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Refinements of Aczél-Popoviciu and Bellman Inequalities

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Abstract Some refinements of the celebrated Aczél-Popoviciu and Bellman inequalities in both discrete and Lebesgue integral forms are provided. We also express Hölder and Aczél-Popoviciu inequalities in a monotonous sequence $S(m)$, and the special case $S(0) \geq S(+\infty)$ is our refinement.

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1 Introduction

The following theorem is well known in the literature as Hölder inequality [6].

Theorem 1.1. *Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $a_i, b_i \geq 0$, $i = 1, ..., n$. Then*

$$\left(\sum_{i=1}^{n} a_i^p \right)^{1/p} \left(\sum_{i=1}^{n} b_i^q \right)^{1/q} \geq \sum_{i=1}^{n} a_i b_i, \tag{1.1}$$

abbreviating as $A \geq B$.

The theorem below is known as Aczél-Popoviciu inequality, see [1], [7].

Theorem 1.2. *Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $a_i, b_i \geq 0$, $i = 1, ..., n$ such that $a_1^p - \sum\limits_{i=2}^{n} a_i^p > 0$, $b_1^q - \sum\limits_{i=2}^{n} b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p \right)^{1/p} \left(b_1^q - \sum_{i=2}^{n} b_i^q \right)^{1/q} \leq a_1 b_1 - \sum_{i=2}^{n} a_i b_i, \tag{1.2}$$

abbreviating as $A' \leq B'$.

The case when $p = q = 2$, provides Aczél inequality.

The following theorem is the famous Minkowski inequality [6].

Theorem 1.3. *Let $p > 1$ and $a_i, b_i \geq 0$, $i = 1, ..., n$. Then*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p} \geq \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{1/p}, \tag{1.3}$$

abbreviating as $C \geq D$.

A converse of this result is incorporated in the theorem below and is known as Bellman inequality [2]:

Theorem 1.4. *Let $p > 1$ and $a_i, b_i \geq 0$, $i = 1, ..., n$ such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$, $b_1^p - \sum_{i=2}^{n} b_i^p > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^{n} b_i^p\right)^{1/p} \leq \left((a_1 + b_1)^p - \sum_{i=2}^{n} (a_i + b_i)^p\right)^{1/p}, \tag{1.4}$$

abbreviating as $C' \leq D'$.

The comparison between two sides of Aczél inequality and CBS inequality was first considered in [3]. A similar comparison for Aczél-Popoviciu inequality and Hölder inequality is established in [5]. Theorem 5 in [5] can also be seen as a comparison between the two sides of Bellman inequality and Minkowski inequality to some extent.

Motivated by these results, in the following, we prove refinements of Aczél-Popoviciu inequality and of Bellman inequality in both discrete and integral forms.

2 The discrete case

The following inequality is a refinement of Aczél-Popoviciu inequality:

Theorem 2.1. *Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $a_i, b_i \geq 0$, $i = 1, ..., n$ be such that*

$$a_1^p - \sum_{i=2}^{n} a_i^p > 0, \quad b_1^q - \sum_{i=2}^{n} b_i^q > 0. \tag{2.1}$$

Then

$$(0 \leq) \frac{\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p} \left(b_1^q - \sum_{i=2}^{n} b_i^q\right)^{1/q}}{a_1 b_1 - \sum_{i=2}^{n} a_i b_i} \leq \frac{\sum_{i=1}^{n} a_i b_i}{\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}} \leq 1. \tag{2.2}$$

Proof. Observe that

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{1/q} \left(\sum_{i=1}^n a_i^p\right)^{1/p} \left(\sum_{i=1}^n b_i^q\right)^{1/q} =$$

$$= \left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} \left(a_1^p + \sum_{i=2}^n a_i^p\right)^{1/p} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{1/q} \left(b_1^q + \sum_{i=2}^n b_i^q\right)^{1/q}$$

$$= \left(a_1^{2p} - \left(\sum_{i=2}^n a_i^p\right)^2\right)^{1/p} \left(b_1^{2q} - \left(\sum_{i=2}^n b_i^q\right)^2\right)^{1/q}$$

$$= \left(a_1^{2p} - \left[\left(\sum_{i=2}^n a_i^p\right)^{2/p}\right]^p\right)^{1/p} \left(b_1^{2q} - \left[\left(\sum_{i=2}^n b_i^q\right)^{2/q}\right]^q\right)^{1/q}.$$

Utilize Aczél-Popoviciu inequality for four numbers to get

$$\left(a_1^{2p} - \left[\left(\sum_{i=2}^n a_i^p\right)^{2/p}\right]^p\right)^{1/p} \left(b_1^{2q} - \left[\left(\sum_{i=2}^n b_i^q\right)^{2/q}\right]^q\right)^{1/q} \leq$$

$$\leq a_1^2 b_1^2 - \left(\sum_{i=2}^n a_i^p\right)^{2/p} \left(\sum_{i=2}^n b_i^q\right)^{2/q} \leq a_1^2 b_1^2 - \left(\sum_{i=2}^n a_i b_i\right)^2$$

$$= \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)\left(a_1 b_1 + \sum_{i=2}^n a_i b_i\right) = \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)\sum_{i=1}^n a_i b_i.$$

This proves the desired result (2.2). □

The following inequality abreviated as $C + C' \leq D + D'$ can be stated as well:

Theorem 2.2. *Let $p > 1$ and $a_i, b_i \geq 0$, $i = 1, ..., n$ be such that*

$$a_1^p - \sum_{i=2}^n a_i^p > 0, \quad b_1^p - \sum_{i=2}^n b_i^p > 0. \tag{2.3}$$

Then

$$\begin{aligned}(0 \leq) \left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p} - \left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p} \leq \\ \leq \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{1/p} - \left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} - \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p}.\end{aligned} \tag{2.4}$$

Proof. Use the proof by contradiction. Assume that the strict inequality

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p} + \left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p} >$$

$$> \left((a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p\right)^{1/p} + \left(\sum_{i=1}^n (a_i + b_i)^p\right)^{1/p}$$

holds for some sequences.

From the fact that $C \geq D \geq D' \geq C' > 0$ and the assumption $C + C' > D + D'$ above, we claim that

$$C^p + C'^p > (D + D' - C')^p + C'^p$$

and

$$(D + D' - C')^p + C'^p \geq D^p + D'^p.$$

The second inequality is due to the fact that $(D + D' - C')$, C' majorizes D, D' and that $f(x) = x^p$ $(p > 1)$ is a convex function on $[0, \infty)$.

If we combine the two inequalities above, we have

$$C^p + C'^p > D^p + D'^p,$$

which is

$$\left[\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p}\right]^p + \left[\left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}\right]^p > 2(a_1 + b_1)^p.$$

However, by reverse Minkowski inequality for positive numbers x_i, y_i, $i = 1, 2$;

$$\left(x_1^{1/p} + x_2^{1/p}\right)^p + \left(y_1^{1/p} + y_2^{1/p}\right)^p \leq ((x_1 + y_1)^{1/p} + (x_2 + y_2)^{1/p})^p, \; p > 1, \tag{2.5}$$

we have that

$$\left[\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{1/p}\right]^p + \left[\left(\sum_{i=1}^n a_i^p\right)^{1/p} + \left(\sum_{i=1}^n b_i^p\right)^{1/p}\right]^p \leq 2(a_1 + b_1)^p.$$

So the assumption is wrong, hence the inequality (2.4) is proved. □

There's another refinement for Theorem 2.1 as follows:

Proposition 2.3. *Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $a_i, b_i \geq 0$, $i = 1, ..., n$ be such that the condition (2.1) is valid. Then*

$$\begin{aligned}
(0 \leq) &\frac{\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{1/p} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{1/q}}{a_1 b_1 - \sum_{i=2}^n a_i b_i} \\
&\leq \frac{\left(\sum_{i=1}^n a_i b_i\right)\left(a_1^2 b_1^2 + \left(\sum_{i=2}^n a_i b_i\right)^2\right)}{\left(\sum_{i=1}^n a_i^p\right)^{1/p}\left(\sum_{i=1}^n b_i^q\right)^{1/q}\left(a_1^{2p} + \left(\sum_{i=2}^n a_i^p\right)^2\right)^{1/p}\left(b_1^{2q} + \left(\sum_{i=2}^n b_i^q\right)^2\right)^{1/q}} \qquad (2.6) \\
&\leq \frac{\sum_{i=1}^n a_i b_i}{\left(\sum_{i=1}^n a_i^p\right)^{1/p}\left(\sum_{i=1}^n b_i^q\right)^{1/q}} \leq 1.
\end{aligned}$$

Proof. For the second inequality, it's obvious that

$$\left(a_1^{2p}+\left(\sum_{i=2}^n a_i^p\right)^2\right)^{1/p}\left(b_1^{2q}+\left(\sum_{i=2}^n b_i^q\right)^2\right)^{1/q}\geq$$
$$\geq a_1^2b_1^2+\left(\sum_{i=2}^n a_i^p\right)^{2/p}\left(\sum_{i=2}^n b_i^q\right)^{2/q}\geq a_1^2b_1^2+\left(\sum_{i=2}^n a_ib_i\right)^2.$$

For the first inequality, the proof is similar to that of Theorem 2.1. Observe that

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q}\left(\sum_{i=1}^n a_i^p\right)^{1/p}\left(\sum_{i=1}^n b_i^q\right)^{1/q}$$
$$\times\left(a_1^{2p}+\left(\sum_{i=2}^n a_i^p\right)^2\right)^{1/p}\left(b_1^{2q}+\left(\sum_{i=2}^n b_i^q\right)^2\right)^{1/q}=$$
$$=\left(a_1^{2p}-\left(\sum_{i=2}^n a_i^p\right)^2\right)^{1/p}\left(b_1^{2q}-\left(\sum_{i=2}^n b_i^q\right)^2\right)^{1/q}$$
$$\times\left(a_1^{2p}+\left(\sum_{i=2}^n a_i^p\right)^2\right)^{1/p}\left(b_1^{2q}+\left(\sum_{i=2}^n b_i^q\right)^2\right)^{1/q}$$
$$=\left(a_1^{4p}-\left(\sum_{i=2}^n a_i^p\right)^4\right)^{1/p}\left(b_1^{4q}-\left(\sum_{i=2}^n b_i^q\right)^4\right)^{1/q}.$$

Utilize Aczél-Popoviciu inequality for four numbers to get

$$\left(a_1^{4p}-\left[\left(\sum_{i=2}^n a_i^p\right)^{4/p}\right]^p\right)^{1/p}\left(b_1^{4q}-\left[\left(\sum_{i=2}^n b_i^q\right)^{4/q}\right]^q\right)^{1/q}\leq$$
$$\leq a_1^4b_1^4-\left(\sum_{i=2}^n a_i^p\right)^{4/p}\left(\sum_{i=2}^n b_i^q\right)^{4/q}\leq a_1^4b_1^4-\left(\sum_{i=2}^n a_ib_i\right)^4$$
$$=\left[a_1^2b_1^2+\left(\sum_{i=2}^n a_ib_i\right)^2\right]\left[a_1^2b_1^2-\left(\sum_{i=2}^n a_ib_i\right)^2\right]$$
$$=\left[a_1^2b_1^2+\left(\sum_{i=2}^n a_ib_i\right)^2\right]\left(\sum_{i=1}^n a_ib_i\right)\left(a_1b_1-\sum_{i=2}^n a_ib_i\right).$$

This proves the desired result (2.6). □

Furthermore, defining

$$S(m)=\frac{\prod\limits_{k=0}^{m}\left((a_1b_1)^{2^k}+\left(\sum\limits_{i=2}^{n}a_ib_i\right)^{2^k}\right)}{\prod\limits_{k=0}^{m}\left(a_1^{p2^k}+\left(\sum\limits_{i=2}^{n}a_i^p\right)^{2^k}\right)^{1/p}\left(b_1^{q2^k}+\left(\sum\limits_{i=2}^{n}b_i^q\right)^{2^k}\right)^{1/q}},$$

we have the following conclusions.

Lemma 2.4. *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $a_i,b_i\geq 0$, $i=1,...,n$ be such that the condition (2.1) is valid. Then*

$$S(+\infty)=\frac{\left(a_1^p-\sum\limits_{i=2}^{n}a_i^p\right)^{1/p}\left(b_1^q-\sum\limits_{i=2}^{n}b_i^q\right)^{1/q}}{a_1b_1-\sum\limits_{i=2}^{n}a_ib_i}.$$

Proof. Note that

$$\begin{aligned}
&S(m)\cdot\frac{a_1b_1-\sum\limits_{i=2}^{n}a_ib_i}{\left(a_1^p-\sum\limits_{i=2}^{n}a_i^p\right)^{1/p}\left(b_1^q-\sum\limits_{i=2}^{n}b_i^q\right)^{1/q}}=\\
&=\frac{(a_1b_1)^{2^{m+1}}-\left(\sum\limits_{i=2}^{n}a_ib_i\right)^{2^{m+1}}}{\left(a_1^{p2^{m+1}}-\left(\sum\limits_{i=2}^{n}a_i^p\right)^{2^{m+1}}\right)^{1/p}\left(b_1^{q2^{m+1}}-\left(\sum\limits_{i=2}^{n}b_i^q\right)^{2^{m+1}}\right)^{1/q}}\\
&=L(m).
\end{aligned}$$

Let $m\to+\infty$, $L(m)=1$, thus the conclusion is proved. □

Theorem 2.5. *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $a_i,b_i\geq 0$, $i=1,...,n$ be such that the condition (2.1) is valid. Then*

$$\frac{\sum\limits_{i=1}^{n}a_ib_i}{\left(\sum\limits_{i=1}^{n}a_i^p\right)^{1/p}\left(\sum\limits_{i=1}^{n}b_i^q\right)^{1/q}}=S(0)\geq S(1)\geq...\geq S(+\infty)=\frac{\left(a_1^p-\sum\limits_{i=2}^{n}a_i^p\right)^{1/p}\left(b_1^q-\sum\limits_{i=2}^{n}b_i^q\right)^{1/q}}{a_1b_1-\sum\limits_{i=2}^{n}a_ib_i}.$$

Proof. Note that

$$\frac{S(m)}{S(m+1)}=\frac{\left(a_1^{p2^{m+1}}+\left(\sum\limits_{i=2}^{n}a_i^p\right)^{2^{m+1}}\right)^{1/p}\left(b_1^{q2^{m+1}}+\left(\sum\limits_{i=2}^{n}b_i^q\right)^{2^{m+1}}\right)^{1/q}}{(a_1b_1)^{2^{m+1}}+\left(\sum\limits_{i=2}^{n}a_ib_i\right)^{2^{m+1}}},$$

while

$$
\begin{aligned}
&\left(a_1^{p2^{m+1}}+\left(\sum_{i=2}^n a_i^p\right)^{2^{m+1}}\right)^{1/p}\left(b_1^{q2^{m+1}}+\left(\sum_{i=2}^n b_i^q\right)^{2^{m+1}}\right)^{1/q}\geq\\
&\geq (a_1b_1)^{2^{m+1}}+\left(\sum_{i=2}^n a_i^p\right)^{2^{m+1}/p}\left(\sum_{i=2}^n b_i^q\right)^{2^{m+1}/q}\\
&\geq (a_1b_1)^{2^{m+1}}+\left(\sum_{i=2}^n a_ib_i\right)^{2^{m+1}},
\end{aligned}
$$

so $\frac{S(m)}{S(m+1)}\geq 1$, and the inequality chain is proved. □

As can be seen, there are infinite terms between Hölder and Aczél-Popoviciu inequalities, and Proposition 2.3 is one case.

3 Integral version

The integral forms of Aczél-Popoviciu type inequalities are considered in [4], [8], [9], [10] for Riemann integrable functions or Lebesgue integrable functions. The theorem below is the Lebesgue integral form of Aczél-Popoviciu inequality.

Theorem 3.1. *Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $f\in L^p[a,b]$, $g\in L^q[a,b]$ such that*

$$\int_a^b |f(x)|^p\,dx\leq A^p,\quad \int_a^b |g(x)|^q\,dx\leq B^q \tag{3.1}$$

holds for some positive constants A, B. Then

$$(0\leq)\left(A^p-\int_a^b |f(x)|^p\,dx\right)^{1/p}\left(B^q-\int_a^b |g(x)|^q\,dx\right)^{1/q}\leq AB-\int_a^b |f(x)g(x)|\,dx. \tag{3.2}$$

With similar idea, we consider the integral version of Theorem 2.1 as follows.

Theorem 3.2. *With the assumptions of Theorem 3.1 we have*

$$
\begin{aligned}
(0\leq)&\frac{\left(A^p-\int_a^b |f(x)|^p\,dx\right)^{1/p}\left(B^q-\int_a^b |g(x)|^q\,dx\right)^{1/q}}{AB-\int_a^b |f(x)g(x)|\,dx}\\
&\leq\frac{AB+\int_a^b |f(x)g(x)|\,dx}{\left(A^p+\int_a^b |f(x)|^p\,dx\right)^{1/p}\left(B^q+\int_a^b |g(x)|^q\,dx\right)^{1/q}}\leq 1.
\end{aligned} \tag{3.3}
$$

Proof. The second inequality is obvious due to the discrete and integral version of Hölder inequality. For the first inequality, observe that

$$\begin{aligned}
&\left(A^p - \int_a^b |f(x)|^p\,dx\right)^{1/p} \left(B^q - \int_a^b |g(x)|^q\,dx\right)^{1/q} \\
&\times \left(A^p + \int_a^b |f(x)|^p\,dx\right)^{1/p} \left(B^q + \int_a^b |g(x)|^q\,dx\right)^{1/q} = \\
&= \left(A^{2p} - \left(\int_a^b |f(x)|^p\,dx\right)^2\right)^{1/p} \left(B^{2q} - \left(\int_a^b |g(x)|^q\,dx\right)^2\right)^{1/q} \\
&= \left(A^{2p} - \left[\left(\int_a^b |f(x)|^p\,dx\right)^{2/p}\right]^p\right)^{1/p} \left(B^{2q} - \left[\left(\int_a^b |g(x)|^q\,dx\right)^{2/q}\right]^q\right)^{1/q}.
\end{aligned}$$

Utilize Aczél-Popoviciu inequality for four numbers to get

$$\begin{aligned}
&\left(A^{2p} - \left[\left(\int_a^b |f(x)|^p\,dx\right)^{2/p}\right]^p\right)^{1/p} \left(B^{2q} - \left[\left(\int_a^b |g(x)|^q\,dx\right)^{2/q}\right]^q\right)^{1/q} \le \\
&\le A^2B^2 - \left(\int_a^b |f(x)|^p\,dx\right)^{2/p} \left(\int_a^b |g(x)|^q\,dx\right)^{2/q} \\
&\le A^2B^2 - \left(\int_a^b |f(x)g(x)|\,dx\right)^2 \\
&= \left(AB + \int_a^b |f(x)g(x)|\,dx\right)\left(AB - \int_a^b |f(x)g(x)|\,dx\right).
\end{aligned}$$

This proves the desired result (3.3). □

Similarly, we also consider integral version for Theorem 3.1.

Theorem 3.3. *Let $p > 1$ and $f,\ g \in L^p[a,b]$ such that*

$$\int_a^b |f(x)|^p\,dx \le A^p, \quad \int_a^b |g(x)|^p\,dx \le B^p \tag{3.4}$$

holds for some positive constants A, B. Then

$$\begin{aligned}
(0 \le) &\left(A^p + \int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(B^p + \int_a^b |g(x)|^p\,dx\right)^{1/p} \\
&- \left((A+B)^p + \int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p} \\
&\le \left((A+B)^p - \int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p} \\
&- \left(A^p - \int_a^b |f(x)|^p\,dx\right)^{1/p} - \left(B^p - \int_a^b |g(x)|^p\,dx\right)^{1/p}.
\end{aligned} \tag{3.5}$$

Proof. The proof is similar to that of Theorem 2.2. Use the proof by contradiction. Assume that

$$\begin{aligned}&\left(A^p-\int_a^b |f(x)|^p\,dx\right)^{1/p}+\left(B^p-\int_a^b |g(x)|^p\,dx\right)^{1/p}\\&+\left(A^p+\int_a^b |f(x)|^p\,dx\right)^{1/p}+\left(B^p+\int_a^b |g(x)|^p\,dx\right)^{1/p}>\\&>\left((A+B)^p+\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p}\\&+\left((A+B)^p-\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/p}\end{aligned}$$

holds for some functions. Based on this assumption and similar discussion in Theorem 2.2, we infer that

$$\begin{aligned}&\left[\left(A^p-\int_a^b |f(x)|^p\,dx\right)^{1/p}+\left(B^p-\int_a^b |g(x)|^p\,dx\right)^{1/p}\right]^p\\&+\left[\left(A^p+\int_a^b |f(x)|^p\,dx\right)^{1/p}+\left(B^p+\int_a^b |g(x)|^p\,dx\right)^{1/p}\right]^p\\&>2(A+B)^p.\end{aligned}$$

However, by reverse Minkowski inequality (2.5) we have

$$\begin{aligned}&\left[\left(A^p-\int_a^b |f(x)|^p\,dx\right)^{1/p}+\left(B^p-\int_a^b |g(x)|^p\,dx\right)^{1/p}\right]^p\\&+\left[\left(A^p+\int_a^b |f(x)|^p\,dx\right)^{1/p}+\left(B^p+\int_a^b |g(x)|^p\,dx\right)^{1/p}\right]^p\\&\le 2(A+B)^p.\end{aligned}$$

So the assumption is wrong, hence the inequality (3.5) is proved. □

Remark 3.1. In [3], [5], the difference between two sides of Aczél-Popoviciu inequality and Hölder inequality is compared, in this paper, we compare the ratio of two sides in Aczél-Popoviciu inequality and Hölder inequality in Theorem 2.1 and Proposition 2.3. Inequality $0\le C^p-D^p\le D'^p-C'^p$ is proved in [5] and we prove $0\le C-D\le D'-C'$ in Theorem 2.2.

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