Gamma(X, \mathcal{L})$ and $H = H^1(X, \mathcal{L})$. We know that $X' = U' \cup V'$ with $U'$ and $V'$ as in Lemma 2.7. So, writing $\Gamma' = \Gamma(X', \mathcal{L}'), H' = H^1(X, \mathcal{L}'), (C^0)' = C^0 \otimes_A k(A)$ and $(C^1)' = C^1 \otimes_A k(A)$, we also have

$$0 \to \Gamma' \to (C^0)' \xrightarrow{d^0\otimes k(A)} (C^1)' \xrightarrow{t'} H' \to 0. \tag{3.2}$$

1.) (3.1) gives:

$$\to (C^0)' \xrightarrow{d^0\otimes k(A)} (C^1)' \xrightarrow{t\otimes k(A)} H \otimes k(A) \to 0. \tag{3.3}$$

With (3.2), we have :

$$\to (C^0)' \to (C^1)' \to H' \to 0. \tag{3.4}$$

It is straightforward that

$$H' \simeq H \otimes k(A).$$

2.) The sequence can split in two sequences:

$$0 \to Ker\, d^0 \to C^0 \to Im\, d^0 \to 0 \tag{3.5}$$

and

$$0 \to Ker\, t \to C^1 \to Im\, t \to 0 \tag{3.6}$$

with $Ker\, t = Im\, d^0$; $Ker\, d^0 = Im\, i$. So from (3.5) and (3.6), we have the following sequence:

$$0 \to Tor_1(Im\, d^0, k(A)) \to (Ker\, d^0) \otimes k(A) \to (C^0)' \to (Im\, d^0) \otimes k(A) \to 0, \tag{3.7}$$

and one has

$$0 \to Tor_1(Im\, t, k(A)) \to (Ker\, t) \otimes k(A) \to (C^1)' \to (Im\, t) \otimes k(A) \to 0. \tag{3.8}$$

We have

$$Ker\, d^0 = \Gamma, \tag{3.9}$$

and we also have

$$Tor_1(Im\, d^0, k(A)) \simeq Tor_2(H, k(A)). \tag{3.10}$$

From (3.5) and (3.6) we respectively have the following exact sequences:

$$0 \to Tor_1(Im\, d^0, k(A)) \to \Gamma \otimes k(A) \overset{i\otimes k(A)}{\to} (C^0)' \overset{\beta}{\to} (Im\, d^0) \otimes k(A) \to 0 \tag{3.11}$$

and

$$0 \to Tor_1(H, k(A)) \to (Im\, d^0) \otimes k(A) \overset{\tau}{\to} (C^1)' \overset{t\otimes k(A)}{\to} H \otimes k(A) \to 0. \tag{3.12}$$

From (3.3), we have $Im(d^0 \otimes k(A)) \simeq Ker(t \otimes k(A))$. Thus, the following holds:

$$Im(d^0 \otimes k(A)) \simeq \frac{(Im\, d^0) \otimes k(A)}{Tor_1(H, k(A))}. \tag{3.13}$$

$\beta$ being a surjective map, the map (projection)

$$(Im\, d^0) \otimes k(A) \overset{p}{\to} \frac{(Im\, d^0) \otimes k(A)}{Tor_1(H, k(A))} \tag{3.14}$$

is also surjective and

$$Ker\, p = Tor_1(H, k(A)). \tag{3.15}$$

From (3.11),(3.13) and (3.14),

$$(C^0)' \overset{\beta}{\to} (Im\, d^0) \otimes k(A) \overset{p}{\to} Im(d^0 \otimes k(A)) \tag{3.16}$$

is a surjective composition. It follows that

$$Ker\beta = Im(i \otimes k(A)) \simeq \frac{\Gamma \otimes k(A)}{Tor_1(Im\, d^0, k(A))} \simeq \frac{\Gamma \otimes k(A)}{Tor_2(H, k(A))}. \tag{3.17}$$

$(C^0)'$ and $(C^1)'$ being vector spaces, the image $Im(d^0 \otimes k(A))$ of $(C^0)' \overset{d^0\otimes k(A)}{\to} (C^1)'$ is also a vector space. Thus one has the two following exact sequences of vectors spaces:

$$0 \to \Gamma' \to (C^0)' \to Im(d^0 \otimes k(A)) \to 0$$

and

$$0 \to Ker(p \circ \beta) \to (C^0)' \to Im(d^0 \otimes k(A)) \to 0.$$

So, applying Proposition 2.9, we obtain

$$Ker(p \circ \beta) = \Gamma'. \tag{3.18}$$

From (3.16), one has the following exact sequence: $0 \to Ker\, \beta \to Ker(p \circ \beta) \to Ker\, p \to 0$, which with Proposition 2.10, finally gives the expected result. □

The main consequence of the above theorem is the following and interesting:

**Corollary 3.5.** *Let $X, X', \mathcal{L}, \mathcal{L}'$ as above.*

1. $\Gamma(X', \mathcal{L}') \simeq \Gamma(X, \mathcal{L}) \otimes_A k(A)$ *if and only if* $H^1(X, \mathcal{L})$ *is a free* $A-$*module.*
2. *If* $H^1(X, \mathcal{L})$ *is a flat* $A-$*module, then* $\Gamma(X, \mathcal{L})$ *is a flat* $A-$*module.*
3. $\Gamma(X, \mathcal{L})$ *is a flat* $A-$*module if and only if the projective dimension of* $H^1(X, \mathcal{L})$ *is at most* 2.

*Proof.* For these 3 items, with some calculations, combine equation (3.10) of the proof of Theorem 3.4 and the exact sequence of the same Theorem. □

### 3.2 Algebraic geometric codes over a local Artinian ring

In the following lines, we characterize WAG codes over rings. So we generalize the results obtained by Walker in [20] (see page 44, with the assumption that $H^1(X, \mathcal{L}) = 0$). Here $H^1(X, \mathcal{L})$ can or cannot be 0. We first give the link between the codes $C \otimes_A k(A)$ and $C'$ by using of exact sequences under some conditions on $deg\, \mathcal{L}$. But giving the general relation between $C \otimes_A k(A)$ and $C'$ using exact sequences is difficult to handle without these conditions. So, adopting at the end the dimensional approach, we give, without any condition, the general link between $dim_{k(A)} C \otimes_A k(A)$ and $dim_{k(A)} C'$. In the sequel, $A$ is considered as any finite local Artinian ring with residue field denoted $k(A)$ or $\mathbb{F}_q$, the Galois field of $q$ elements.

**Proposition 3.6.** *Let ev be given by* $ev : \Gamma(X, \mathcal{L}) \to \oplus \Gamma(Z_i, \mathcal{L}|_{Z_i}) \to A^n$ *(see Definition 2.1). Let be* $K'_{ev}$ *defined by* $K'_{ev} = \Gamma(X', \phi^\star(\mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n))) = \Gamma(X', \mathcal{L}' \otimes \phi^\star \mathcal{O}_X(-Z_1 - \cdots - Z_n))$. *The following sequence is exact:*

$$0 \to \frac{Ker\, ev \otimes \mathbb{F}_q}{Tor_2(H^1(X, \mathcal{L} \otimes \mathcal{O}_X(\sum_{i=1}^{n} -Z_i)), \mathbb{F}_q)} \to K'_{ev} \to Tor_1(H^1(X, \mathcal{L} \otimes \mathcal{O}_X(\sum_{i=1}^{n} -Z_i)), \mathbb{F}_q) \to 0.$$

*Proof.* From Proposition 2.5,

$$Ker\, ev = \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n)),$$

and

$$K'_{ev} = \Gamma(X', \phi^\star(\mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n))) = \Gamma(X', \mathcal{L}' \otimes \phi^\star \mathcal{O}_X(-Z_1 - \cdots - Z_n)).$$

$X$ being a curve over $A$ and $\mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n))$ an invertible sheaf on $X$, use Theorem 3.4 to conclude. □

**Corollary 3.7.** *Let ev be as above. If* $deg\, \mathcal{L} < n$ *, then* $Ker\, ev = 0$.

*Proof.* If $deg\, \mathcal{L} = deg\, \mathcal{L}' < n$, then $K'_{ev} = \Gamma(X', \phi^\star(\mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n)) = 0$. From Proposition 3.6, it follows that $Tor_1(H^1(X, \mathcal{L} \otimes \mathcal{O}_X(\sum_{i=1}^{n} -Z_i)), \mathbb{F}_q) = 0$, which means $H^1(X, \mathcal{L} \otimes \mathcal{O}_X(\sum_{i=1}^{n} -Z_i))$ is free module, so the projective dimension of $H^1(X, \mathcal{L} \otimes \mathcal{O}_X(\sum_{i=1}^{n} -Z_i))$ is 0. Thus $Tor_2(H^1(X, \mathcal{L} \otimes \mathcal{O}_X(\sum_{i=1}^{n} -Z_i)), \mathbb{F}_q) = 0$. So $Ker\, ev \otimes \mathbb{F}_q = 0$. By Nakayama's Lemma, $Ker\, ev = 0$. □

**Lemma 3.8.** *[20] If $C$ is a free module or free code over a finite chain ring, then*

$$\pi(C) = C \otimes_A \mathbb{F}_q.$$

The following gives an answer of the question in [18] (page 67) for AG code.

**Theorem 3.9.** *Let $X, X', \mathcal{L}$ and $\mathcal{L}'$ be as above. Write $C = C_A(X, \mathcal{L}, \mathcal{Z})$ and $C' = C_{\mathbb{F}_q}(X', \mathcal{L}', \mathcal{P})$. If $deg\,\mathcal{L} < n$, then the following sequence is exact:*

$$0 \longrightarrow \frac{C \otimes_A \mathbb{F}_q}{Tor_2(H^1(X, \mathcal{L}), \mathbb{F}_q)} \longrightarrow C' \longrightarrow Tor_1(H^1(X, \mathcal{L}), \mathbb{F}_q) \to 0.$$

*Proof.* Since $deg\,\mathcal{L} < n$, $\Gamma(X, \mathcal{L}) \simeq C = C_A(X, \mathcal{L}, \mathcal{Z})$ and $\Gamma(X', \mathcal{L}') \simeq C' = C_{\mathbb{F}_q}(X', \mathcal{L}', \mathcal{P})$. Apply Theorem 3.4 and Corollary 3.7 to conclude. □

Theorem 3.9 is given with $deg\,\mathcal{L} < n$, which is the characterization of AG-codes. So to obtain the general view, that is to characterize the WAG-codes, it is welcome to ask: what can be said when $deg\,\mathcal{L} \geq n$? Before the answer of this question, let us mention the two following facts:

1. Given any set $G$, we denote by $|G|$ its cardinality.
2. The set $X$ can be partitioned in three sets $\mathcal{Z}$, $suppG$ and $\mathcal{R}$ such $X = \mathcal{Z} \cup \mathcal{R} \cup suppG$ where $\mathcal{L} = \mathcal{O}_X(G)$ and $\mathcal{Z} = \{Z_1, \ldots, Z_n\}$.

   Now, if $X$ has many $A-$points such that $|\mathcal{Z} \cup \mathcal{R}| > deg\,\mathcal{L}$, then consider $deg\,\mathcal{L} + 1$ distinct $A-$points of $X$ taking in account the first $n$ distinct points $Z_1, \ldots, Z_n$ used to construct the code $C$. Now consider the code given by the image of the evaluation

   $$ev : \Gamma(X, \mathcal{L}) \to A^{deg\,\mathcal{L}+1},$$

   with

   $$ev(f) = (f(Z_1), \ldots, f(Z_n), f(Z_{n+1}), \ldots, f(Z_{deg\,\mathcal{L}+1})).$$

   Let $T$ be the code defined as the subset of $A^{deg\,\mathcal{L}+1}$ by putting 0 in all the first $n$ coordinates as follows: $T = \{(0, \ldots, 0, f(Z_{n+1}), \ldots, f(Z_{deg\,\mathcal{L}+1})), f \in \Gamma(X, \mathcal{L})\}$. It is direct that $C$ can be seen as $C = \{(f(Z_1), \ldots, f(Z_n), 0, \ldots, 0), f \in \Gamma(X, \mathcal{L})\}$. We see that $Im\,ev = C \oplus T$. Since $deg\,\mathcal{L} < deg\,\mathcal{L} + 1$, with Theorem 3.9, we obtain

   $$0 \longrightarrow \frac{(C \oplus T) \otimes_A \mathbb{F}_q}{Tor_2(H^1(X, \mathcal{L}), \mathbb{F}_q)} \longrightarrow (C' \oplus T') \longrightarrow Tor_1(H^1(X, \mathcal{L}), \mathbb{F}_q) \to 0.$$

We can resume the above by the following

**Theorem 3.10.** *Assume that $X$ is partitioned in three sets $\mathcal{Z}, suppG$ and $\mathcal{R}$ such that $X = \mathcal{Z} \cup \mathcal{R} \cup suppG$. If $deg\,\mathcal{L} \geq n$ and $X$ has many $A-$points such that $|\mathcal{Z} \cup \mathcal{R}| > deg\,\mathcal{L}$, then there exists a linear code $T$ such that*

$$0 \longrightarrow \frac{(C \oplus T) \otimes_A \mathbb{F}_q}{Tor_2(H^1(X, \mathcal{L}), \mathbb{F}_q)} \longrightarrow (C' \oplus T') \longrightarrow Tor_1(H^1(X, \mathcal{L}), \mathbb{F}_q) \to 0.$$

When $deg\,\mathcal{L} \geq n$ and $|\mathcal{Z} \cup \mathcal{R}| < deg\,\mathcal{L}$, it is difficult to handle the relation between $C'$ and $C \otimes_A k(A)$ by the use of exact sequences. So we initiate the dimensional approach that gives the general relation between $dim\,C'$ and $dim\,C \otimes_A \mathbb{F}_q$ since $C'$ and $C \otimes_A \mathbb{F}_q$ are $\mathbb{F}_q-$vector spaces. Before, let us give this remark.

**Remark 3.1.** Let $ev'$ be the evaluation $ev' : \Gamma(X', \mathcal{L}') \to \mathbb{F}_q$ which associates each $f \in \Gamma(X', \mathcal{L}')$ to $(f(P_1), \ldots, f(P_n))$. Using the relation between the points $Z_i$ and $P_i$ as in Theorem 2.8 and Definition 2.4, we see that $Ker\,ev' = K'_{ev}$ where $K'_{ev}$ is given by Proposition 3.6 and $Ker\,ev' = \Gamma(X', \mathcal{L}' \otimes \mathcal{O}_{X'}(-P_1 - \cdots - P_n))$. In fact, $P_i$ being a specialization of an $A-$point $Z_i$ (see [16]), it comes that $\phi^\star \mathcal{O}_X(Z_i) = \mathcal{O}_{X'}(P_i)$. Since $\phi^\star(\mathcal{L}_1 \otimes \mathcal{L}_2) = \phi^\star(\mathcal{L}_1) \otimes \phi^\star(\mathcal{L}_2)$, we are done.

We prove the content of this remark in the following lines by giving more details. Let us recall the following both central results.

**Lemma 3.11.** *[18, 20] For any Cartier divisor $D$, $\mathcal{O}_X(D)$ is an invertible sheaf and the application $D \mapsto \mathcal{O}_X(D)$ is a one to one correspondence between the Cartier divisors and the invertible sheaves of $X$.*

**Lemma 3.12.** *[18, 20] $Hom_{spec\,A}(spec\,A, X) \to Hom_{spec\,k}(spec\,k, X')$ is surjective map. This means that every $k-$rational point of $X'$ is a specialization a certain $A-$point of $X$.*

More explicitly, given an $A-$point $Z \in Hom_{spec\,A}(spec\,A, X)$, we have

$$Z \times_{spec\,A} spec\,k \in Hom_{spec\,A}(spec\,A, X) \times_{spec\,A} spec\,k.$$

But $Hom_{spec\,A}(spec\,A, X) \times_{spec\,A} spec\,k$ is canonically isomorphic to $Hom_{spec\,k}(spec\,k, X')$. Thus the $k-$rational points of $X'$ can be identified with the points $Z \times_{spec\,A} spec\,k$. This means that the $k-$rational points of $X'$ are obtained from the $A-$points $Z$ of $X$ by the reduction modulo the maximal ideal $m$ of $A$. This was mentioned in [15]. Furthermore, in the proof of Lemma 35 in [3], we observe that the local parameter of $Z : spec\,A \to X$ is given by the closed immersion $Z = spec(B/J) \to U$ defined by $Z$ (in fact one can identify $Z$ to $spec(B/J)$) for a certain well defined $A-$module $B$ where $U = spec\,B$ is a open affine in the covering of $X$. Furthermore this local paramater $t$ is the generator of this $J$ which is principal ideal.

Consider $P = Z \times_{spec\,A} spec\,k = Z\ mod\ m$ as we observed above. Consider $B' = B \otimes_A k$, $U' = spec\,B'$ and $J' = J \otimes_A k$. Then $t\ mod\ m$ generates $J'$ and $P = spec(B'/J')$ is the specialization $Z$ in $X'$. This means that the Cartier divisor of $P = \{(U', t\ mod\ m), (V', 1)\}$ where the Cartier divisor of $Z$ is $Z = \{(U, t), (V, 1)\}$. This induces the following:

**Lemma 3.13.** *The Cartier divisor of of $P$ is $P = \{(U', \bar{t}), (V', 1)\}$ where the Cartier divisor of $Z$ is $Z = \{(U, t), (V, 1)\}$ and $\bar{t} = t\ mod\ m$.*

**Theorem 3.14.** *Given a Cartier divisor $Z$ associated to an $A-$point $Z$, consider the unique closed point $P$ contained in $Z$ which is rational over $X'$. Denote also by $P$ the Cartier divisor associated to $P$ in $X'$. Then we have $\phi^\star \mathcal{O}_X(-Z) = \mathcal{O}_{X'}(-P)$.*

*Proof.* Since $\{U', V'\}$ is an open cover of $X'$, it suffices to prove this equality in each of the both open subsets of $X'$. With the Lemma 3.11, $\mathcal{O}_X(-Z)$ is an invertible sheaf.

So with Lemma 2.7, we have $\Gamma(U', \phi^\star \mathcal{O}_X(-Z)) = \Gamma(U, \mathcal{O}_X(-Z)) \otimes_A k$. Since $-Z = \{(U, 1/t), (V, 1)\}$, this last equality means that

$$.\phi^\star \mathcal{O}_X(-Z)(U') = \mathcal{O}_X(-Z)(U) \otimes_A k = t\mathcal{O}_X(U) \otimes_A k.$$

With the same Lemma 2.7, for $\mathcal{L} = \mathcal{O}_X$, and knowing that $\phi^\star \mathcal{O}_X = \mathcal{O}_{X'}$, we have

$$\mathcal{O}_X(U) \otimes_A k = \mathcal{O}_{X'}(U').$$

So if we set $\bar{t} = t \ mod \ m$, we clearly have $t\mathcal{O}_X(U) \otimes_A k = \bar{t}\mathcal{O}_{X'}(U')$. This implies that

$$\phi^\star \mathcal{O}_X(-Z)(U') = \bar{t}\mathcal{O}_{X'}(U').$$

But Lemma 3.13 induces that $-P = \{(U', 1/\bar{t}), (V', 1)\}$ because $-Z = \{(U, 1/t), (V, 1)\}$. By definition of the Cartier divisor, we clearly have

$$\mathcal{O}_{X'}(-P)(U') = \bar{t}\mathcal{O}_{X'}(U').$$

This implies that

$$\phi^\star \mathcal{O}_X(-Z)(U') = \mathcal{O}_{X'}(-P)(U').$$

In the same way, one proves that

$$\phi^\star \mathcal{O}_X(-Z)(V') = \mathcal{O}_{X'}(-P)(V').$$

Therefore

$$\phi^\star \mathcal{O}_X(-Z) = \mathcal{O}_{X'}(-P)$$

□

Now we can give the full proof of what was stated in Remark 3.1.

**Corollary 3.15.** *Given a Cartier divisor $Z_i$ associated to an $A-$point $Z_i$, consider the unique closed point $P_i$ contained in $Z_i$ which is rational over $X'$. Denote also by $P_i$ the Cartier divisor associated to $P_i$ in $X'$. Then for any invertible sheaf $\mathcal{L}'$ of $X'$, we have*

$$\Gamma(X', \mathcal{L}' \otimes \phi^\star \mathcal{O}_X(-Z_1 - \cdots - Z_n)) = \Gamma(X', \mathcal{L}' \otimes \mathcal{O}_{X'}(-P - \cdots - P_n)).$$

*Proof.* With Theorem 3.14 we know that $\mathcal{L}' \otimes \phi^\star \mathcal{O}_X(-Z) = \mathcal{L}' \otimes \mathcal{O}_{X'}(-P)$. So both invertible sheaves have the same global section. □

Now, with another approach, we are ready to give the complete relation between $dim_{\mathbb{F}_q} C'$ and $dim_{\mathbb{F}_q} C \otimes_A \mathbb{F}_q$.

**Theorem 3.16.** *Let $C$ and $C'$ be defined as above. For all $j \geq 1$, we have the following relation. Set*

$$\Delta Tor H^1(\mathcal{L}) = dim\, Tor_1(H^1(X, \mathcal{L})) - dim\, Tor_2(H^1(X, \mathcal{L}))$$

*and*

$$\begin{aligned}\Delta Tor H^1(\mathcal{L}, \mathcal{Z}) =& dim\, Tor_1(H^1(X, \mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n)) \\ & - dim\, Tor_2(H^1(X, \mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n))).\end{aligned}$$

*Then*

$$dim_{\mathbb{F}_q} C' - dim_{\mathbb{F}_q} C \otimes_A \mathbb{F}_q = \sum_{l=1}^{j} (-1)^{l-1} dim\, Tor_l(C) + \sum_{l=1}^{j} (-1)^l dim\, Tor_l(\Gamma) + \sum_{l=1}^{j} (-1)^{l-1} dim\, Tor_l(Ker\, ev) + \Delta TorH^1(\mathcal{L}) - \Delta TorH^1(\mathcal{L}, \mathcal{Z}) + (-1)^j dim\, Im\, d_j^3,$$

*where*

$$\overset{d_j^3}{\to} Tor_j(Ker\, ev) \overset{d_j^2}{\to} Tor_j(\Gamma) \overset{d_j^1}{\to} Tor_j(C) \to \cdots \overset{d_1^1}{\to} Tor_1(C) \to \to (Ker\, ev) \otimes_A \mathbb{F}_q \to \Gamma \otimes_A \mathbb{F}_q \to C \otimes_A \mathbb{F}_q \to 0$$

*is an exact sequence with* $Ker\, ev = \Gamma(X, \mathcal{L} \otimes \mathcal{O}_X(-Z_1 - \cdots - Z_n))$ *and* $\Gamma = \Gamma(X, \mathcal{L})$

*Proof.* For any field $\mathbb{F}_q$, it is not difficult to see that for any long exact sequence of $\mathbb{F}_q-$vector spaces

$$\ldots\ldots A_{j+1} \overset{d_j}{\to} A_j \overset{d_{j-1}}{\to} A_{j-1} \overset{d_{j-2}}{\to} A_{j-2} \ldots A_2 \overset{d_2}{\to} A_1 \overset{d_1}{\to} A_0 \to 0,$$

the dimensions of these $A_k$ are related by $\sum_{k=0}^{j} (-1)^k dim_{\mathbb{F}_q} A_k + (-1)^{j+1} dim_{\mathbb{F}_q} Imd_j = 0$. Now we have the following exact sequences

$$0 \to Ker\, ev' \to \Gamma(X', \mathcal{L}') \to C' \to 0,$$
$$0 \to Ker\, ev \to \Gamma(X, \mathcal{L}) \to C \to 0.$$

The short sequence $0 \to Ker\, ev \to \Gamma(X, \mathcal{L}) \to C \to 0$ induces a long exact sequence

$$\overset{d_j^3}{\to} Tor_j(Ker\, ev) \overset{d_j^2}{\to} Tor_j(\Gamma) \overset{d_j^1}{\to} Tor_j(C) \to \cdots \overset{d_1^1}{\to} Tor_1(C) \to \to (Ker\, ev) \otimes_A \mathbb{F}_q \to \Gamma \otimes_A \mathbb{F}_q \to C \otimes_A \mathbb{F}_q \to 0.$$

Since $K'_{ev} = Ker\, ev'$, combine with dimension theory applied to the sequences of Theorem 3.4 and Proposition 3.6. □

Finally we obtain the following

**Corollary 3.17.** *1. If $C$ is a free code over $A$, then*

$$dim\, C' - dim\, C \otimes_A \mathbb{F}_q = \Delta TorH^1(\mathcal{L}) - \Delta TorH^1(\mathcal{L}, \mathcal{Z}).$$

*2. If $\Gamma(X, \mathcal{L})$ is a free module over $A$, then*

$$dim\, C' - dim\, C \otimes_A \mathbb{F}_q = dim\, Tor_1(C) + \Delta TorH^1(\mathcal{L}) - \Delta TorH^1(\mathcal{L}, \mathcal{Z}).$$

*3. If $H^1(X, \mathcal{L})$ is a free $A-$module, then*

$$dim\, C' - dim\, C \otimes_A \mathbb{F}_q = dim\, Tor_1(C) - \Delta TorH^1(\mathcal{L}, \mathcal{Z}).$$

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