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A note on Deaconescu's conjecture

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Abstract Hasanalizade [5] studied Deaconescu's conjecture for positive composite integer n. A positive composite integer $n \geq 4$ is said to be a Deaconescu number if $S_2(n) \mid \phi(n) - 1$. In this paper, we improve Hasanalizade's result by proving that a Deaconescu number n must have at least seventeen distinct prime divisors, i.e., $\omega(n) \geq 17$ and must be strictly larger than $5.86 \cdot 10^{22}$. Further, we prove that if any Deaconescu number n has all prime divisors greater than or equal to 11, then $\omega(n) \geq p^*$, where p^* is the smallest prime divisor of n and if $n \in D_3$ then all the prime divisors of n must be congruent to 2 modulo 3 and $\omega(n) \geq 48$.

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1 Introduction

Lehmer [7] conjectured that if $\phi(n) \mid (n-1)$ (where $\phi(n)$ is the Euler's totient function), then n must be a prime number. A positive composite integer that satisfies the condition $\phi(n) \mid (n-1)$ is called a Lehmer number. Many interesting properties of Lehmer numbers can be found in [1–3, 9]. Let $S_2(n)$ denote Schemmel's totient function, which is a multiplicative function defined by

$$S_2(p^k) := \begin{cases} 0, & \text{if } p = 2 \\ p^{k-1}(p-2), & \text{if } p > 2 \end{cases}$$

where p is a prime number and $k \in \mathbb{Z}^+$. Deaconescu [4], probably inspired by Lehmer, considered Schemmel's totient function and conjectured that for $n \geq 2$

$$S_2(n) \mid \phi(n) - 1$$

if and only if n is prime. A positive composite integer $n \geq 4$ is said to be a Deaconescu number if $S_2(n) \mid \phi(n)-1$. Hasanalizade [5] proved that a Deaconescu number n must be an odd, squarefree positive integer such that $\omega(n) \geq 7$. Further, Hasanalizade provided an upper bound for n using Nielsen's Lemma ([8], Lemma 1.4) and also proved that for a monic non-constant polynomial $P(X) \in \mathbb{Z}[X]$, there are at most finitely many composite integers n such that $S_2(n) \mid (\phi(n)-1)$ and $P(S_2(n)) \equiv 0 \pmod{\phi(n)}$, an analogous result due to Hernandez and Luca ([6], Theorem 1).

Observe that Deaconescu's conjecture says that for every $M \geq 1$ the set D_M (we are using notations used by Hasanalizade) of integers satisfying $MS_2(n) = \phi(n) - 1$ contains only prime numbers. Hasanalizade proved that the conjecture is true for $M = 1$, that is, the set D_1 contains only prime numbers (see [5], Lemma 1). Therefore, we consider $M \geq 3$.

The following theorem describes the nature of a positive composite integer n which belongs to D_3.

Theorem 1. *If n is a Deaconescu number such that $n \in D_3$, then all the prime divisors of n must be congruent to 2 modulo 3 and $\omega(n) \geq 48$.*

We improve the lower bound for $\omega(n)$ for a Deaconescu number n by proving the following theorem.

Theorem 2. *If n is a Deaconescu number, then the number of distinct prime divisors of n is at least seventeen, that is, $\omega(n) \geq 17$.*

As a consequence of Theorem 2, we can produce a lower bound for a Deaconescu number n.

Theorem 3. *A Deaconescu number must be strictly larger than $5.86 \cdot 10^{22}$.*

Throughout this paper, we use $p_1, p_2, \ldots, p_{\omega(n)}$ to denote prime numbers.

2 Preliminaries

In this section, we prove some useful lemmas concerning Deaconescu numbers.

Lemma 4. *If n is a Deaconescu number having all prime divisors greater than or equal to 11, then $\omega(n) \geq p^*$, where p^* is the smallest prime divisor of n.*

Proof. Let us consider a function $\tau : [7, \infty) \to \mathbb{R}$ defined by $\tau(t) = \left(1 + \frac{1}{t}\right)^{1+t}$. Then

$$\tau'(t) = \left(1 + \frac{1}{t}\right)^{1+t} \left(\log\left(1 + \frac{1}{t}\right) - \frac{1}{t}\right) < 0 \quad \text{for} \quad t \geq 7,$$

therefore τ is a strictly decreasing function of t on $[7, \infty)$ and thus for all $t \in [7, \infty)$

$$\tau(t) = \left(1 + \frac{1}{t}\right)^{1+t} \leq \left(1 + \frac{1}{7}\right)^{8} = \tau(7) < 3. \tag{2.1}$$

Let n be a Deaconescu number with all prime divisors greater than or equal to 11 and also let p^* be the smallest prime divisor of n. Then

$$\begin{aligned} 3 \leq M = \frac{\phi(n)-1}{S_2(n)} < \frac{\phi(n)}{S_2(n)} &= \prod_{p|n}\left(\frac{p-1}{p-2}\right) \\ &= \prod_{p|n}\left(1+\frac{1}{p-2}\right) \\ &< \prod_{p|n}\left(1+\frac{1}{p^*-2}\right) = \left(1+\frac{1}{p^*-2}\right)^{\omega(n)}. \end{aligned}$$

If possible, suppose that $\omega(n) < p^*$. Then, setting $t = p^* - 2$ in (2.1), we get

$$\left(1+\frac{1}{p^*-2}\right)^{\omega(n)} \leq \left(1+\frac{1}{p^*-2}\right)^{p^*-1} < \tau(7) < 3.$$

It follows that $3 \leq M < 3$, which is a contradiction. Hence $\omega(n) \geq p^*$. □

Lemma 5. *If n is a Deaconescu number, then no prime divisor of n is congruent to 1 modulo M.*

Proof. Let p be a prime which divides n and suppose that $p \equiv 1 \pmod{M}$. Note that since $\phi(n) \equiv 1 \pmod{M}$ and $(p, n/p) = 1$, it follows that

$$0 \equiv \phi(p) \cdot \phi(n/p) = \phi(n) \equiv 1 \pmod{M},$$

a contradiction. □

3 Proof of Theorem 1

Let n be a Deaconescu number such that $n \in D_3$. Then, from Lemma 5 we have $p \not\equiv 1 \pmod{M}$ for all prime divisors p of n, where $M = \frac{\phi(n)-1}{S_2(n)} \geq 3$, thus in particular if $M = 3$ then the prime divisors of n belong to the set $\mathcal{P} = \{3, p : p \text{ is a prime}, \ p \equiv 2 \pmod{3}\}$. If possible, assume that $3 \mid n$. Then

$$n = 3 \cdot \prod_{i=2}^{\omega(n)} p_i,$$

where $p_i \geq 5$. Since $M = 3$, $p_i = 3k_i + 2$, for some $k_i \in \mathbb{Z}^+$, for all $2 \leq i \leq \omega(n)$. Therefore

$$3 \cdot 1 \cdot \prod_{i=2}^{\omega(n)} (3k_i) = 2 \cdot \prod_{i=2}^{\omega(n)} (3k_i + 1) - 1,$$

then

$$2 \cdot \prod_{i=2}^{\omega(n)} (3k_i + 1) \equiv 1 \pmod{3},$$

that is

$$2 \equiv 1 \pmod 3,$$

but this is impossible. Therefore $3 \nmid n$.

Now suppose that $\omega(n) \leq 47$, then since $M = 3$, if we write

$$n = \prod_{i=1}^{\omega(n)} p_i,$$

then $p_i = 3k_i + 2$, for some $k_i \in \mathbb{Z}^+$, for all $1 \leq i \leq \omega(n)$. Therefore

$$3 \cdot \prod_{i=1}^{\omega(n)} (3k_i) = \prod_{i=1}^{\omega(n)} (3k_i + 1) - 1,$$

which implies that

$$3 = \prod_{i=1}^{\omega(n)} \left(1 + \frac{1}{3k_i}\right) - \prod_{i=1}^{\omega(n)} (3k_i)^{-1}. \tag{3.1}$$

Note that

$$\begin{aligned}
\prod_{i=1}^{\omega(n)} \left(1 + \frac{1}{3k_i}\right) - \prod_{i=1}^{\omega(n)} (3k_i)^{-1} &< \prod_{i=1}^{\omega(n)} \left(1 + \frac{1}{3k_i}\right) \\
&\leq \prod_{i=1}^{47} \left(1 + \frac{1}{3k_i}\right) \\
&< \prod_{\substack{k \in \mathcal{A} \\ \mathcal{A} = \{k:\ 5 \leq 3k+2 \leq 71 \text{ is prime}\} \\ |\mathcal{A}| = 10}} \left(1 + \frac{1}{3k}\right) \cdot \left(1 + \frac{1}{81}\right)^{37} < 3,
\end{aligned}$$

which clearly contradicts (3.1). Therefore, we must have $\omega(n) \geq 48$. This completes the proof. □

4 Proof of Theorem 2

Assume that $n = p_1 p_2 \cdots p_{\omega(n)}$ with $2 < p_1 < p_2 < \cdots < p_{\omega(n)}$ is a Deaconescu number and $\omega(n) \leq 16$. Let $\mathcal{A} = \{p \ : \ p \text{ is a prime, } 3 \leq p \leq 59\}$ and $\mathcal{A}^* = \{p \ : \ p \text{ is a prime, } p \not\equiv 1 \pmod 5, \text{ and } 3 \leq p \leq 79\}$. Note that $\mathcal{A}$ and $\mathcal{A}^*$ each have the 16 primes in the appropriate intervals. Now we see that

$$3 \leq M = \frac{\phi(n) - 1}{S_2(n)} < \frac{\phi(n)}{S_2(n)} = \prod_{i=1}^{\omega(n)} \left(\frac{p_i - 1}{p_i - 2}\right) = \prod_{i=1}^{\omega(n)} \left(1 + \frac{1}{p_i - 2}\right) \leq \prod_{p \in \mathcal{A}} \left(1 + \frac{1}{p - 2}\right) < 6.$$

It follows that $M = 3$ or $M = 5$. If $M = 5$ then we have

$$5 = M = \frac{\phi(n) - 1}{S_2(n)} < \frac{\phi(n)}{S_2(n)} = \prod_{i=1}^{\omega(n)} \left(\frac{p_i - 1}{p_i - 2} \right) = \prod_{i=1}^{\omega(n)} \left(1 + \frac{1}{p_i - 2} \right) \leq \prod_{p \in \mathcal{A}^*} \left(1 + \frac{1}{p - 2} \right) < 5,$$

a contradiction. It follows that $M = 3$, and thus $n \in D_3$. But in this case from Theorem 1, we must have $\omega(n) \geq 48$, again a contradiction. Therefore, our assumption that $\omega(n) \leq 16$ is wrong, and hence we must have $\omega(n) \geq 17$. □

5 Proof of Theorem 3

Let n be a Deaconescu number. Then the number of distinct prime divisors of n is greater than or equal to 17 by Theorem 2. Therefore

$$n \geq \prod_{j=1}^{\omega(n)} P_{j+1},$$

where P_j is the j-th prime number (e.g. $P_2 = 3, P_5 = 11$). Note that

$$n \geq \prod_{j=1}^{\omega(n)} P_{j+1} \geq \prod_{j=1}^{17} P_{j+1} > 5.86 \cdot 10^{22}.$$

This completes the proof. □

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