sciendo
DOI: 10.2478/awutm-2025-0004

Annals of West University of Timisoara
Mathematics and Computer Science
61 (2025), 41–54

Discrete characterizations of h-dichotomy for linear discrete-time systems in Banach spaces

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Abstract The present paper is focused of some concepts of dichotomy with growth rates (h-dichotomy, weak h-dichotomy) for linear discrete systems. These concepts use two types of dichotomy projections sequences (invariant and strongly invariant) and generalize some well-known dichotomy concepts. More precisely, necessary and sufficient conditions of Datko type are given using both invariant and strongly invariant projections sequences for h-dichotomy and weak h-dichotomy.

AMS Subject Classification (2020) 34D05; 34D09
Keywords discrete time systems; (weak) uniform h-dichotomy; growth rate

1 Introduction

The notions of (uniform) exponential dichotomy was introduced by Perron in [17], and considered afterwards by Ta Li in [12] in the discrete variant. Remarkable results on exponential dichotomy were obtained in [3, 5, 9, 13]. In addition, the exponential dichotomy was studied from different perspectives in [11] and [18]. Connections between admissibility and various dichotomy notions for evolution families are given in [7, 8, 21, 23, 24]. Important methods in the asymptotic theory of evolutionary equations are the so-called Datko, Rolewicz, Zabczyk characterizations (see [2, 6, 27] and the presentation in [22]). The study in this paper is in this direction. The present paper considers three concepts of uniform h-dichotomy, weak uniform h-dichotomy and provides new characterizations of Datko type. We study h-dichotomy with growth rates for linear discrete time systems in Banach spaces with respect to invariant and strongly invariant projections sequences. The properties presented in the first part of the preliminary section have been previously obtained in [2] and [4], where the authors considered the notion of h-dichotomy with invariant sequence projections. After that, we present new criteria. We mention that various concepts of polynomial dichotomies have been explored by Dragičević [7], Dragičević, Sasu and Sasu [8], Seimeanu [25], and Seimeanu and Megan [26]. In addition, this work is motivated by the recent progress made in the dichotomy theory (see [1, 2, 4, 7, 8, 10, 14–16, 19, 20, 23–26]).

2 Preliminaries

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X into itself. The norms on both these spaces will be denoted by $\|\cdot\|$. Let $\mathbb{N}$ denote the set of all positive intergers and we denote by Δ and T the following sets:

$$\Delta = \{(m, n) \in \mathbb{N}^2 : m \geq n\} \quad T = \{(m, n, p) \in \mathbb{N}^3 : m \geq n \geq p\}.$$

In this paper we consider linear discrete-time systems of the form

$$(\mathcal{A}) \qquad x_{n+1} = A_n x_n, \quad n \in \mathbb{N}$$

where (A_n) is a sequence in $\mathcal{B}(X)$.
Then every solution $x = (x_n)$ of system $(\mathcal{A})$ is given by

$$x_m = A_m^n x_n, \quad \text{for all } (m, n) \in \Delta,$$

where

$$A_m^n = \begin{cases} A_{m-1}A_{m-2}...A_n, & m > n \\ I, & m = n \end{cases}$$

and I is the identity operator on X.

Remark 2.1. We have the following properties:

(i) $A_{n+1}^n = A_n$, for all $n \in \mathbb{N}$;

(ii) $A_m^n A_n^p = A_m^p$, for all $(m, n, p) \in T$.

Definition 2.1. *A nondecreasing sequence (h_n) on $[1, \infty)$ is called a growth rate sequence if* $\lim\limits_{n\to\infty} h_n = \infty$.

Definition 2.2. *A sequence (P_n) on $\mathcal{B}(X)$ is called a sequence of projections on X if $P_n^2 = P_n$, for all $n \in \mathbb{N}$.*

Remark 2.2. If (P_n) is a sequence of projections on X, then the sequence $Q_n = I - P_n$ is also a projections sequence on X, which is called the *complementary projections sequence of P_n*. Then $Ker(Q_n) = Range(P_n)$ and $Range(Q_n) = Ker(P_n)$ and $P_nQ_n = Q_nP_n = 0$ for all $n \in \mathbb{N}$.

Definition 2.3. *The sequence (P_n) is said to be invariant for the linear system $(\mathcal{A})$, if*

$$A_n P_n = P_{n+1} A_n, \quad \text{for all } n \in \mathbb{N}.$$

Remark 2.3. If (P_n) is invariant for $(\mathcal{A})$ then

$$A_m^n P_n = P_m A_m^n, \quad A_m^n Q_n = Q_m A_m^n,$$

for all $(m, n) \in \Delta$.

Remark 2.4. If the sequence of projections (P_n) is invariant for the linear system $(\mathcal{A})$, then the complementary sequence of projections (Q_n) is also invariant for the linear system $(\mathcal{A})$.

Definition 2.4. *The sequence of projections (P_n) is said to be strongly invariant for the linear system $(\mathcal{A})$ if it is invariant for $(\mathcal{A})$ and the restriction of A_m^n is an isomorphism from $Range(Q_n)$ to $Range(Q_m)$.*

Remark 2.5. If the sequence of projections (P_n) is a strongly invariant for the system $(\mathcal{A})$, then there exists $B : \Delta \to \mathcal{R}$ with $B(m,n) = B_m^n : Range(Q_m) \to Range(Q_n)$ isomorphism from $Range(Q_m)$ to $Range(Q_n)$ with the following properties:

(i) $A_m^n B_m^n Q_m = Q_m$,

(ii) $B_m^n A_m^n Q_n = Q_n$,

(iii) $B_m^p Q_m = B_n^p B_m^n Q_m$,

for all $(m, n), (n, p) \in \Delta$.

Proof. See [2]. □

2.1 h-Dichotomy with invariant sequence of projections

Definition 2.5. *Let (P_n) be a sequence projections invariant for the linear system $(\mathcal{A})$. We say that the pair $(\mathcal{A}, P)$ is uniformly h-dichotomic (u.h.d.) if there exist $N \geq 1$ and $\nu > 0$ with:*

(uhd$_1$) $h_m^\nu ||A_m^n P_n x|| \leq N h_n^\nu ||P_n x||$,

(uhd$_2$) $h_m^\nu ||Q_n x|| \leq N h_n^\nu ||A_m^n Q_n x||$,

for all $(m, n, x) \in \Delta \times X$.

Remark 2.6. Let (P_n) be invariant. We note that the pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exist $N \geq 1$ and $\nu > 0$ such that:

(uhd$'_1$) $h_m^\nu ||A_m^p P_p x|| \leq N h_n^\nu ||A_n^p P_p x||$,

(uhd$'_2$) $h_m^\nu ||A_n^p Q_p x|| \leq N h_n^\nu ||A_m^p Q_p x||$,

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity. For $(uhd_1) \implies (uhd_1')$ we have

$$||A_m^p P_p x|| = ||A_m^n P_n A_n^p P_p x|| \leq N\left(\frac{h_n}{h_m}\right)^{\nu} ||P_n A_n^p P_p x|| \leq N\left(\frac{h_n}{h_m}\right)^{\nu} ||A_n^p P_p x||$$

and for $(uhd_2) \implies (uhd_2')$ we have

$$||A_m^p Q_p x|| = ||A_m^n Q_n A_n^p Q_p x|| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^{\nu} ||Q_n A_n^p Q_p x|| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^{\nu} ||A_n^p Q_p x||.$$

Sufficiency. It is obvious, talking $p = n$ both in (uhd_1') and (uhd_2').

□

Remark 2.7. 1. For $h_m = e^m$, for all $m \in \mathbb{N}$ in Definition 2.5, one obtains the *uniform exponential dichotomy* property, in short (u.e.d.).

2. For $h_m = m+1$, for all $m \in \mathbb{N}$ in Definition 2.5, one obtains the *uniform polynomial dichotomy* property, in short (u.p.d.).

Definition 2.6. *We say that the pair $(\mathcal{A}, P)$ has uniform h-growth (u.h.g) if there exist $M \geq 1$ and $\omega > 0$ such that:*

(uhg_1) $h_n^{\omega}||A_m^n P_n x|| \leq M h_m^{\omega}||P_n x||$,

(uhg_2) $h_n^{\omega}||Q_n x|| \leq M h_m^{\omega}||A_m^n Q_n x||$,

for all $(m, n, x) \in \Delta \times X$.

Remark 2.8. We note that the pair $(\mathcal{A}, P)$ has uniform h-growth if and only if there exist $M \geq 1$ and $\omega > 0$ such that:

(uhg_1') $h_n^{\omega}||A_m^p P_p x|| \leq M h_m^{\omega}||A_n^p P_p x||$,

(uhg_2') $h_n^{\omega}||A_n^p Q_p x|| \leq M h_m^{\omega}||A_m^p Q_p x||$,

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity: For $(uhg_1) \implies (uhg_1')$ we have

$$h_n^{\omega}||A_m^p P_p x|| = h_n^{\omega}||A_m^n P_n A_n^p P_p x|| \leq M h_m^{\omega}||P_n A_n^p P_p x|| \leq M h_m^{\omega}||A_n^p P_p x||$$

and for $(uhg_2') \implies (uhg_2')$ we have

$$||A_m^p Q_p x|| = ||A_m^n Q_n A_n^p Q_p x|| \geq \frac{1}{M}\left(\frac{h_n}{h_m}\right)^{\omega} ||Q_n A_n^p Q_p x|| \geq \frac{1}{M}\left(\frac{h_n}{h_m}\right)^{\omega} ||A_n^p Q_p x||.$$

Sufficiency: It is obvious for $p = n$ in (uhg_1') and (uhg_2'). □

Remark 2.9. The particular cases for the concept of uniform h-growth are:

1. If $h_m = e^m$, the pair $(\mathcal{A}, P)$ has uniform exponential growth, in short (u.e.g.).
2. If $h_m = m + 1$, the pair $(\mathcal{A}, P)$ has uniform polynomial growth, in short (u.p.g.).

Definition 2.7. *Let (P_n) be a projections sequence which is invariant for the linear system $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is said to be weakly uniformly h-dichotomic (w.u.h.d.) if there exist $N \geq 1$ and $\nu > 0$ with:*

(wuhd$_1$) $h_m^\nu ||A_m^n P_n|| \leq N h_n^\nu ||P_n||$,

(wuhd$_2$) $h_m^\nu ||Q_n|| \leq N h_n^\nu ||A_m^n Q_n||$,

for all $(m,n) \in \Delta$.

Remark 2.10. Let (P_n) be a projections sequence which is invariant for the linear system $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is weakly uniformly h-dichotomic if and only if there exist $N \geq 1$ and $\nu > 0$ such that:

$(wuhd_1')$ $h_m^\nu ||A_m^p P_p|| \leq N h_n^\nu ||A_n^p P_p||$,

$(wuhd_2')$ $h_m^\nu ||A_n^p Q_p|| \leq N h_n^\nu ||A_m^p Q_p||$,

for all $(m,n,p) \in T$.

Proof. Necessity. For $(wuhd_1) \implies (wuhd_1')$ we have

$$h_m^\nu ||A_m^p P_p|| = h_m^\nu ||A_m^n P_n A_n^p P_p|| \leq N h_n^\nu ||P_n A_n^p P_p|| \leq N h_n^\nu ||A_n^p P_p||$$

for all $(m,n,p) \in T$.

For $(wuhd_2) \implies (wuhd_2')$ we have

$$||A_m^p Q_p|| = ||A_m^n Q_n A_n^p Q_p|| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^\nu ||Q_n A_n^p Q_p|| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^\nu ||A_n^p Q_p||.$$

Sufficiency. It is obvious, talking $p = n$ both in $(wuhd_1')$ and $(wuhd_2')$. □

Remark 2.11. 1. If $h_m = e^m$ for all $m \in \mathbb{N}$, we obtain the concept of *weak uniform exponential dichotomy* (w.u.e.d.).

2. If $h_m = m + 1$ for all $m \in \mathbb{N}$, we obtain the concept of *weak uniform polynomial dichotomy* (w.u.p.d.).

Definition 2.8. *We say that the pair $(\mathcal{A}, P)$ has weak uniform h-growth (w.u.h.g) if there exist $M \geq 1$ and $\omega > 0$ such that:*

(wuhg$_1$) $h_n^\omega ||A_m^n P_n|| \leq M h_m^\omega ||P_n||$,

(wuhg$_2$) $h_n^\omega ||Q_n|| \leq M h_m^\omega ||A_m^n Q_n||$,

for all $(m,n) \in \Delta$.

Remark 2.12. We note that the pair $(\mathcal{A}, P)$ has weak uniform h-growth if and only if there exist $M \geq 1$ and $\omega > 0$ such that:

$(wuhg_1')$ $h_n^\omega ||A_m^p P_p|| \leq M h_m^\omega ||A_n^p P_p||$,

$(wuhg_2')$ $h_n^\omega ||A_n^p Q_p|| \leq M h_m^\omega ||A_m^p Q_n||$, for all $(m, n, p) \in T$.

Proof. Necessity: For $(wuhg_1) \implies (wuhg_1')$ we have

$$h_n^{\omega}||A_m^p P_p|| = h_n^{\omega}||A_m^n P_n A_n^p P_p|| \leq M h_m^{\omega}||A_n^p P_p||$$

and for $(wuhg_2) \implies (wuhg_2')$ we have

$$||A_m^p Q_p|| = ||A_m^n Q_n A_n^p Q_p|| \geq \frac{1}{M}\left(\frac{h_n}{h_m}\right)^{\omega}||Q_n A_n^p Q_p|| \geq \frac{1}{M}\left(\frac{h_n}{h_m}\right)^{\omega}||A_n^p Q_p||.$$

Sufficiency: It is obvious for $p = n$ in $(wuhg_1')$ and $(wuhg_2')$. □

Remark 2.13. The particular cases for the concept of weak uniform h-growth are:

1. If $h_m = e^m$, we obtain the concept of weak uniform exponential growth, in short (w.u.e.g.).
2. If $h_m = m+1$, we obtain the concept of weak uniform polynomial growth, in short (w.u.p.g.).

2.2 h-Dichotomy with strongly invariant sequence of projections

Proposition 2.1. *Let (P_n) be a strong invariant sequence projections for the linear system $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exist $N \geq 1$, $\nu > 0$ with:*

(uhd_1^s) $h_m^{\nu}||A_m^p P_p x|| \leq N h_n^{\nu}||A_n^p P_p x||$,

(uhd_2^s) $h_n^{\nu}||B_m^p Q_m x|| \leq N h_p^{\nu}||B_m^n Q_m x||$,

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity. If the pair $(\mathcal{A}, P)$ is u.h.d. with the respect to the sequence of projections (P_n), then

$$h_m^{\nu}||A_m^p P_p x|| = h_m^{\nu}||A_m^n P_n A_n^p P_p x|| \leq N h_n^{\nu}||A_n^p P_p x||$$

and

$$\begin{aligned} h_n^{\nu}||B_m^p Q_m x|| &= h_n^{\nu}||B_n^p B_m^n Q_m x|| = h_n^{\nu}||B_n^p Q_n B_m^n Q_m x|| \\ &\leq N h_p^{\nu}||Q_n B_m^n Q_m x|| = N h_p^{\nu}||B_m^n Q_m x||, \end{aligned}$$

for all $(m, n, p, x) \in T \times X$.

Sufficiency. This is straightforward, for $p = n$ in (uhd_1^s) and $p = m$ in (uhd_2^s). □

Proposition 2.2. *Let (P_n) be a strong invariant sequence projections for the linear system $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is uniformly h- dichotomic if and only if there exist $N \geq 1$, $\nu >$ 0 such that:*

$(uhd_1^{s'})$ $h_m^{\nu}||A_m^n P_n x|| \leq N h_n^{\nu}||P_n x||$,

$(uhd_2^{s'})$ $h_m^{\nu}||B_m^n Q_m x|| \leq N h_n^{\nu}||Q_m x||$,

for all $(m, n, x) \in \Delta \times X$.

Proof. We only have to prove the equivalence between the $(uhd_2^s) \iff (uhd_2^{s'})$.

For the *necessity*, from (uhd_2) and the equalities (i) and (ii) from Remark 2.5 we have that

$$\begin{aligned} h_m^\nu ||B_m^n Q_m x|| &= h_m^\nu ||Q_n B_m^n Q_m x|| \le N h_n^\nu ||A_m^n Q_n B_m^n Q_m x|| \\ &= N h_n^\nu ||Q_m A_m^n B_m^n Q_m x|| = N h_n^\nu ||Q_m x||. \end{aligned}$$

The *sufficiency* follows similarly, by using (ii) from Remark 2.5 and (uhd_2^s) in the following manner:

$$h_m^\nu ||Q_n x|| = ||B_m^n A_m^n Q_n x|| = ||B_m^n Q_m A_m^n Q_n x|| \le N h_m^\nu ||A_m^n Q_n x||.$$

□

Definition 2.9. *Let (P_n) be a sequence of projections which is strongly invariant for $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is said to be weakly uniformly h-dichotomic if and only if there exist $N \ge 1$ and $\nu > 0$ such that:*

$(wuhd_1^s)$ $h_m^\nu ||A_m^n P_n|| \le N h_n^\nu ||P_n||$,

$(wuhd_2^s)$ $h_m^\nu ||B_m^n Q_m|| \le N h_n^\nu ||Q_m||$,

for all $(m, n) \in \Delta$.

Remark 2.14. Let (P_n) be strongly invariant for $(\mathcal{A})$. The pair $(\mathcal{A}, P)$ is weakly uniformly h-dichotomic if there exist $N \ge 1$ and $\nu > 0$ with:

$(wuhd_1^{s'})$ $h_m^\nu ||A_m^p P_p|| \le N h_n^\nu ||A_n^p P_p||$,

$(wuhd_2^{s'})$ $h_n^\nu ||B_m^p Q_m|| \le N h_p^\nu ||B_m^n Q_m||$,

for all $(m, n, p) \in T$.

3 Main results

In this paper, we will consider $\mathcal{H}$ the set of growth rates (h_n) considered in [2], Section 3, which satisfy the following properties:

(i) $\exists H > 1 : h_{n+1} \le H h_n, \forall n \in \mathbb{N}$;

(ii) $\forall \alpha \in (-1, 0),\ \exists\ H_1 > 1 : \sum_{k=m}^{\infty} h_k^\alpha \le H_1 h_m^\alpha,\ \forall m \in \mathbb{N}$;

(iii) $\forall \alpha \in (0, 1),\ \exists\ H_2 > 1 : \sum_{j=0}^{m} h_j^\alpha \le H_2 h_m^\alpha,\ \forall m \in \mathbb{N}$.

Theorem 3.1. *[2] If the pair $(\mathcal{A}, P)$ has uniform h-growth with $h \in \mathcal{H}$, then:*

(i) the pair $(\mathcal{A}, P)$ is uniformly h-dichotomic;

(ii) there exists constants $D \geq 1$ and $d > 0$ such that the following properties hold:

$$(uhD_1') \sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^d ||A_n^p P_p x||, \quad \forall\ (n,p,x) \in \Delta \times X,$$

$$(uhD_2') \sum_{k=n}^{\infty} \frac{h_k^d}{||A_k^p Q_p x||} \leq \frac{D h_n^d}{||A_n^p Q_p x||}, \quad \forall\ (n,p,x) \in \Delta \times X \text{ and } A_n^p Q_p x \neq 0.$$

Proof. See (uhD_1^1) in [2, Theorem 2, (uhD_1)] and (uhD_1^2) in [2, Theorem 3, (uhD_2')]. □

Theorem 3.2. *[2] If $(\mathcal{A}, P)$ has uniform h-growth, $h \in \mathcal{H}$, then the pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exists $D \geq 1$, $d > 0$ with:*

$$(uhD_1^2) \sum_{j=p}^{m} \frac{h_j^{-d}}{||A_j^p P_p x||} \leq \frac{D h_m^{-d}}{||A_m^p P_p x||}, \text{ for all } (m,n,p,x) \in T \times X \text{ and } A_m^p P_p x \neq 0,$$

$$(uhD_2^2) \sum_{j=p}^{m} \frac{||A_j^p Q_p x||}{h_j^d} \leq \frac{D||A_m^p Q_p x||}{h_m^d}, \text{ for all } (m,n,p,x) \in T \times X.$$

Proof. See (uhD_2^1) in [2, Theorem 2, (uhD_1')] and (uhD_2^2) in [2, Theorem 3, (uhD_2)]. □

We consider (P_n) a sequence of projections strongly invariant for $(\mathcal{A})$ and (Q_n) the complementary projections sequence of (P_n).

Theorem 3.3. *If $(\mathcal{A}, P)$ has uniform h-growth and $h \in \mathcal{H}$, the pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exists $D \geq 1$, $d > 0$ with*

$$(uhD_1^s) \sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^d ||A_n^p P_p x|| \text{ for all } (m,n,p,x) \in T \times X,$$

$$(uhD_2^s) \sum_{k=p}^{\infty} \frac{h_k^d}{||B_m^k Q_m x||} \leq \frac{D h_p^d}{||B_m^p Q_m x||}, \text{ for all } x \in X \text{ with } B_m^p Q_m \neq 0 \text{ and } (m,n,p) \in T.$$

Proof. For the *necessity*, it is obvious using Theorem 3.1 that we have to prove only the second condition. Since the pair $(\mathcal{A}, P)$ is uniformly h-dichotomic, we have $N \geq 1$, $\nu > 0$ such that we have (uhD_2^s)

$$\sum_{k=p}^{\infty} \frac{h_k^d}{||B_m^k Q_m x||} \leq \sum_{k=p}^{\infty} N h_k^d \left(\frac{h_p}{h_k}\right)^{\nu} \cdot \frac{1}{||B_m^p Q_m x||}$$
$$\leq \frac{N h_p^{\nu}}{||B_m^p Q_m x||} \sum_{k=p}^{\infty} h_k^{d-\nu} \leq \frac{N H_1 h_p^d}{||B_m^p Q_m x||} = \frac{D h_p^d}{||B_m^p Q_m x||},$$

where $D = N H_1 \geq 1$ for all $x \in X$, $B_m^p Q_m \neq 0$.

Suppose there exists $k_0 \geq p$ such that $B_m^{k_0} Q_m x = 0$. Since Q_m is a projection, we have $Q_m x \in Range(Q_m)$. By hypothesis, $B_m^{k_0}$ is a linear isomorphism. In particular, it is injective, so from $B_m^{k_0} Q_m x = 0$ we get $Q_m x = 0$. But this contradicts the assumption $B_m^p Q_m x \neq 0$. Indeed, since B_m^p is linear, it would follow that $B_m^p Q_m x = B_m^p(0) = 0$ which contradicts $B_m^p Q_m x \neq 0$. Therefore, our assumption must be false, and we conclude that $B_m^k Q_m x \neq 0$ for all $k \geq p$.

For the *sufficiency*, if we consider $k = n$ in (uhD_2^s), we have

$$\frac{h_n^d}{||B_m^n Q_m x||} \leq \frac{D h_p^d}{||B_m^p Q_m x||}$$

which is equivalent to

$$h_n^d ||B_m^p Q_m x|| \leq D h_p^d ||B_m^n Q_m x||.$$

Thus, we obtain (uhd_2^s), for $D = N, d = \nu$. □

Theorem 3.4. *If $(\mathcal{A}, P)$ has uniform h-growth, $h \in \mathcal{H}$, then the pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exist $D \geq 1,\ d > 0$ with:*

$$(uhD_2^{s_1}) \quad \sum_{j=n}^{m} \frac{h_j^{-d}}{||A_j^n P_n x||} \leq \frac{D h_m^{-d}}{||A_m^n P_n x||}, \text{ for all } (m,n,x) \in \Delta \times X \text{ and } A_m^n P_n x \neq 0,$$

$$(uhD_2^{s_2}) \quad \sum_{j=n}^{m} \frac{||B_m^j Q_m x||}{h_j^d} \leq \frac{D ||Q_m x||}{h_m^d}, \text{ for all } (m,n,p,x) \in T \times X.$$

Proof. See $(uhD_1^{s_1})$ in [2, Theorem 2, (uhD_1')] and $(uhD_2^{s_2})$ in [2, Theorem 4, (uhD_2^s)]. □

Theorem 3.5. *If the pair $(\mathcal{A}, P)$ has weak uniform h-growth with $M \geq 1,\ \omega > 0$ and $h \in \mathcal{H}$, then we have:*

(i) The pair $(\mathcal{A}, P)$ is weakly uniformly h-dichotomic.

(ii) There exists constants $D \geq 1$ and $d > 0$ such that the following properties hold:

$$(wuhD_1^1) \quad \sum_{k=n}^{\infty} h_k^d ||A_k^p P_p|| \leq D h_n^d ||A_n^p P_p||, \ \forall\ (n,p) \in \Delta,$$

$$(wuhD_1^2) \quad \sum_{k=n}^{\infty} \frac{h_k^d}{||A_k^p Q_p||} \leq \frac{D h_n^d}{||A_n^p Q_p||}, \ \forall\ (n,p) \in \Delta \text{ and } A_n^p Q_p \neq 0.$$

(iii) There exists constants $D \geq 1$ and $d > 0$ such that the following properties hold:

$$(wuhD_1^3) \quad \sum_{k=n}^{\infty} h_k^d ||A_k^n P_n|| \leq D h_n^d ||P_n||, \ \forall\ (m,n) \in \Delta,$$

$$(wuhD_1^4) \quad \sum_{k=n}^{\infty} \frac{h_k^d}{||A_k^n Q_n||} \leq \frac{D h_n^d}{||Q_n||}, \ \forall\ n \in \mathbb{N} \text{ and } Q_n \neq 0.$$

Proof. $(i) \implies (ii)$ The pair $(\mathcal{A}, P)$ is weakly uniformly h-dichotomic, so we have that there exist two constants $N \geq 1$, $d \in (0, \nu)$, such that we have

$(wuhD_1^1)$

$$\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p|| \leq \sum_{k=n}^{\infty} h_k^d N \left(\frac{h_k}{h_n}\right)^{-\nu} ||A_n^p P_p|| \leq N h_n^{\nu} ||A_n^p P_p|| \sum_{k=n}^{\infty} h_k^{d-\nu} \leq D h_n^d ||A_n^p P_p||$$

where $D = NH_1 \geq 1$, and

$(wuhd_1^2)$

$$\sum_{k=n}^{\infty} \frac{h_k^d}{||A_k^p Q_p||} \leq \sum_{k=n}^{\infty} h_k^d \cdot N \cdot \left(\frac{h_k}{h_n}\right)^{\nu} \cdot \frac{1}{||A_n^p Q_p||} \leq N h_n^{\nu} \cdot \frac{1}{||A_n^p Q_p||} \cdot \sum_{k=n}^{\infty} h_k^{d-\nu} \leq \frac{N H_1 h_n^d}{||A_n^p Q_p||} = \frac{D h_n^d}{||A_n^p Q_p||},$$

where $D = NH_1 \geq 1$ with $A_n^p Q_p \neq 0$. We have that $A_k^p = A_k^n A_n^p Q_p$. Since both $A_n^p Q_p \neq 0$ and A_k^n is a linear operator, it follows that $A_k^p Q_p \neq 0$ for $k \geq n \geq p$.

$(ii) \implies (iii)$ For $p = n$ in $(wuhD_1^1)$ we have:

$$\sum_{k=n}^{\infty} h_k^d ||A_k^n P_n|| \leq D h_n^d ||A_n^n P_n|| = D h_n^d ||P_n||$$

for all $n \in \mathbb{N}$.
For $p = n$ in $(wuhD_1^2)$ we have:

$$\sum_{k=n}^{\infty} \frac{h_k^d}{||A_k^n Q_n||} \leq \frac{D h_n^d}{||A_n^n Q_n||} = \frac{D h_n^d}{||Q_n||}$$

for $Q_n \neq 0$.

$(iii) \implies (i)$ We have to prove $(wuhD_1^3) \implies (wuhd_1)$. The pair $(\mathcal{A}, P)$ has $(u.h.g)$ with $M \geq 1$ and $\omega > 0$ such that we have:

$$\begin{aligned} ||A_m^p P_p|| = ||A_m^n A_n^p P_p|| &\leq M \left(\frac{h_m}{h_n}\right)^{\omega} ||A_n^p P_p|| \leq M \left(\frac{h_m}{h_n}\right)^{\omega} \cdot \left(\frac{h_m}{h_n}\right)^{d} \left(\frac{h_m}{h_n}\right)^{-d} ||A_n^p P_p|| \\ &\leq M H^{\omega+d} \cdot \left(\frac{h_m}{h_n}\right)^{-d} \cdot ||A_n^p P_p|| = N \left(\frac{h_m}{h_n}\right)^{-d} ||A_n^p P_p||, \end{aligned}$$

where $N = MH^{\omega+d} \geq 1$ and $\nu = d$.

Secondly, we have to prove $(wuhD_1^4) \implies (wuhd_2)$. Note that

$$||A_m^p Q_p|| = ||A_m^n A_n^p Q_p|| \geq \frac{1}{M} \left(\frac{h_n}{h_m}\right)^{\omega} ||A_n^p Q_p||,$$

which is equivalent to

$$\begin{aligned} ||A_n^p Q_p|| &\leq M \left(\frac{h_m}{h_n}\right)^{\omega+d} \cdot \left(\frac{h_m}{h_n}\right)^{-d} ||A_m^p Q_p|| \\ &\leq M H^{\omega+d} \left(\frac{h_m}{h_n}\right)^{-d} ||A_m^p Q_p|| = N \left(\frac{h_m}{h_n}\right)^{-d} ||A_m^p Q_p||, \end{aligned}$$

where $N = MH^{\omega+d} \geq 1$ and $\nu = d$.

$\square$

Theorem 3.6. *If $(\mathcal{A}, P)$ has weak uniform h-growth, $h \in \mathcal{H}$, then the pair $(\mathcal{A}, P)$ is weakly uniformly h-dichotomic if and only if exists $D \geq 1,\ d > 0$ with:*

$(wuhD_2^1)$ $\displaystyle\sum_{j=p}^{m} \frac{h_j^{-d}}{||A_j^p P_p||} \leq \frac{D h_m^{-d}}{||A_m^p P_p||}$, *for all $(m,n,p) \in T$ and $A_m^p P_p \neq 0$,*

$(wuhD_2^2)$ $\displaystyle\sum_{j=p}^{m} \frac{||A_j^p Q_p||}{h_j^d} \leq \frac{D||A_m^p Q_p||}{h_m^d}$, *for all $(m,n,p) \in T$.*

Proof. Necessity. The pair $(\mathcal{A}, P)$ is weakly uniformly h-dichotomic, so we have $N \geq 1$, $d \in (0, \nu)$, such that we have:

$(wuhD_2^1)$

$$\sum_{j=p}^{m} \frac{h_j^{-d}}{||A_j^p P_p||} \leq \sum_{j=p}^{m} N h_j^{-\nu} \cdot \left(\frac{h_j}{h_m}\right)^{\nu} \cdot \frac{1}{||A_m^p P_p||} \leq \frac{N h_m^{-\nu}}{||A_m^p P_p||} \cdot \sum_{j=p}^{m} h_j^{\nu-d}$$
$$\leq \frac{N H_2 h_m^{-d}}{||A_m^p P_p||} = \frac{D h_m^{-d}}{||A_m^p P_p|||},$$

where $D = NH_2 \geq 1$ and $A_m^p P_p \neq 0$. Suppose that for some $j \in \{p, p+1, ..., m\}$, we have $A_j^p P_p = 0$. Then, for any $k \geq j$, we can write $A_k^p P_p = A_k^j A_j^p P_p = A_j^k \cdot 0 = 0$. In particular, this implies $A_m^p P_p = 0$, which contradicts the assumption $A_m^p P_p \neq 0$. Therefore, it must be that $A_j^p P_p \neq 0$ for all $j \in \{p, ..., m\}$.

Moreover, we have

$(wuhD_2^2)$

$$\sum_{j=p}^{m} \frac{||A_j^p Q_p||}{h_j^d} \leq \sum_{j=p}^{m} N h_j^{-d} \left(\frac{h_m}{h_j}\right)^{-d} ||A_m^p Q_p|| \leq N h_m^{-\nu} ||A_m^p Q_p|| \sum_{j=p}^{m} h_j^{\nu-d} \leq D h_m^{-d} ||A_m^p Q_p||,$$

where $D = NH_2 \geq 1$.

Sufficiency. We have to prove $(wuhD_2^1) \implies (wuhd_1)$ and $(wuhD_2^2) \implies (wuhd_2)$.

For $j = n$ in $(wuhD_2^1)$ we have

$$\frac{h_n^{-d}}{||A_n^p P_p||} \leq \frac{D h_m^{-d}}{||A_m^p P_p||},$$

which is equivalent to

$$h_m^d ||A_m^p P_p|| \leq D h_n^d ||A_n^p P_p||,$$

where $D = N$ and $\nu = d$.

For $j = n$ in $(wuhD_2^2)$ we have

$$\frac{||A_n^p Q_p||}{h_n^d} \leq \frac{D||A_m^p Q_p||}{h_m^d},$$

which is equivalent to

$$h_m^d ||A_n^p Q_p|| \leq D h_n^d ||A_m^p Q_p||,$$

where $D = N$ and $\nu = d$.

□

4 Conclusions

The present paper gives properties and characterizations for the general concept of uniform h-dichotomy, with respect to invariant projection sequences and with respect to strongly invariant projection sequences. The results are a continuation of the study performed in [2]. Also, using the idea of the characterizations of Datko-Zabczyk type for the general concept of uniform h-dichotomy ([2]), we obtain some characterizations for the concept of weak uniform h-dichotomy with respect to invariant projection sequences and with respect to strongly invariant projection sequences. More precisely Theorem 3.5 and Theorem 3.6 are novel.

Acknowledgement

The author would like to express deep gratitude to Profesor Emeritus Mihai Megan for his valuable and constructive suggestions and useful critiques during the planning and development of this research work.

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Received: 28.05.2024

Revised: 30.11.2024, 15.03.2025

Accepted: 15.06.2025